ISSN 0078 - 2017

NOTAS DE LOGICA MATEMATICA

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PROCEEDINGS OF THE IX LATIN AMERICAN SYMPOSIUM ON MATHEMATICAL LOGIC

(Part 2)

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INMABB - CONICET
UNIVERSIDAD NACIONAL DEL SUR
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(*) *La publicación de este volumen ha sido subsidiada por el Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.*

INDICE

Alternatives to the Consequence-theoretic Approach to Metalogic.

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ABSTRACT. On the common approach, logic seems to be understood as an intuitively primary and utmost general conceptual framework which can be used, in turn, as a basis for the analysis and explanation of a large variety of different but related metalogical ideas such as (i) consequence, provability or deduction; (ii) deductive theory or dedúctively-closed system; (iii) consistency or compatibility; (iv) maximality or completeness of a kind, etc.
Yet, on the other hand, it may be interesting and even fruitful to think of logic as a complex construct to be analyzed into some simpler pieces which can be constructed from some other intuitively simpler and more primitive ideas. In fact, any of the ideas (i) - (v) can serve such purpose. To be more precise, a set of conditions to be imposed on any one of the ideas (i) - (v) can be found such that the logic which it determines is the classical logic. The sets of conditions characteristic of each of these ideas are shown to be pairwise equivalent (*i.e.* expressively equivalent, to be sure). In each case, the description is carried out in two stages or levels. At the first, set-theoretic or language-independent stage the object language under consideration is simply taken to be a non-empty set of completely unstructured sentences. At the second, logical or language-dependent stage the underlying language is assumed to be structured so that all of its logical constants are made explicit. A rather welcome by-product here is a separable characterization of each logical operator or of each set of mutually independent logical operators.

The use of the idea of consequence (i) is the most obvious option here. All the standard frameworks for metalogic are, in fact, consequence-theoretic. The best known abstract setting for this framework is A. Tarski's general consequence theory. The next framework for metalogic, mentioned above, is based on the idea of deductive theory (ii). The treatment of the set-theoretic component of this programme is motivated by Tarski's theory of systems and the lattice-theoretic characterization of closure systems but it departs from the two sources in a few details. A consistency-theoretic framework for metalogic, based on consistency (iii) as a primitive idea, is also intuitively obvious. However, an implementation of this programme inevitably requires that a definition of consistency be freed from negation-dependence and, indeed, from dependence on any logical constant. Within this programme, logical constants are explained in terms of consistency and it is for this reason that they cannot be used in the definition of the latter. In this paper it is shown that the equivalence of the consistency-theoretic framework with that based on the idea of consequence can, indeed, be established at the set-theoretic level, *i.e.* in the absence of any logical constant. A maximality-theoretic or completeness-theoretic framework appears to be intuitively less obvious. The only certain intuition here seems to be that completeness is a property of large sets. Besides, an explanation of logic in terms of maximality or completeness is affected by the fact that the idea can be categorized in more than one way, sometimes as a maximal extension mapping of a kind and sometimes as a family of maximally extended sets.

1. INTRODUCTION.

On the common approach, logic seems to be understood as an intuitively primary and utmost general conceptual framework which can be used, in turn, as a basis for the analysis and explanation of a variety of different but related metalogical ideas such as
(i) consequence, provability or deducibility;
(ii) deductive theory or deductively-closed system;

(iii) consistency or compatibility;
(iv) maximality or completeness of a kind, etc.

Yet, on the other hand, it may be interesting and even fruitful to think of logic as a complex construct to be analyzed into simpler pieces which can be constructed from some other intuitively simpler and more primitive ideas. In fact, it can be shown that any of the above mentioned ideas (i) - (v) can, in fact, serve such purpose. To be more precise, a set of conditions to be imposed on any one of these ideas (i) - (v) can be found such that the logic which it determines is the classical logic. The sets of conditions characteristic of each of these ideas are shown to be pairwise equivalent (expressively equivalent, to be sure). Also of interest is the problem of how, in each case, the conditions found can be modified in order to make the logic, which they determine, coincide with a non-classical logic, for instance, with the intuitionistic logic, the Johansson-minimal logic or the three-valued logic of Lukasiewicz. However, because of lack of space, the discussion of this problem is postponed.

In each case, the description is carried out in two stages or levels. At the first, set-theoretic or language-independent stage the language under consideration, or the object language as it is also called, is taken to be, simply, a non-empty set of completely unstructured sentences. At the second, logical or language-dependent stage the underlying language is assumed to be structured so that all of its logical connectives and/or quantifiers, *i.e.* all the logical constants in question are made explicit. A rather welcome by-product here is a separable characterization of each logical constant or of each set of mutually independent logical constants.

The use of the idea of deducibility (i) is the most obvious option here. The standard frameworks for metalogic are, in fact, consequence-theoretic. The best known abstract setting for this framework is A. Tarski's general consequence theory. (cf [8, 9, 10], [3], [11] and [1]). A variant of this theory is briefly discussed in this paper.

The next framework for metalogic, mentioned above, is based on the idea of deductive theory (ii). The treatment of the set-theoretic component of this programme is motivated by Tarski's theory of systems (cf [10]) and the lattice-theoretic characterization of closure systems (cf for instance, [2]) and it only departs from the two sources in a few details.

A consistency-theoretic framework for metalogic, based on consistency (iii) as a primitive idea, is also intuitively obvious. However, an implementation of this programme inevitably requires that a definition of consistency be freed from negation-dependence and, indeed, from dependence on any other logical constant. Logical constants are to be explained in terms of consistency and it is for this reason that they cannot be used in the definition of the latter. In this paper it is shown that the equivalence of the consistency-theoretic framework with that based on the idea of consequence can, indeed, be established at the set-theoretic level, *i.e.* in the absence of any logical constant.

A completeness-theoretic framework appears to be intuitively less obvious. The only certain intuition here seems to be that completeness is a property of large sets. Besides, an explanation of logic in terms of completeness is affected by the fact the idea can be categorized in more than one way, sometimes as a maximal extension mapping of a kind and sometimes as a family of maximally extended sets.

It seems that the above mentioned metalogical frameworks and their mutual relationships have not been subjected to a systematic study as yet. Only a few works going in this direction can be mentioned. They include papers [5], [12] and [7] from the REFERENCE LIST.

2. LANGUAGE.

Symbol S is used to denote the set of all sentences of some object language. It is assumed that S is non-empty. Members of S and subsets of S are symbolized as A, B, C,..., and as X, Y, Z,..., respectively, with or without subscripts. The expression $X \in \mathbf{Fin}(Y)$ means that X is a finite subset of Y. To simplify notation, expressions of the kind $\phi(X \cup \{A_1, A_2, ..., A_n\})$ are rendered as $\phi(X, A_1, A_2, ..., A_n)$. The set of all subsets of a set X and the set of all supersets of X (within S) are denoted and defined as $2^X = \{Y \subseteq S: Y \subseteq X\}$ and $2_X = \{Y \subseteq S: X \subseteq Y\}$, respectively. Clearly, $X \in 2^Y$ iff $Y \in 2_X$, $X \in 2_X$ and $2^S \subseteq 2_\emptyset$. A non-empty class **K** of subsets of S is directed iff for any $\mathbf{L} \in \mathbf{Fin}(\mathbf{K})$ there is $Y \in \mathbf{K}$ such that $\cup\mathbf{L} \subseteq Y$, or in other words, iff each finite subunion of the union $\cup\mathbf{K}$ is included into some member of **K**.

If # is a n-ary connective and $A_1, A_2, ..., A_n \in S$, then $\#A_1A_2...A_n \in S$. Thus, whenever a connective # is specified, S is taken to be closed under #. If # is binary, it is customary to write A#B instead of #AB. In what follows we are concerned, mainly, with the standard connectives:

$\neg$ (negation), $\rightarrow$ (conditional or implication), $\wedge$ (conjunction), $\vee$ (disjunction), and $\equiv$ (biconditional).

We say that X is <u>($\rightarrow$)-complete</u> iff, for any A and B, the fact that $A \rightarrow B \in X$ is equivalent to the fact that if $A \in X$, then $B \in X$. It proves convenient to associate with each standard connective of the object language a metalanguage correlate $\neg^*$, $\rightarrow^*$, $\wedge^*$, $\vee^*$, and $\equiv^*$, respectively. Thus we can use $\rightarrow^*$ to rewrite the condition "if $A \in X$, then $B \in X$" as "$(A \in X) \rightarrow^* (B \in X)$". By a similar convention, we may also say that X is <u>($\neg$)-complete</u> iff, for any A and B, the fact that $\neg A \in X$ is equivalent to the fact that $\neg^*(A \in X)$, *i.e.* to the fact that $A \notin X$. The above passages suggest the following general definition.

DEFINITION (Dn-1). Let # be a n-ary connective of our object language. X is called <u>(#)-complete</u> iff, for any $A_1, A_2, ..., A_n$, the conditions below are equivalent.

i. $\#A_1A_2...A_n \in X$,

ii. $\#^*(A_1 \in X)(A_2 \in X)...(A_n \in X)$.

To simplify the notation, the superscript * is omitted hereafter so that condition (ii) above becomes now

ii'. $\#(A_1 \in X)(A_2 \in X)...(A_n \in X)$.

It may be remarked that, intuitively, a set X is complete in respect of a connective iff the connective does in X precisely what it says it does there.

In what follows it is assumed that # is a n-ary connective.

3. CONSEQUENCE OPERATIONS.

3.1. Definition of the operation of consequence and its basic properties. This operation was introduced and studied by A. Tarski. In a series of papers published in the thirties Tarski developed a general consequence theory which he used in his extensive study of a variety of syntactic properties of deductive systems including properties such as consistency, completeness, axiomatizability, independence, definability, and decidability.

If $X \subseteq S$, we want to be able to talk about the set of all possible conclusions which can be inferred from members of X using the logic **L** which underlies the set S of all sentences of our object language. We denote this set by **Pr**(X). The expression "$A \in$ **Pr**(X)" stands for "A is provable from X" or else for "A is a consequence of X". Here the set X can be treated, informally, as a set of extra-logical axioms, hypotheses or otherwise specified assumptions or suppositions. This context suggests that **Pr**: $2^S \rightarrow 2^S$, *i.e.* that **Pr** is a unary operation on the power set of S which maps subsets of S to subsets of S.

Several equivalent definitions of the operation of consequence can be found in the literature (cf, for instance, [4]). They differ in referring to different defining concepts as well as in providing different intuitive motivations to the defined operation. The standard, most common definition involves the concept of proof. Other definitions make the operation of consequence dependent on the concept of logical law or that of rule of deduction. The definition in terms of proof runs as follows.

DEFINITION (Dn-2). $A \in$ **Pr**(X) iff there are $A_1, A_2, \ldots, A_n$ such that $A = A_n$ and, for any $i \leq n$, either $A_i \in X$, or A can be proved from $A_1, A_2, \ldots, A_{i-1}$.

Properties of the operation of consequence, implied by this definition fall into two natural groups. The first group includes properties whose formulation only involves (some) set-theoretic notions but does not depend on a particular structure of the underlying language S. For this reason we call them set-theoretic properties of **Pr**. Here are some examples of such set-theoretic properties.

i. **Pr** is reflexive in the sense that each sentence is a consequence of itself. In other words, $X \subseteq$ **Pr**(X), for any X.

ii. **Pr** is inclusion-monotonic. In other words, if $X \subseteq Y$, then **Pr**(X) $\subseteq$ **Pr**(Y), for any X, Y.

iii. **Pr** is idempotent in the sense that consequences of the consequences of a set of sentences are already consequences of this set. In other words, **Pr**(**Pr**(X)) $\subseteq$ **Pr**(X), for any X.

iv. **Pr** is finite in the sense that **Pr**(X) $\subseteq \bigcup\{$**Pr**(Y): $Y \in$ **Fin**(X)$\}$, for any X.

iv. **Pr** is compact in the sense that, for any X, the fact that **Pr**(X) = S implies that there is $Y \in$ **Fin**(X) such that **Pr**(Y) = S. **Pr** is weakly compact in the sense that there is $X \in$ **Fin**(S) such that **Pr**(X) = S.

v. **Pr** is regular in the sense that, for any A and X, the fact that A $\notin$ **Pr**(X) implies that there exists Y $\in 2_X$ such that A $\notin$ **Pr**(Y) and A $\in$ **Pr**(Y, B) for any B $\notin$ Y. **Pr** is strongly regular in the following sense. For any A and X the fact that A $\notin$ **Pr**(X) implies that there exists Y $\in 2_X$ such that A $\notin$ **Pr**(Y) and **Pr**(Y, B) = **S** for any B $\notin$ Y.

vi. **Pr** is normal in the sense that the set of all consequences of the empty set of (extra-logical) assumptions is identical with the set **L** of all theorems of the logic underlying language **S**. In other words, **Pr**($\emptyset$) = **L**.

The other group includes properties of **Pr** whose formulation depends on the structure of **S**. To be more specific, it depends on logical constants such as connectives or quantifiers. For this reason we refer to them as logical properties of **Pr**. Here are examples of such properties.

vii. **Pr** is ($\neg$)-compact in the sense that **Pr**(A, $\neg$A) = **S**, for any A.

viii. **Pr** admits ($\neg$)-cuts in the sense that **Pr**(X, A)$\cap$**Pr**(X, $\neg$A) $\subseteq$ **Pr**(X), for any A and X.

ix. **Pr** has the ($\rightarrow$)-deduction property in the sense that the fact that a conditional sentence is provable from a set X of sentences is equivalent to the fact that its consequent is provable from its antecedent on the ground of X. In other words, A$\rightarrow$B $\in$ **Pr**(X) iff B $\in$ **Pr**(X, A), for any A, B and X.

x. **Pr** admits ($\rightarrow$)-cuts in the sense that **Pr**(X, A)$\cap$**Pr**(X, A$\rightarrow$B) $\subseteq$ **Pr**(X), for any A, B and X.

It will be seen later that the properties of **Pr** referred to in theorems (i) - (x) are among the basic properties of this operation in the sense that all the other properties of **Pr** we may be interested in can, in fact, be inferred from them without any explicit or implicit reference to definition (Dn-2). The precise meaning of these remarks will be clarified in the next section which is devoted to the problem of axiomatization of **Pr**.

3.2. "Bottom Up" and "Top Down" descriptions of consequence operation. The next definition of the consequence operation, although it is closely related to definition (Dn-2), differs from the latter in making an appeal not only to the concept of proof but also to that of logical law. However, the concept of law involved in this context is set-theoretic and not logical because it does not presuppose a structured underlying language. This definition introduces the set of all consequences of a set X by working "from the bottom up", to borrow the suggestive expression from H. Enderton's Logic Text. The set X together with the set of all logical laws constitute the first or the initial stage. At each consecutive stage we add all the sentences which can be proved directly from all the members of the preceding stage, and so we move up and up until, ideally, we get all the possible consequences of X.

In order to avoid confusion with the consequence operation **Pr**, as introduced by definition (Dn-2), the consequence operation to be defined now is denoted as **Fn**.

DEFINITION (Dn-3). ("Bottom-up" consequence operation).

i. $A \in \mathbf{Fn}^{[1]}(X)$ iff $A \in X$ or A is a logical law;

ii. $A \in \mathbf{Fn}^{[k+1]}(X)$ iff either $A \in \mathbf{Fn}^{[k]}(X)$ or there are $n \geq 1$ and $A_1, A_2, ..., A_n \in \mathbf{Fn}^{[k]}(X)$ such that A can be proved from $A_1, A_2, ..., A_n$;

iii. $A \in \mathbf{Fn}(X)$ iff $A \in \mathbf{Fn}^{[k]}(X)$ for some $k \geq 1$.

Clearly, the concept of proof and that of logical law can be characterized in such a way that definitions (Dn-2) and (Dn-3) become equivalent in the sense that $\mathbf{Pr}(X) = \mathbf{Fn}(X)$, for any X.

Yet another definition of the consequence operation, which is based on the concept of deductive closure, may be seen as a way of introducing the set of all consequences of X by working "from the top down". To avoid confusion with the previously introduced operations of **Pr** and **Fn**, the operation to be defined now is symbolized as **Gn**.

DEFINITION (Dn-4). ("Top-down" consequence operation). $\mathbf{Gn}(X)$ is the least extension of X closed under all the deduction rules in question. In other words, for any A and X, $A \in \mathbf{Gn}(X)$ iff $A \in Y$, for any $Y \in 2_X$ such that Y is closed under all the rules of deduction in question.

One can easily provide such a characterization of the concepts involved here that $\mathbf{Gn}(X) = \mathbf{Fn}(X)$ for any X.

3.3. Definition of consequence operation in terms of logic of some structured language. The last definition of the operation of consequence, to be mentioned in this section, depends on the intuitive idea of a structured logical law. A formal rendering of an individual logical law of some language presupposes that the language is structured so that the presence of some logical constants is made explicit. By way of example, we assume below that our object language contains the connective of conditional $\rightarrow$.

DEFINITION (Dn-5). $A \in \mathbf{Hn}(X)$ iff there are $A_1, A_2, ..., A_n \in X$ such that $A_1 \rightarrow (A_2 \rightarrow (...(A_n \rightarrow A)...))$ is a logical law.

The concepts involved in this definition can be easily characterized in such a way that definitions (Dn-2) and (Dn-5) are equivalent. In other words, $\mathbf{Hn}(X) = \mathbf{Pr}(X)$, for any X.

4. AN AXIOMATIC APPROACH TO THE OPERATION OF CONSEQUENCE.

4.1. Closure operators and their set-theoretic properties. The consequence operation **Pr**, dealt with so far, is a particular operation on the power set of the set S of all sentences of some object language. Its meaning is determined entirely by definition (Dn-2) or, equivalently, by one of definitions (Dn-3), (Dn-4) or (Dn-5). Some of the theorems involving the notions of S and **Pr**, provable with the help of definition (Dn-2), may be singled out as basic (S, **Pr**)-statements and used to derive from them other (S, **Pr**)-statements without any recourse to definition (Dn-2). In order to emphasize the fact that these basic statements may be considered independently of definition (Dn-2), we refer to them as, say, (S, **Cn**)-statements where **Cn** stands for an arbitrary unary operation on the power set of the set S. It is such basic (S, **Cn**)-statements and their relative strength that we want to look into now. To be more specific, the problem of this section consists in (i) finding a (finite or recursive) set of (S, **Cn**)-statements called axioms for the primitive notions S and **Cn**; and (ii) proving that the (S, **Cn**)-axioms not only become true after each occurrence of the symbol **Cn** in these axioms is substituted by the symbol of **Pr**, as introduced by definition (Dn-2) or any of its equivalents (Dn-3), (Dn-4) or (Dn-5), but also entail precisely what is entailed by these definitions. We proceed now to a systematic discussion and presentation of a solution to this problem. We begin with a series of definitions which prove helpful in this respect.

DEFINITION (Dn-6). Let $\mathbf{Cn} \subseteq S \times 2^S$.

i. **Cn** is <u>idle on X</u> iff $\mathbf{Cn}(X) \subseteq S$, for any X.

ii. **Cn** is <u>trivial on X</u> iff $\mathbf{Cn}(X) = S$, for any X. **Cn** is <u>trivial</u> iff it is trivial on X for any X.

iii. **Cn** is <u>reflexive</u> iff $X \subseteq \mathbf{Cn}(X)$, for any X.

iv. **Cn** is <u>monotonic</u> iff $X \subseteq Y$ implies that $\mathbf{Cn}(X) \subseteq \mathbf{Cn}(Y)$, for any X, Y.

v. **Cn** is <u>idempotent</u> iff $\mathbf{Cn}(\mathbf{Cn}(X)) \subseteq \mathbf{Cn}(X)$, for any X.

vi. **Cn** is a <u>closure operator</u> iff **Cn** is idle on S, reflexive, monotonic and idempotent. In case **Cn** is a closure operator and X a subset of S, we call the set **Cn**(X) the <u>closure</u> of X.

vii. X is <u>Cn-maximal,</u> (or <u>maximal,</u> for short) <u>in respect of omitting A</u> iff $A \notin \mathbf{Cn}(X)$ and $A \in \mathbf{Cn}(X, B)$ for any $B \notin X$, *i.e.* iff X omits A but has no A-omitting proper superset. X is <u>strongly maximal</u> iff $\mathbf{Cn}(X) \neq S$ and $\mathbf{Cn}(X, B) = S$ for any $B \notin X$.

viii. **Cn** is <u>regular</u> iff, for any A and X, the fact that $A \notin \mathbf{Cn}(X)$ implies that there is $Y \in 2_X$ such that $A \notin \mathbf{Cn}(Y)$ and $A \in \mathbf{Cn}(Y, B)$ for any $B \notin Y$, *i.e.* iff, for any A, every A-omitting set can be extended to a set which has no A-omitting proper superset. **Cn** is <u>strongly regular</u> iff, for any A and X, the fact that $A \notin \mathbf{Cn}(X)$ implies that there is $Y \in 2_X$ such that $A \notin \mathbf{Cn}(Y)$ and $\mathbf{Cn}(Y, B) = S$ for any $B \notin Y$.

ix. **Cn** is <u>finite</u> iff $\mathbf{Cn}(X) \subseteq \cup\{\mathbf{Cn}(Y)\colon Y \in \mathbf{Fin}(X)\}$.

x. **Cn** is <u>compact</u> iff, for any X, the fact that $\mathbf{Cn}(X) = S$ implies that there is Y ∈ **Fin**(X) such that **Cn**(Y) = S. **Cn** is <u>weakly compact</u> iff there is X ∈ **Fin**(S) such that **Cn**(X) = S.

The Remark below gives an insight into some immediate relationships among the concepts introduced by definition (Dn-6).

REMARK 1.

i. If **Cn** is monotonic, then the fact that **Cn** is idle on S is equivalent to the fact that it is idle on X, for any X, *i.e.* to the fact that **Cn**: $2^S \to 2^S$.

ii. If **Cn** is monotonic, then the fact that **Cn** is non-trivial is equivalent to the fact that it is non-trivial on Ø.

iii. Every strongly maximal set is maximal in respect of omitting some sentence A.

iv. If **Cn** is monotonic, then, for any X, the fact that X is strongly maximal is equivalent to the fact that **Cn**(X)≠ S and Y = X for any Y ∈ 2_X such that **Cn**(Y) ≠ S.

v. If **Cn** is monotonic, then, for any A and X, the fact that X is maximal in respect of omitting A is equivalent to the fact that A ∉ **Cn**(X) and Y = X for any Y ∈ 2_X such that A ∉ **Cn**(Y).

vi. If **Cn** is a closure operator and X a set maximal in respect of omitting some sentence A, then **Cn**(X) = X.

4.2. **Logical properties of closure operators.** As we said earlier, there are two levels of study of closure operators, set-theoretic and logical. At the set-theoretic level there are just two primitive notions, a set S of meaningful sentences of some object language, and a unary operator **Cn** from the power set of S to itself. At this level no structure is imposed on the set S. Further study of the properties of **Cn** presupposes specification of the syntactic structure of sentences in S and when it comes to that level we presuppose that there are in S some compound sentences made up of some fixed connectives such as ¬, →, ∧, ∨ and ≡. Properties of **Cn** relative to a connective # are called (#)-relative properties of this operator or else logical properties. We adopt the following definition.

DEFINITION (Dn-7).

i. **Cn** is <u>(#)-saturated</u> iff, for any A and X, the fact that X is maximal in respect of omitting A implies that X is (#)-complete. **Cn** is <u>(Δ)-saturated</u> iff **Cn** is (#)-saturated for any # ∈ Δ, where Δ is a set of connectives.

ii. **Cn** is <u>(¬)-compact</u> iff **Cn**(A, ¬A) = S, for any A.

iii. **Cn** <u>admits (¬)-cuts</u> iff **Cn**(X, A)∩**Cn**(X, ¬A) ⊆ **Cn**(X), for any A and X.

iv. **Cn** is <u>(¬)-classical</u> iff the fact that A ∈ **Cn**(X) is equivalent to the fact that **Cn**(X, ¬A) = S, for any A and X.

v. Cn has the ($\rightarrow$)-deduction property iff, for any A, B and X, the fact that $A \rightarrow B \in Cn(X)$ is equivalent to the fact that $B \in Cn(X, A)$.

vi. Cn admits ($\rightarrow$)-cuts iff $Cn(X, A) \cap Cn(X, A \rightarrow B) \subseteq Cn(X)$, for any A, B and X.

vii. Cn is ($\rightarrow$)-classical iff Cn has the ($\rightarrow$)- deduction property and admits ($\rightarrow$)-cuts.

viii. Cn is ($\rightarrow$, $\neg$)-classical iff, for any $m \geq 0$, any $A_1, A_2, \ldots, A_m$, B, and any X, the fact that $A_1 \rightarrow (A_2 \rightarrow (\ldots(A_m \rightarrow B)\ldots)) \in Cn(X)$ is equivalent to the fact that $Cn(X, A_1, A_2, \ldots, A_m, \neg B) = S$.

ix. Cn is ($\wedge$)-classical iff $Cn(X, A \wedge B) = Cn(X, A, B)$, for any A, B and X.

x. Cn is ($\vee$)-classical iff $Cn(X, A \vee B) = Cn(X, A) \cap Cn(X, B)$, for any A, B and X.

xi. Cn has the ($\equiv$)-deduction property iff, for any A, B and X, the fact that $A \equiv B \in Cn(X)$ is equivalent to the fact that both $A \in Cn(X, B)$ and $B \in Cn(X, A)$.

xii. Cn admits ($\equiv$)-cuts iff $Cn(X, A) \cap Cn(X, B) \cap Cn(X, A \equiv B) \subseteq Cn(X)$, for any A, B and X.

xiii. Cn is ($\equiv$)-classical iff Cn has the ($\equiv$)-deduction property and admits ($\equiv$)-cuts.

The consequence-theoretic framework has thus been described and is ready for further study. Yet, a systematic study of consequence theory is not what we concentrate on in the present paper. Rather, its purpose is to focus on aspects of the relationship between consequence theory, on the one hand, and some alternative metalogical frameworks based on ideas, different from that of consequence, such as deductive theory, consistency and maximality, on the other hand. Therefore, we will close this section with only a brief illustration and discussion, without proofs, of the saturation property of the classical consequence operations, i.e. the consequence operations characteristic of the classical propositional logic.

REMARK 2. If Cn is a regular closure operator and $\Delta \subseteq \{\neg, \rightarrow, \wedge, \vee, \equiv\}$, then the conditions below are equivalent.

i. Cn is (Δ)-saturated;

ii. Cn is (Δ)-classical.

To be more specific, the statements the above theorem is a short for are included among the theorems below, where Cn stands for a regular closure operator on S.

i. The conditions below are equivalent.

Cn is $(\neg)$-saturated;

Cn is $(\neg)$-compact and admits $(\neg)$-cuts;

Cn is $(\neg)$-classical.

ii. The conditions below are equivalent.

Cn has the $(\rightarrow)$-deduction property;

The fact that $B_1 \rightarrow (B_2 \rightarrow (\ldots(B_n \rightarrow A)\ldots)) \in Cn(X)$ is equivalent to the fact that $A \in Cn(X, B_1, B_2, \ldots, B_n)$, for any X any $n \geq 0$, any A, B_1, $B_2, \ldots, B_n$, and any X.

iii. The conditions below are equivalent.

Cn is $(\rightarrow)$-saturated;

Cn has the $(\rightarrow)$-deduction property and admits $(\rightarrow)$-cuts.

iv. The conditions below are equivalent.

Cn is $(\rightarrow, \neg)$-saturated;

Cn has the $(\rightarrow, \neg)$-deduction property.

v. The conditions below are equivalent.

Cn is $(\wedge)$-saturated;

Cn is $(\wedge)$-classical;

$Cn(A \wedge B) = Cn(A, B)$, for any A, B.

vi. The conditions below are equivalent.

Cn is $(\vee)$-saturated;

Cn is $(\vee)$-classical.

vii. The conditions below are equivalent.

Cn is $(\equiv)$-saturated;

Cn $(\equiv)$-classical.

5. DEDUCTIVE THEORIES.

5.1. Definition of a deductive theory. From a practical viewpoint, a concrete deductive theory proves to be a finite set of statements about a fixed domain or universe of discourse. This view normally includes description of ways such statements have been arrived at, justified in one way or another, and systematized up to any given stage of development. However, a systematic study of deductive theories seems to require that a deductive theory include not only the actual statements already known and accepted but also all the potential statements which may yet to be discovered and justified in some way or another within the framework of the theory under consideration. On this idealization a deductive theory is comprehended as an accomplished, completed and closed organic unit. It is this type of idealization of concrete deductive theories which is accommodated under the name of a deductive theory in this section using the operation of consequence. We denote by $\mathbf{Th}_{\mathbf{Pr}}$ the class of all deductive theories which can be constructed within the set S of all sentences of some language, and we define it as follows.

DEFINITION (Dn-8). If **Pr** is an operator on S, as defined by definition (Dn-2), then $X \in \mathbf{Th}_{\mathbf{Pr}}$ iff $\mathbf{Pr}(X) = X$, for any X.

Thus deductive theories are deductively closed subsets of S. Defined in this way they play an important role in logical and metamathematical investigations.

5.2. An axiomatic treatment of the class of deductive theories. The class $\mathbf{Th}_{\mathbf{Pr}}$, which we introduced and discussed in the previous paragraph, is a particular class of subsets of the set S of all sentences of some object-language. Its meaning is determined entirely by definition (Dn-8). Some theorems involving the notions of S and $\mathbf{Th}_{\mathbf{Pr}}$, provable in the framework of consequence theory enriched by definition (Dn-8), may be designated as basic (S, $\mathbf{Th}_{\mathbf{Pr}}$)-statements from which other (S, $\mathbf{Th}_{\mathbf{Pr}}$)-statements can be derived directly, *i.e.* without any recourse to definition (Dn-8) or to any theorem of consequence theory. In order to emphasize the fact that these basic statements can be considered independently of the context of consequence theory and definition (Dn-8), we refer to them as (S, **Th**)-statements, say, where **Th** stands for an arbitrary class of subsets of S. It is such (S, **Th**)-statements and their relative strength that we want to look into now. As previously, there are two levels of study of such statements, set-theoretic and logical. At the set-theoretic level we have only two primitive notions, a set S of meaningful sentences of a fixed language and a class **Th** of subsets of S. At this level no structure is imposed on the set S. At the logical level we study the properties of **Th** which presuppose specification of the syntactic structure of sentences in S. At this level it is assumed that there are in S some compound sentences which correspond to the usual connectives of the ordinary language. The properties of class **Th**, which depend on these connectives, are called logical properties of this class.

We precede the discussion of properties of **Th** with the following definition.

DEFINITION (Dn-9). Let $\mathbf{Th} \subseteq 2^S$ and $X \in \mathbf{Th}$.

i. **Th** is <u>proper</u> iff $S \in \mathbf{Th}$.

ii. **Th** is <u>trivial</u> iff $\bigcap\mathbf{Th} = S$.

iii. **Th** is a <u>closure system</u> iff **Th** is proper and, for any non-empty $\mathbf{K} \subseteq 2^S$, the fact that $\mathbf{K} \subseteq \mathbf{Th}$ implies that $\cap\mathbf{K} \in \mathbf{Th}$, *i.e.* iff **Th** is closed under arbitrary (non-empty) intersections.

iv. **Th** is inductive iff $\mathbf{Th} \neq \emptyset$ and, for any non-empty $\mathbf{K} \subseteq \mathbf{Th}$, the fact that **K** is directed implies that $\cup\mathbf{K} \in \mathbf{Th}$, *i.e.* iff **Th** is a non-empty class closed under arbitrary directed unions.

v. X is maximal in respect of omitting A iff $A \notin X$ and, for any $Y \in \mathbf{Th}\cap 2_X$, the fact that $A \notin Y$ implies that $Y = X$, *i.e.* iff X is an inclusion-maximal member of **Th** which omits A. X is strongly maximal in respect of omitting A iff $A \notin X$ and, for any $Y \in \mathbf{Th}\cap 2_X$, the fact that $Y \neq S$ implies that $Y = X$. X is maximal iff there is A such that X is maximal in respect of omitting A. X is strongly maximal iff there is A such that X is strongly maximal relative to A.

vi. **Th** is (#)-saturated iff, for any $X \in \mathbf{Th}$, the fact that X is maximal implies that X is (#)-complete. **Th** is (Δ)-saturated iff **Th** is (#)-saturated, for any $\# \in \Delta$, where Δ is a set of connectives.

In connection with definition (Dn-9;iii) it may be remarked that the fact that a closure system Th is proper is equivalent to the fact that it is closed under the empty intersection as well.

THEOREM 1. ([6] and [2]).

i. If **Cn** is a closure operator and

$$\mathbf{Th}_{\mathbf{Cn}} = \{X: \mathbf{Cn}(X) = X\},$$

then $\mathbf{Th}_{\mathbf{Cn}}$ is a closure system. If Th is a closure system and

$$\mathbf{Cn}_{\mathbf{Th}}(X) = \cap(\mathbf{Th}\cap 2_X),$$

for any X, then $\mathbf{Cn}_{\mathbf{Th}}$ is a closure operator.

ii. If **Cn** is a closure operator, then $\mathbf{Cn}_{\mathbf{Th}_{\mathbf{Cn}}} = \mathbf{Cn}$.
If **Th** is a closure system, then $\mathbf{Th}_{\mathbf{Cn}_{\mathbf{Th}}} = \mathbf{Th}$.

iv. If the operator **Cn** is finite, then $\mathbf{Th}_{\mathbf{Cn}}$ is inductive.
If the class Th is inductive, then $\mathbf{Cn}_{\mathbf{Th}}$ is finite.

COROLLARY. The definitions of $\mathbf{Th}_{\mathbf{Cn}}$ in terms of **Cn** and of $\mathbf{Cn}_{\mathbf{Th}}$ in terms of **Th** establish a one-to-one correspondence between closure operators and closure systems.

THEOREM 2.

i. If a closure operator **Cn** is non-trivial, then $\mathbf{Th}_{\mathbf{Cn}}$ is non-trivial.
If a closure system **Th** is non-trivial, then $\mathbf{Cn}_{\mathbf{Th}}$ is non-trivial.

ii. If the operator **Cn** is (#)-saturated, then $\mathbf{Th}_{\mathbf{Cn}}$ is (#)-saturated.
If the class **Th** is (#)-saturated, then $\mathbf{Cn}_{\mathbf{Th}}$ is (#)-saturated.

PROOF. **Proof of Theorem 2;i.** Suppose that a closure operator **Cn** is non-trivial. By Remark 1, this gives that

1. $Cn(\emptyset) \neq S$
Also assume that
2. $\bigcap Th_{Cn} = S$
It follows from 1 that there is $A \in S$ such that
3. $A \notin Cn(\emptyset)$
Since $A \in S$, we may conclude from 2 that
4. $A \in X$, for any $X \in Th_{Cn}$
By the hypothesis, **Cn** is a closure operator so $Cn(Cn(\emptyset)) = Cn(\emptyset)$. Hence by 4
5. $A \in Cn(\emptyset)$
contrary to 3. Now suppose that **Th** is non-trivial, *i.e.*
6. $\bigcap Th \neq S$
and
7. $Cn_{Th}(\emptyset) = S$
By 6 there is A such that
8. $A \in S$
and
9. $A \notin \bigcap Th$
From 7 and 8
10. $A \in Cn_{Th}(\emptyset)$
From 10 by the definition of Cn_{Th}
11. $A \in \bigcap Th$
contrary to 9.

Proof of Theorem 2;ii. Suppose that **Cn** is a (#)-saturated closure operator and that
1. $X \in Th_{Cn}$, $A \notin X$ and $Y = X$ for any $Y \in Th_{Cn} \cap 2_X$
Then, by the definition of Th_{Cn}
2. $A \notin Cn(X)$
From 1 and Remark 1 we obtain that
3. $A \in Cn(X, B)$ for any $B \notin X$
Cn is (#)-saturated. Hence by 2 and 3
4. X is (#)-complete.
Now suppose that **Th** is a (#)-saturated closure system and that
1. $A \notin Cn_{Th}(X)$ and $A \in Cn_{Th}(X,B)$ for any $B \notin X$
Hence, by the definition of Cn_{Th}, there is $Y \in Th \cap 2_X$ such that $A \notin Y$ so
2. $A \notin X$
It remains to be verified that
3. for any $Z \in Th \cap 2_X$, if $A \notin Z$, then $Z = X$
Suppose not, *i.e.* that $Z \in Th \cap 2_X$, $A \notin Z$ and $Z \neq X$ for some Z. Then there is B such that $B \notin X$ and $B \in Z$. Then, by 1, $A \in Cn_{Th}(X,B)$. So, by the definition of Cn_{Th}, $A \in Z$ for any $Z \in Th \cap 2_{X \cup \{B\}}$. But $X \subseteq Z$ and $A \notin Z$. So $B \notin Z$, a contradiction. This proves line 3. Now, using the definition of a (#)-saturated closure system, we get from 2 and 3 that
4. X is (#)-complete.
Q.E.D.

The following remark shows that the "translating" definitions of Th_{Cn} in terms of **Cn** and of Cn_{Th} in terms of **Th** are faithful. This shows that the contents of the theory of **Th** can be "read off" inside the theory of **Cn**, and conversely, the contents of the theory of **Cn** can be "read off" inside the theory of **Th**.

REMARK 3.

i. If **Cn** is a closure operator, then , for any A and X, the fact that $A \in \mathbf{Cn}(X)$ is equivalent to the fact that, for any $Y \in 2_X$, the condition that $\mathbf{Cn}(Y) = Y$ implies that $A \in Y$.

ii. If **Th** is a closure system, then, for any X, the fact that $X \in \mathbf{Th}$ is equivalent to the fact that $\cap(\mathbf{Th} \cap 2_X) = X$.

6. CONSISTENCY.

6.1. Definition of consistency. The aim of this section is to show that the theory of strongly regular consequence operations, *i.e.* strongly regular closure operators on the set of all sentences of some language, on the one hand, and the theory of regular consistency property, on the other hand, are equivalent. The role played by the concept of regularity, which appears on each side of this equivalence, is also discussed and conditions under which it can be dropped are provided. We denote by $\mathbf{Cons}_{\mathbf{Pr}}$ the class of all consistent sets of sentences which can be constructed within the set S of all sentences of the language using the concept of provability **Pr**. The definition of $\mathbf{Cons}_{\mathbf{Pr}}$ runs as follows.

DEFINITION (Dn-10). If **Pr** is a consequence operation, as described by definition (Dn-2), then $\mathbf{Cons}_{\mathbf{Pr}} = \{X\colon \mathbf{Pr}(X) \neq S\}$.

On this definition, $\mathbf{Cons}_{\mathbf{Pr}}$ is the class of all **Pr**-closed proper subsets of S. The definition seems to accommodate very well the most basic intuition of consistency as a property of small sets. Clearly, on definition (Dn-10), $\mathbf{Cons}_{\mathbf{Pr}}$ depends on no logical constants. The most common definition of consistency, call it $\mathbf{Cons}_{\neg}$, depends on the connective of negation and runs as follows.

DEFINITION (Dn-11). If **Pr** is a consequence operation, as defined by definition (Dn-2), then $\mathbf{Cons}_{\neg} = \{X\colon \neg A \notin \mathbf{Cn}(X, A)$, for any $A\} = \{X\colon$ the fact that $A \in \mathbf{Cn}(X)$ implies that $\neg A \notin \mathbf{Cn}(X)$, for any $A\}$.

6.2. An axiomatization of the notion of consistency. Clearly, if a (S, $\mathbf{Cons}_{\mathbf{Pr}}$)-statement is picked up as an axiom, meant to characterize S and $\mathbf{Cons}_{\mathbf{Pr}}$ as primitive ideas, the statement must not only be separated from the context of consequence theory but also treated independently of definition (Dn-10). To avoid possible confusion between two such usages of the notion of consistency, we replace symbol $\mathbf{Cons}_{\mathbf{Pr}}$ by that of **Cons**. We need a number of definitions dealing with various aspects of **Cons**.

DEFINITION (Dn-12). Let $\mathbf{Cons} \subseteq 2^S$.

i. **Cons** is <u>trivial</u> iff $S \in \mathbf{Cons}$.

ii. Cons is proper iff $\emptyset \in$ **Cons**.

iii. **Cons** is hereditary iff the fact that $Y \in \mathbf{Cons} \cap 2_X$ implies that $X \in \mathbf{Cons}$, for any X and Y.

iv. **Cons** is a consistency property iff **Cons** is both hereditary and non-trivial.

v. X is Cons-maximal, or maximal for short, iff X is an inclusion-maximal member of **Cons**, *i.e.* iff $X \in \mathbf{Cons}$ and $Y = X$ for any $Y \in \mathbf{Cons} \cap 2_X$.

vi. **Cons** is regular iff, for any X, the fact that $X \in \mathbf{Cons}$ implies that there exists a **Cons**-maximal extension of X, *i.e.* iff all members of **Cons** are inclusion-maximalizable.

vii. **Cons** is compact iff the fact that $X \notin \mathbf{Cons}$ implies that there is $Y \in \mathbf{Fin}(X)$ such that $Y \notin \mathbf{Cons}$, for any X. **Cons** is weakly compact iff there is $X \in \mathbf{Fin}(S)$ such that $X \notin \mathbf{Cons}$.

viii. **Cons** is a finite character property iff **Cons** is a compact consistency property.

ix. **Cons** is (#)-saturated iff, for any X, the fact that X is **Cons**-maximal implies that it is (#)-complete. **Cons** is (Δ)-saturated iff **Cons** is (#)-saturated, for any $\# \in \Delta$, where Δ is a set of connectives.

x. **Cons** is ($\neg$)-compact iff $\{A, \neg A\} \notin \mathbf{Cons}$, for any A.

xi. **Cons** admits ($\neg$)-expansion iff the fact that $X \in \mathbf{Cons}$ implies that $X \cup \{A\} \in \mathbf{Cons}$ or $X \cup \{\neg A\} \in \mathbf{Cons}$, for any A, and X.

xii. **Cons** is ($\neg$)-classical iff **Cons** is ($\neg$)-compact and admits ($\neg$)-expansion.

xiii. **Cons** is ($\rightarrow$)-analytic iff the fact that $X \cup \{A \rightarrow B, A\} \in \mathbf{Cons}$ implies that $X \cup \{A, B\} \in \mathbf{Cons}$, for any A, B and X.

xiv. **Cons** is ($\rightarrow$)-synthetic iff the fact that $X \cup \{B\} \in \mathbf{Cons}$ implies that $X \cup \{A \rightarrow B\} \in \mathbf{Cons}$, for any A, B and X.

xv. **Cons** admits ($\rightarrow$)-expansion iff the fact that $X \in \mathbf{Cons}$ implies that either $X \cup \{A\} \in \mathbf{Cons}$ or $X \cup \{A \rightarrow B\} \in \mathbf{Cons}$, for any A, B and X.

xvi. **Cons** is ($\rightarrow$)-classical iff **Cons** is ($\rightarrow$)-analytic, ($\rightarrow$)-synthetic and admits ($\rightarrow$)-expansion.

xvii. **Cons** is ($\rightarrow$, $\neg$)-classical iff the fact that that $X \cup \{A_1 \rightarrow (A_2 \rightarrow (\ldots(A_m \rightarrow B)\ldots))\} \in \mathbf{Cons}$ is equivalent to the fact that $X \cup \{A_1, A_2, \ldots, A_m, \neg B\} \notin \mathbf{Cons}$, for any $m \geq 0$, any $A_1, A_2, \ldots, A_m$, B, and any **Cons**-maximal set X.

xviii. **Cons** is <u>(∧)-classical</u> iff the fact that $X \cup \{A \wedge B\} \in$ **Cons** is equivalent to the fact that $X \cup \{A, B\} \in$ **Cons**, for any A, B and X.

xix. **Cons** is <u>(∨)-classical</u> iff the fact that $X \cup \{A \vee B\} \in$ **Cons** is equivalent to the fact that $X \cup \{A\} \in$ **Cons** or $X \cup \{B\} \in$ **Cons**, for any A, B and X.

xx. **Cons** is <u>(≡)-analytic</u> iff the fact that $X \cup \{A, A \equiv B\} \in$ **Cons** or $X \cup \{B, A \equiv B\} \in$ **Cons** implies that $X \cup \{A, B\} \in$ **Cons**, for any A, B and X.

xxi. **Cons** is <u>(≡)-synthetic</u> iff the fact that $X \cup \{A, B\} \in$ **Cons** implies that $X \cup \{A, B, A \equiv B\} \in$ **Cons**, for any A, B and X.

xxii. **Cons** <u>admits (≡)-expansion</u> iff the fact that $X \in$ **Cons** implies that either $X \cup \{A\} \in$ **Cons** or $X \cup \{B\} \in$ **Cons** or $X \cup \{A \equiv B\} \in$ **Cons**, for any A, B and X.

xxiii. **Cons** is <u>(≡)-classical</u> iff **Cons** is (≡)-analytic, (≡)-synthetic and admits (≡)-expansion.

REMARK 4. If **Cons** is hereditary, then **Cons** is proper iff **Cons** $\neq \emptyset$.

REMARK 5. If **Cons** is a regular consistency property and $\Delta \subseteq \{\neg, \rightarrow, \wedge, \vee, \equiv\}$, then the conditions below are equivalent.

i. **Cons** is (Δ)-saturated;

ii. **Cons** is (Δ)-classical.

6.3. Regular consistency property versus strongly regular closure operator. Here we describe conditions under which a consistency theory is expressively equivalent to the classical consequence theory.

THEOREM 3.

i. If **Cn** is a strongly regular closure operator and

$$\mathbf{Cons}_{\mathbf{Cn}} = \{X \subseteq S: \mathbf{Cn}(X) \neq S\},$$

then $\mathbf{Cons}_{\mathbf{Cn}}$ is a regular consistency property. If **Cons** is a regular consistency property and, for any A and X, we put that

$A \in \mathbf{Cn}_{\mathbf{Cons}}(X)$ iff, for any Y, the fact that $X \cup Y \in$ **Cons** implies that $X \cup Y \cup \{A\} \in$ **Cons**,

then $\mathbf{Cn}_{\mathbf{Cons}}$ is a strongly regular closure operator.

ii. If **Cn** is a strongly regular closure operator, then $\mathbf{Cn}_{\mathbf{Cons}_{\mathbf{Cn}}} = \mathbf{Cn}$. If **Cons** is a regular consistency property, then $\mathbf{Cons}_{\mathbf{Cn}_{\mathbf{Cons}}} = \mathbf{Cons}$.

iii. If a strongly regular closure operator **Cn** is non-trivial, then $\mathbf{Cons}_{\mathbf{Cn}}$ is proper. If a regular consistency property **Cons** is proper, then $\mathbf{Cn}_{\mathbf{Cons}}$ is non-trivial.

iv. If the operator **Cn** is (#)-saturated, then $\mathbf{Cons}_{\mathbf{Cn}}$ is (#)-saturated.
If the property **Cons** is (#)-saturated, then $\mathbf{Cn}_{\mathbf{Cons}}$ is (#)-saturated.

PROOF. Proof of Theorem 3;i. Suppose **Cn** is a strongly regular closure operator. Then, clearly, $\mathbf{Cons}_{\mathbf{Cn}}$ is non-trivial because $\mathbf{Cn}(S) = S$. It is also hereditary because **Cn** is monotonic. It remains to be checked that $\mathbf{Cons}_{\mathbf{Cn}}$ is regular. Suppose $X \in \mathbf{Cons}_{\mathbf{Cn}}$, *i.e.* $A \notin \mathbf{Cn}(X)$ for some A. By the regularity of **Cn**, there is $Y \in 2_X$ such that $A \notin \mathbf{Cn}(Y)$ but $\mathbf{Cn}(Y, B) = S$ for any $B \notin Y$. So $Y \in \mathbf{Cons}_{\mathbf{Cn}}$. To complete the proof, we claim now that Y is an inclusion-maximal member of $\mathbf{Cons}_{\mathbf{Cn}}$. For, if it is not, then there is a proper superset Z of Y such that $Z \in \mathbf{Cons}_{\mathbf{Cn}}$, *i.e.* there is $B \in Z$ such that $B \notin Y$. So $\mathbf{Cn}(Y, B) = S$ and, with it, $\mathbf{Cn}(Z) = S$ because **Cn** is monotonic. Hence $Z \notin \mathbf{Cons}_{\mathbf{Cn}}$, a contradiction. Now suppose **Cons** is a regular consistency property. Claim: $\mathbf{Cn}_{\mathbf{Cons}}$ is reflexive. Indeed, if $A \notin \mathbf{Cn}_{\mathbf{Cons}}(X)$, then $X \cup Y \in \mathbf{Cons}$ but $X \cup Y \cup \{A\} \notin \mathbf{Cons}$ for some Y so $A \notin X$. Claim: $\mathbf{Cn}_{\mathbf{Cons}}$ is monotonic. Suppose that $X \subseteq Y$, $A \in \mathbf{Cn}_{\mathbf{Cons}}(X)$ and $A \notin \mathbf{Cn}_{\mathbf{Cons}}(Y)$. By the definition of $\mathbf{Cn}_{\mathbf{Cons}}$, $Y \cup Z \in \mathbf{Cons}$ but $Y \cup Z \cup \{A\} \notin \mathbf{Cons}$. Hence $X \cup Y \cup Z \in \mathbf{Cons}$. So by the definition of $\mathbf{Cn}_{\mathbf{Cons}}$ and the hereditary character of **Cons**, $Y \cup Z \cup \{A\} \in \mathbf{Cons}$, a contradiction. Claim: $\mathbf{Cn}_{\mathbf{Cons}}$ is idempotent. Suppose that $A \in \mathbf{Cn}_{\mathbf{Cons}}(\mathbf{Cn}_{\mathbf{Cons}}(X))$ and $A \notin \mathbf{Cn}_{\mathbf{Cons}}(X)$. Then $X \cup Y \in \mathbf{Cons}$ but $X \cup Y \cup \{A\} \notin \mathbf{Cons}$ for some Y. So by another use of the definition of $\mathbf{Cn}_{\mathbf{Cons}}$

1. if $\mathbf{Cn}_{\mathbf{Cons}}(X) \cup Y \in \mathbf{Cons}$, then $\mathbf{Cn}_{\mathbf{Cons}}(X) \cup Y \cup \{A\} \in \mathbf{Cons}$

Cons is hereditary and $\mathbf{Cn}_{\mathbf{Cons}}$ reflexive. So by 1

2. if $\mathbf{Cn}_{\mathbf{Cons}}(X) \cup Y \in \mathbf{Cons}$, then $X \cup Y \cup \{A\} \in \mathbf{Cons}$

But $X \cup Y \cup \{A\} \notin \mathbf{Cons}$. Hence by 2

3. $\mathbf{Cn}_{\mathbf{Cons}}(X) \cup Y \notin \mathbf{Cons}$

Cons is regular and $X \cup Y \in \mathbf{Cons}$ so there is in **Cons** an inclusion-maximal extension Z of $X \cup Y$. On the other hand, $\mathbf{Cn}_{\mathbf{Cons}}$ is reflexive and monotonic. So

4. $\mathbf{Cn}_{\mathbf{Cons}}(X) \cup Y \subseteq \mathbf{Cn}_{\mathbf{Cons}}(Z)$

We will show now that

5. $\mathbf{Cn}_{\mathbf{Cons}}(Z) \subseteq Z$

Suppose $C \in \mathbf{Cn}_{\mathbf{Cons}}(Z)$. Hence $Z \cup \{C\} \in \mathbf{Cons}$ by the definition of $\mathbf{Cn}_{\mathbf{Cons}}$. But Z is an inclusion-maximal member of **Cons** so $C \in Z$. From 4 and 5

6. $\mathbf{Cn}_{\mathbf{Cons}}(X) \cup Y \subseteq Z$

Cons is hereditary and $Z \in \mathbf{Cons}$ so $\mathbf{Cn}_{\mathbf{Cons}}(X) \cup Y \in \mathbf{Cons}$, contrary to 3. Claim: $\mathbf{Cn}_{\mathbf{Cons}}$ is strongly regular. Suppose

1. $A \notin \mathbf{Cn}_{\mathbf{Cons}}(X)$ for some A and X

Hence by the definition of $\mathbf{Cn}_{\mathbf{Cons}}$

2. $X \cup Y \in \mathbf{Cons}$ but $X \cup Y \cup \{A\} \notin \mathbf{Cons}$ for some Y

Cons is regular. So by 2

3. there is an inclusion-maximal member Z in **Cons** such that $X \cup Y \subseteq Z$

We will show now that

4. $A \notin \mathbf{Cn}_{\mathbf{Cons}}(Z)$

Suppose that, on the contrary, $A \in Cn_{Cons}(Z)$. Then $Z \cup \{A\} \in$ **Cons**, by the definition of Cn_{Cons} and the fact that $Z \in$ **Cons**. **Cons** is hereditary so $X \cup Y \cup \{A\} \in$ **Cons**, contrary to 2. It remains to be shown that

5. $Cn_{Cons}(Z, B) = S$ for any $B \notin Z$

Suppose, on the contrary, that $B \in Z$ but $Cn_{Cons}(Z, B) \neq S$. Then $C \notin Cn_{Cons}(Z, B)$ for some C. Hence by the definition of Cn_{Cons}, there is Z' such that $Z \cup \{B\} \cup Z' \in$ **Cons** but $Z \cup \{B\} \cup Z' \cup \{C\} \notin$ **Cons**. **Cons** is hereditary so $Z \cup \{B\} \in$ **Cons**. **Cons** is inclusion-maximal so $B \in Z$, a contradiction.

Proof of Theorem 3;ii. Claim: $Cn_{Cons_{Cn}} \subseteq Cn$ for any strongly regular closure operator Cn. By Theorem 3;i, $\mathbf{Cons}_{Cn}$ is a regular consistency property. Also by Theorem 3;i, $Cn_{Cons_{Cn}}$ is a regular closure operator. It remains to be shown that $Cn_{Cons_{Cn}}(X) \subseteq Cn(X)$ for any X. Suppose not, *i.e.*

1. $A \in Cn_{Cons_{Cn}}(X)$ and $A \notin Cn(X)$ for some A and X

From 1 and the definition of $\mathbf{Cons}_{Cn}$

2. if $X \cup Y \in \mathbf{Cons}_{Cn}$, then $X \cup Y \cup \{A\} \in \mathbf{Cons}_{Cn}$ for any Y

Cn is strongly regular. Hence by 1, there is $Y \supseteq X$ such that

3. $A \notin Cn(Y)$ but $Cn(Y, B) = S$ for any $B \notin Y$

$X \subseteq Y$. Hence $X \cup Y \in \mathbf{Cons}_{Cn}$, by 3 and the definition of $\mathbf{Cons}_{Cn}$. Hence $X \cup Y\{A\} \in \mathbf{Cons}_{Cn}$, *i.e.* $Cn(X \cup Y, A) \neq S$, by 2 and the definition of $\mathbf{Cons}_{Cn}$. Hence by 3

4. $A \in Y$

Cn is reflexive. So by 3

5. $A \in Cn(Y)$

contrary to 3. Claim: $Cn \subseteq Cn_{Cons_{Cn}}$ for any strongly regular closure operator Cn. By Theorem 3;i, $\mathbf{Cons}_{Cn}$ is a regular consistency property and $Cn_{Cons_{Cn}}$ is a strongly regular closure operator. Now we must show that $Cn(X) \subseteq Cn_{Cons_{Cn}}(X)$ for any X. Suppose not, *i.e.*

1. $A \in Cn(X)$ and $A \notin Cn_{Cons_{Cn}}(X)$ for some A and X

From 1 and the definition of $Cn_{Cons_{Cn}}$

2. $X \cup Y \in \mathbf{Cons}_{Cn}$ and $X \cup Y \cup \{A\} \notin \mathbf{Cons}_{Cn}$ for some Y

$\mathbf{Cons}_{Cn}$ is regular. Hence by 2

3. there is an inclusion-maximal member Z in $\mathbf{Cons}_{Cn}$ such that $X \cup Y \subseteq Z$

Also from 1 and the definition of $Cn_{Cons_{Cn}}$

4. if $X \cup Z' \in \mathbf{Cons}_{Cn}$, then $X \cup Z' \cup \{A\} \in \mathbf{Cons}_{Cn}$ for any Z'

From 3 and 4 by taking Z' to be $Y \cup Z$

5. $X \cup Y \cup Z \cup \{A\} \in \mathbf{Cons}_{Cn}$

$\mathbf{Cons}_{Cn}$ is hereditary. Hence by 5

6. $X \cup Y \cup \{A\} \in \mathbf{Cons}_{Cn}$

contrary to 2. Claim: $\mathbf{Cons}_{Cn_{Cons}} \subseteq$ **Cons** for any regular consistency property **Cons**. By Theorem 3;i, Cn_{Cons} is a strongly regular closure operator and $\mathbf{Cons}_{Cn_{Cons}}$ is a regular consistency property. We will show now that the fact that $X \in \mathbf{Cons}_{Cn_{Cons}}$ implies that $X \in$ **Cons** for any X. Suppose $X \in \mathbf{Cons}_{Cn_{Cons}}$. By the definition of $\mathbf{Cons}_{Cn_{Cons}}$, $Cn_{Cons}(X) \neq S$, *i.e.* $A \notin Cn_{Cons}(X)$ for some A.

Hence by the definition of Cn_{Cons}, $X \cup Y \in$ **Cons** for some Y. Hence $X \in$ **Cons** because **Cons** is hereditary. Claim: **Cons** $\subseteq$ $\mathbf{Cons}_{Cn_{Cons}}$ for any regular consistency property **Cons**. To show that if $X \in$ **Cons**, then $X \in \mathbf{Cons}_{Cn_{Cons}}$ for any X, suppose that, on the contrary,

1. $X \in$ **Cons** and $X \notin \mathbf{Cons}_{Cn_{Cons}}$ for some X

Cons is regular. Hence by 1

2. there is an inclusion-maximal member Y of **Cons** such that $X \subseteq Y$

From 1 and the definition of $\mathbf{Cons}_{Cn_{Cons}}$

3. $Cn_{Cons}(X) = S$

Clearly

4. $Cn_{Cons}(X) \subseteq \cap\{Y \in 2_X$: Y is inclusion-maximal in **Cons**$\}$

For suppose not, *i.e.* that

4.1. $A \in Cn_{Cons}(X)$ but $A \notin \cap\{Y \in 2_X$: Y is inclusion-maximal in **Cons**$\}$,

i.e. that $A \notin Z$ for some inclusion-maximal $Z \in$ **Cons**$\cap 2_X$

Hence by the definition of Cn_{Cons}

4.2. if $X \cup Z \in$ **Cons**, then $X \cup Z \cup \{A\} \in$ **Cons** for any Z

Hence

4.3. $X \cup Z \cup \{A\} \in$ **Cons**

because $X \subseteq Z \in$ **Cons**. From 4.3

4.4. $A \in Z$

because Z is inclusion-maximal by 4.1. But this is a contradiction so line 4 is proved.
From 3 and 4

5. $S \subseteq \cap\{Y \in 2_X$: Y is inclusion-maximal in **Cons**$\}$

From 2 and 5

6. $Y = S$

Cons is non-trivial by the hypothesis. Hence by 6

7. $Y \notin$ **Cons**

contrary to 2.

Proof of Theorem 3;iii. Clearly, if Cn is non-trivial, *i.e.* $Cn(X) \neq S$, for some X, then, by the definition of $\mathbf{Cons}_{Cn}$, $X \in \mathbf{Cons}_{Cn}$. Since $\mathbf{Cons}_{Cn}$ is hereditary, this gives that $\emptyset \in$ **ConsCn**, *i.e.* that $\mathbf{Cons}_{Cn}$ is proper. Now assume that a regular consistency property **Cons** is proper, *i.e.* that

1. $\emptyset \in$ **Cons**

and suppose that, on the contrary, Cn_{Cons} is trivial, *i.e.* that, for any X,

2. $Cn_{Cons}(X) = S$

Cons is regular. Hence, by 1, there is X' such that

3. $X' \in$ **Cons**

and

4. $Y = X'$, for any $Y \in$ **Cons**$\cap 2_{X'}$

We want to show now that

5. $S \subseteq X'$

Suppose $A \in S$. By 2, $A \in Cn_{Cons}(\emptyset)$. By the definition of **CnCons**,

for any Z, if $X'' \cup Z \in$ **Cons**, then $X'' \cup Z \cup \{A\} \in$ **Cons**

Hence by 3, $X \cup \{A\} \in$ **Cons**. Hence by 4, $A \in X$ so line 5 is proved. **Cons** is hereditary so by 3 and 5

6. $S \in$ **Cons**

contrary to our hypothesis.

Proof of Theorem 3;iv. Suppose **Cn** a (#)-saturated strongly regular closure operator. To verify that $\mathbf{Cons}_{\mathbf{Cn}}$ is (#)-saturated suppose that , for any X, X is an inclusion-maximal member of **Cons**. We must show that under these assumptions, X is (#)-complete. By the definition of $\mathbf{Cons}_{\mathbf{Cn}}$, **Cn** ≠ S, *i.e.*

1. A ∉ Cn(X) for some A

It remains to be shown that

2. A ∈ Cn(X, B) for any B ∉ X

Suppose not, *i.e.* B ∉ X and A ∉ Cn(X, B) for some B. It follows that Cn(X, B) ≠ S. Hence X ∪{B} ∈ $\mathbf{Cons}_{\mathbf{Cn}}$ by the definition of $\mathbf{Cons}_{\mathbf{Cn}}$. But X is inclusion-maximal. So B ∈ X, a contradiction. This proves line 2. From 1, 2 and the definition of a (#)-saturated operator

3. X is (#)-complete

Proof of Theorem 3;v. Suppose **Cons** is a (#)-saturated regular consistency property. Suppose also that

1. A ∉ $\mathbf{Cn}_{\mathbf{Cons}}$(X) and A ∈ $\mathbf{Cn}_{\mathbf{Cons}}$(X, B) for any B ∉ X

Then, by the definition of $\mathbf{Cn}_{\mathbf{Cons}}$, there is Y such that

2. X∪Y ∈ **Cons** and X∪Y∪{A} ∉ **Cons**

We want now to verify that

3. Y is an inclusion-maximal member of **Cons**

Suppose

3.1. X ⊆ Y ∈ **Cons**

We claim now that

3.2. Y ⊆ X

Indeed, if C ∈ Y, then C ∈ $\mathbf{Cn}_{\mathbf{Cons}}$(X), by the definition of $\mathbf{Cn}_{\mathbf{Cons}}$. Then C ∈ X, by line 1, Remark 1 and Theorem 3;ii. This proves line 3.2. From 3.1 and 3.2

3.3. Y = X

so line 3 is proved. Now, **Cons** is (#)-saturated and hereditary. So it follows from 2 and 3 that X is (#)-complete.

Q.E.D.

COROLLARY. The definitions of $\mathbf{Cons}_{\mathbf{Cn}}$ in terms of **Cn** and of $\mathbf{Cn}_{\mathbf{Cons}}$ in terms of **Cons** establish a one-to-one correspondence between strongly regular closure operators and regular consistency properties.

The remark below shows faithfulness of our "translating" definitions of $\mathbf{Cons}_{\mathbf{Cn}}$ in terms of **Cn** and of $\mathbf{Cn}_{\mathbf{Cons}}$ in terms of **Cons**.

REMARK 6.

i. If Cn is a strongly regular closure operator, then, for any A and X, the fact that A ∈ **Cn**(X) is equivalent to the fact that, for any Y ∈ 2_X, the condition that Cn(Y) ≠ S implies that Cn(Y, A) ≠ S.

ii. If **Cons** is a regular consistency property, then, for any X, the fact that X ∈ **Cons** is equivalent to the fact that there are A ∈ S and Y ∈ 2_X such that Y ∈ **Cons** and Y∩{A} ∉ **Cons.**

6.4. Full consistency families versus closure operators. The proof of the equivalence of the consistency-theoretic framework with the consequence-theoretic framework, as presented in the previous paragraph, makes use of a regular consistency property and a strongly regular closure operator. However, the result of Theorem 3 can be adapted to the setting of arbitrary closure operators where no requirement of regularity or strong regularity is imposed on a closure operator any more if a consistency property is replaced by what is called a full consistency family which is defined below. The idea of a full consistency family was suggested to me by Lloyd Humberstone of Monash University, Melbourne, Australia.

DEFINITION (Dn-13). Let $\mathbb{Cons} \subseteq 2^{2^S}$.

i. $\mathbb{Cons}$ is a <u>consistency family</u> iff each member of $\mathbb{Cons}$ is a consistency property.

ii. $\mathbb{Cons}$ is a <u>full consistency family</u> iff, for any $Cons \subseteq 2^S$, the fact that $Cons \in \mathbb{Cons}$ is equivalent to the fact that there is B such that $Cons = \{X:$ there are $Cons' \in \mathbb{Cons}$ and $Y \in Cons' \cap 2_X$ such that $Y \cup \{B\} \notin Cons'\}$.

THEOREM 4.

i. If Cn is a closure operator and

$$\mathbb{Cons}_{Cn} = \{Cons_A : A \in S\} = \{\{X : A \notin Cn(X)\} : A \in S\},$$

then $\mathbb{Cons}_{Cn}$ is a full consistency family.

ii. If $\mathbb{Cons}$ is a full consistency family and, for any A and X, we put that

$A \in Cn_{\mathbb{Cons}}(X)$ iff, for any $Cons \in \mathbb{Cons}$ and any Y,

the fact that $X \cup Y \in Cons$ implies that $X \cup Y \cup \{A\} \in Cons$,

then Cn_{Cons} is a closure operator.

iii. If Cn is a closure operator, then $Cn_{\mathbb{Cons}_{Cn}} = Cn$.
If $\mathbb{Cons}$ is a full consistency family, then $\mathbb{Cons}_{Cn_{\mathbb{Cons}}} = \mathbb{Cons}$.

COROLLARY. The definitions of $\mathbb{Cons}_{Cn}$ in terms of Cn and of $Cn_{\mathbb{Cons}}$ in terms of $\mathbb{Cons}$ establish a one-to-one correspondence between closure operators and full consistency families.

REMARK 7. The definition of Theorem 4 is a generalization of the definition of Theorem 3, *i.e.* $Cn_{\mathbb{Cons}}(X) = \bigcap\{Cn_{Cons}(X) : Cons \in \mathbb{Cons}\}$, for any X.

6.5. Comments on the relationship between consequence and consistency. The first thing Theorem 3 shows is that the definability of consistency in terms of consequence as well as definability of consequence in terms of consistency can be established at the set-theoretic, language-independent level, *i.e.* independently of any logical constants. Without this being feasible, the proof of the expressive equivalence of a consistency-theoretic and a consequence-theoretic frameworks, and the fact that each of these frameworks determines the same logic, become meaningless.

The traditional characterization of consistency in terms of consequence is disturbed - if not contaminated - by the presence of negation. Thus, by the Aristotelian definition,

(i) a set X of sentences of some language is called consistent iff there is no sentence A in that language such that A and ¬A are both consequences of X.

This definition makes consistency parasitic on negation, a fact which severely restricts the universality and application of this concept. To say the least, this traditional definition simply does not apply to negationless fragments of the full language. By a well-known result of E. L. Post, however, this restriction can be cleared away and negation eliminated altogether from the definition of consistency in terms of consequence.

Negation has been portrayed as a catalyst also in the extant definitions of consequence in terms of consistency. By one such definition, attributed to R. Carnap,

(ii) a sentence A is a consequence of a set X of sentences iff ¬A is inconsistent with X, *i.e.* iff the set $X \cup \{\neg A\}$ is inconsistent.

One of the outcomes of this paper is that an adequate characterization of consequence in terms of consistency in the absence of negation, in fact, in the absence of any logical constants, is available. Namely,

(iii) A is a consequence of a set X of sentences iff, for any set Y of sentences, the fact that A is consistent with $X \cup Y$ is implied by the fact that X is consistent with Y.

It may be observed that in this definition the set Y cannot be replaced by a single sentence B or, at least, such a replacement is not possible at the set-theoretic, language-independent level. However, the following definition:

(iv) A is a consequence of X iff A is a member of any inclusion-maximal consistent extension of the set X.

is (set-theoretically) equivalent to definition (iii).

Secondly, an inspection of the proof of the expressive equivalence of the consistency-theoretic and consequence-theoretic frameworks shows that the proof depends on what is called in the paper the regularity of consistency and a correlated strong regularity of consequence. There seems to be no problem with an intuitive grasp of the concept of a regular consistency. As it appears, weak regularity shows inadequate for our purpose. (It may be remembered that, while each finite consequence is regular, only consequences which are both finite and compact are strongly regular).

7. LINDENBAUM EXTENSION OPERATORS.

7.1. Definition of a Lindenbaum operator. By Lindenbaum's extension lemma, each consistent set of sentences can be extended to a consistent and complete deductive system. In general, extensions of this kind are not unique. The passage from a consistent set to a particular deductively closed, consistent and complete extension is not a mapping. One way of making it a mapping is to correlate with each set, whether consistent or inconsistent, the family of all deductively closed consistent and complete extensions of this set rather than correlating with it one particular extension of this kind.

To be sure, the mapping reached in this way is a mapping from 2^S to 2^{2^S} rather than from 2^S to 2^S. We will call a mapping of this kind a Lindenbaum extension operator. Below we provide a formal definition of such an operator using the framework of consequence theory based on the operation **Pr** as defined by definition (Dn-2). We denote such an operator as $\mathbf{Lin}_{\mathbf{Pr}}$ so as to make its dependence on **Pr** explicit.

DEFINITION (Dn-14).
$\mathbf{Lin}_{\mathbf{Pr}}(X) = \{Y \in 2_X: \mathbf{Pr}(Y) \neq S$ and $Z = Y$ for any $Z \in 2_Y$ such that $\mathbf{Pr}(Y) \neq S\}$, for any X.

In the next paragraph we provide an axiomatization of the operator $\mathbf{Lin}_{\mathbf{Pr}}$ thus treating it as a primitive concept.

7.2. **An axiomatization of the concept of Lindenbaum operator.** Operator $\mathbf{Lin}_{\mathbf{Pr}}$ is a particular mapping from the power set 2^S of the set S of all sentences of some language to the power set of the power set 2^{2^S} of S. Its meaning is determined entirely by definition (Dn-14). Some theorems involving the notions of S and $\mathbf{Lin}_{\mathbf{Pr}}$, which can be proved in consequence theory using definition (Dn-14), can be designated as basic (S, $\mathbf{Lin}_{\mathbf{Pr}}$)-statements in the sense that other (S, $\mathbf{Lin}_{\mathbf{Pr}}$)-statements can be derived from them directly, *i.e.* without any recourse to definition (Dn-14) or to any theorem of consequence theory. In order to emphasize the fact that these basic statements are considered independently of the context of consequence theory and definition (Dn-14), we refer to them as, say, (S, **Lin**)-statements, where **Lin** stands for an arbitrary mapping from 2^S to 2^{2^S}. It is such (S, **Lin**)-statements and their relative strength that we want to look into now. Again, there are two levels of study of the operators, set-theoretic and logical. At the set-theoretic level we have just two primitive notions, the set S of all meaningful sentences of some language and a unary mapping **Lin** from the power set of S to the the power set of this power set. At this level no structure is imposed on the set S. Further study of the properties of **Lin** presupposes specification of the syntactic structure of sentences in S. Accordingly, at the second level we assume that there are in S some compound sentences made up of some logical constants. We now proceed to the description of some of the properties of the operator **Lin** on the set S.

DEFINITION (Dn-15). Let **Lin**: $2^S \to 2^{2^S}$.

i. **Lin** is <u>trivial on X</u> iff $S \in \mathbf{Lin}(X)$, for any X. **Lin** is <u>trivial</u> iff there is X such that **Lin** is trivial on X.

ii. **Lin** is <u>proper on X</u> iff $\mathbf{Lin}(X) \neq \emptyset$, for any X. **Lin** is <u>proper</u> iff there is X such that **Lin** is proper on X.

iii. **Lin** is <u>extensive</u> iff $\mathbf{Lin}(X) \subseteq 2_X$, for any X.

iv. **Lin** is <u>antimonotonic</u> iff the fact that $X \subseteq Y$ implies that $\mathbf{Lin}(Y) \subseteq \mathbf{Lin}(X)$, for any X, Y.

v. **Lin** is <u>inclusive</u> iff $\mathbf{Lin}(\emptyset) \cap 2_X \subseteq \mathbf{Lin}(X)$, for any X.

vi. **Lin** is a <u>Lindenbaum operator</u> iff **Lin** is non-trivial, extensive, antimonotonic and inclusive.

vii. **Lin** is <u>regular</u> iff the fact that $Y \in \mathbf{Lin}(X)$ and $X \in \mathbf{Lin}(\emptyset)$ implies that X = Y, for any X, Y.

viii. **Lin** is <u>compact</u> iff the fact that **Lin**(X) = Ø implies that there is Y ∈ **Fin**(X) such that **Lin**(Y) = Ø, for any X. **Lin** is <u>weakly compact</u> iff there is X ∈ **Fin**(S) such that **Lin**(X) = Ø.

ix. **Lin** is (#)-saturated iff the fact that X ∈ **Lin**(Ø) implies that X is (#)-complete, for any X.

x. **Lin** is <u>(¬)-compact</u> iff **Lin**(A)∩**Lin**(¬A) =Ø, for any A.

xi. **Lin** <u>admits (¬)-expansion</u> iff **Lin**(X, A)∪**Lin**(X, ¬A) = **Lin**(X), for any A, and X.

xii. **Lin** is <u>(¬)-classical</u> iff **Lin** is (¬)-compact and admits (¬)-expansion.

xiii. **Lin** is <u>(→)-analytic</u> iff **Lin**(X, A→B)∩**Lin**(X, A) ⊆ **Lin**(X,B), for any A, B and X.

xiv. **Lin** is <u>(→)-synthetic</u> iff **Lin**(X, B) ⊆ **Lin**(X,A→B), for any A, B and X.

xv. **Lin** <u>admits (→)-expansion</u> iff **Lin**(X, A)∪**Lin**(X, A→B) = **Lin**(X), for any A, B and X.

xvi. **Lin** is <u>(→)-classical</u> iff **Lin** is (→)-analytic, (→)-synthetic and admits (→)-expansion.

xvii. **Lin** is <u>(→, ¬)-classical</u> iff
Lin(X, A_1→(A_2→(...(A_m→B)...)))∩**Lin**(X, A_1, A_2,..., A_m, ¬B) =Ø, and
Lin(X, A_1→(A_2→(...(A_m→B)...)))∪**Lin**(X, A_1, A_2,..., A_m, ¬B) = **Lin**(Ø),
for any $m \geq 0$, any A_1, A_2,..., A_m, B, and any **Cons**-maximal set X.

xviii. **Lin** is <u>(∧)-classical</u> iff **Lin**(X, A∧B) = **Lin**(X, A)∩**Lin**(X, B), for any A, B and X.

xix. **Lin** is <u>(∨)-classical</u> iff **Lin**(X, A∧B) = **Lin**(X, A)∪**Lin**(X, B), for any A, B and X.

xx. **Lin** is <u>(≡)-analytic</u> iff **Lin**(X, A, A≡B)∪**Lin**(X, B, A≡B) ⊆ **Lin**(X, A, B), for any A, B and X.

xxi. **Lin** is <u>(≡)-synthetic</u> iff **Lin**(X, A, B) ⊆ **Lin**(X, A, B, A≡B), for any A, B and X.

xxii. **Lin** <u>admits (≡)-expansion</u> iff **Lin**(X, A)∪**Lin**(X, B)∪**Lin**(X, A≡B) = **Lin**(X), for any A, B and X.

xxiii. **Lin** is <u>(≡)-classical</u> iff **Lin** is (≡)-analytic, (≡)-synthetic and admits (≡)-expansion.

Before proceeding to the discussion of the relation between the presented above maximality-theoretical framework, in terms of a mapping **Lin**, and the framework based on the idea of consistency, we note without proof the following remark, which gives some insight into the properties, both set-theoretic and logical, of the theory of **Lin**.

REMARK 8.

i. If **Lin** is antimonotonic, then the fact that **Lin** is trivial is equivalent to the fact that it is trivial on $\emptyset$.

ii. If **Lin** is antimonotonic, then the fact that **Lin** is proper is equivalent to the fact that it is proper on $\emptyset$.

iii. If **Lin** is a Lindenbaum operator, then the conditions below are equivalent.

  a. **Lin** is (Δ)-saturated;

  b. **Lin** is (Δ)-classical.

Now to the relation between the theory of **Lin** and that of **Cons**.

THEOREM 5.

i. If **Cons** is a regular consistency property and

$$Y \in \mathbf{Lin}_{\mathbf{Cons}}(X) \text{ iff } Y \in \mathbf{Cons} \cap 2_Y \text{ and } Z = Y \text{ for any } Z \in \mathbf{Cons} \cap 2_Y,$$

for any X and Y, then $\mathbf{Lin}_{\mathbf{Cons}}$ is a regular Lindenbaum operator. If **Lin** is a regular Lindenbaum operator and

$$X \in \mathbf{Cons}_{\mathbf{Lin}} \text{ iff } \mathbf{Lin}(\emptyset) \cap 2_X \neq \emptyset,$$

for any X, then $\mathbf{Cons}_{\mathbf{Lin}}$ is a regular consistency property.

ii. If **Cons** is a regular consistency property, then $\mathbf{Cons}_{\mathbf{Lin}_{\mathbf{Cons}}} = \mathbf{Cons}$. If **Lin** is a regular Lindenbaum operator, then $\mathbf{Lin}_{\mathbf{Cons}_{\mathbf{Lin}}} = \mathbf{Lin}$.

iii. If a regular consistency property **Cons** is proper, then $\mathbf{Lin}_{\mathbf{Cons}}$ is proper. If a regular Lindenbaum operator **Lin** is proper, then $\mathbf{Cons}_{\mathbf{Lin}}$ is proper.

iv. If the property **Cons** is (#)-saturated, then $\mathbf{Lin}_{\mathbf{Cons}}$ is (#)-saturated. If the operator **Lin** is (#)-saturated, then $\mathbf{Cons}_{\mathbf{Lin}}$ is (#)-saturated.

PROOF. Proof of Theorem 5;i. If **Cons** is a regular consistency property, then, clearly, $\mathbf{Lin}_{\mathbf{Cons}}$ is a regular Lindenbaum operator. Now suppose that **Lin** is a regular Lindenbaum operator. The fact that $\mathbf{Cons}_{\mathbf{Lin}}$ is a consistency property follows immediately from the definition of $\mathbf{Cons}_{\mathbf{Lin}}$ and the hypothesis. To see that $\mathbf{Cons}_{\mathbf{Lin}}$ is regular, assume that $X \in \mathbf{Cons}_{\mathbf{Lin}}$. Under this assumption and following the definition of $\mathbf{Cons}_{\mathbf{Lin}}$, there is X^+ such that $X^+ \in \mathbf{Lin}(\emptyset) \cap 2_X$. This gives also that $X^+ \in \mathbf{Lin}(\emptyset) \cap 2_{X^+}$, for $X^+ \subseteq X^+$. So $X^+ \in \mathbf{Cons}_{\mathbf{Lin}}$ and $X^+ \in \mathbf{Cons}_{\mathbf{Lin}} \cap 2_X$. It remains to be shown that $Z \subseteq X^+$ for any $Z \in \mathbf{Cons}_{\mathbf{Lin}} \cap 2_{X^+}$. Assume $Z \in \mathbf{Cons}_{\mathbf{Lin}} \cap 2_{X^+}$. Then, by the definition of $\mathbf{Cons}_{\mathbf{Lin}}$, $Z^+ \in \mathbf{Lin}(\emptyset) \cap 2_Z$

for some Z^+. Then $Z^+ \in$ **Lin**(X^+), for **Lin** is inclusive and $X^+ \subseteq Z^+$. Then $Z^+ = X^+$, for **Lin** is regular and $X^+ \in$ **Lin**$(\emptyset) \cap 2_{X^+}$.

Proof of Theorem 5;ii. Claim: $\mathbf{Cons}_{\mathbf{Lin}_{\mathbf{Cons}}} \subseteq \mathbf{Cons}$ for any regular consistency property **Cons**. By Theorem 5;i, $\mathbf{Lin}_{\mathbf{Cons}}$ is a regular Lindenbaum operator. By Theorem 5;ii, $\mathbf{Cons}_{\mathbf{Lin}_{\mathbf{Cons}}}$ is a regular consistency property. To show the inclusion, suppose $X \in \mathbf{Cons}_{\mathbf{Lin}_{\mathbf{Cons}}}$. Then $Y \in \mathbf{Lin}_{\mathbf{Cons}}(\emptyset)$ and $X \subseteq Y$, by the definition of $\mathbf{Cons}_{\mathbf{Lin}_{\mathbf{Cons}}}$. Hence $Y \in$ **Cons** but $Z = Y$ for any $Z \in$ **Cons**, by the definition of $\mathbf{Lin}_{\mathbf{Cons}}$. But **Cons** is hereditary so $X \in$ **Cons** by 2 and 3. Claim: $\mathbf{Cons} \subseteq \mathbf{Cons}_{\mathbf{Lin}_{\mathbf{Cons}}}$ for any regular consistency property **Cons**. Suppose $X \in$ **Cons**. By the regularity of **Cons**, there is $Y \in \mathbf{Cons} \cap 2_X$ such that $Z = Y$ for any $Z \in \mathbf{Cons} \cap 2_Y$. So $Y \in \mathbf{Lin}_{\mathbf{Cons}}(\emptyset)$, by the definition of $\mathbf{Lin}_{\mathbf{Cons}}$. Hence $Y \in \mathbf{Lin}_{\mathbf{Cons}}(\emptyset) \cap 2_X$ and $X \in \mathbf{Cons}_{\mathbf{Lin}_{\mathbf{Cons}}}$, by the definition of $\mathbf{Cons}_{\mathbf{Lin}_{\mathbf{Cons}}}$. Claim: $\mathbf{Lin}_{\mathbf{Cons}_{\mathbf{Lin}}} \subseteq \mathbf{Lin}$ for any regular Lindenbaum operator **Lin**. Suppose the inclusion does not hold, *i.e.*

1. $Y \in \mathbf{Lin}_{\mathbf{Cons}_{\mathbf{Lin}}}(X)$ and $Y \notin \mathbf{Lin}(X)$ for some X and Y

By the definition of $\mathbf{Lin}_{\mathbf{Cons}_{\mathbf{Lin}}}$

2. $Y \in \mathbf{Cons}_{\mathbf{Lin}} \cap 2_X$ and $Z = Y$ for any $Z \in \mathbf{Cons}_{\mathbf{Lin}} \cap 2_Y$

By the definition of **Lin**

3. $Y^+ \in \mathbf{Lin}(\emptyset) \cap 2_Y$ for some Y^+

Lin is inclusive so

4. $Y^+ \in \mathbf{Lin}(Y)$

Clearly, $X \subseteq Y^+$, for $X \subseteq Y$. Hence by 1

5. $Y \notin \mathbf{Lin}(Y^+)$

for **Lin** is antimonotonic. On the other hand, it follows from 1 and the definition of $\mathbf{Cons}_{\mathbf{Lin}}$ that

6. $Z = Y$ for any Z and Z^+ such that $Z^+ \in \mathbf{Lin}(\emptyset)$, $Z \subseteq Z^+$ and $Y \subseteq Z$

Taking Z and Z^+ to be Y^+ we get from 3 and 6 that

7. $Y^+ = Y$

so by 5

8. $Y^+ \notin \mathbf{Lin}(Y)$

contrary to 4. Claim: $\mathbf{Lin} \subseteq \mathbf{Lin}_{\mathbf{Cons}_{\mathbf{Lin}}}$ for any regular Lindenbaum operator. Suppose that $Y \in \mathbf{Lin}(X)$. **Lin** is extensive and antimonotonic so $Y \in \mathbf{Lin}(\emptyset)$ and $X \subseteq Y$. By the definition of $\mathbf{Cons}_{\mathbf{Lin}}$, $Y \in \mathbf{Cons}_{\mathbf{Lin}} \cap 2_Y$, for $Y \subseteq Y$. Hence $Y \in \mathbf{Cons}_{\mathbf{Lin}}$ and $Y \in \mathbf{Cons}_{\mathbf{Lin}} \cap 2_X$ because $X \subseteq Y$. It remains to be shown that

$Z = Y$ for any $Z \in \mathbf{Cons}_{\mathbf{Lin}} \cap 2_Y$

Assume

1. $Z \in \mathbf{Cons}_{\mathbf{Lin}}$ and $Y \subseteq Z$

By the definition of $\mathbf{Cons}_{\mathbf{Lin}}$

2. $Z^+ \in \mathbf{Lin}(\emptyset) \cap 2_Z$ for some Z^+

Lin is inclusive so

3. $Z+ \in \mathbf{Lin}(Z)$

Lin is antimonotonic and $Y \subseteq Z$ so

4. $Z^+ \in \mathbf{Lin}(Y)$

Lin is regular and $Y \in$ **Lin**($\emptyset$) so

5. $Z^+ = Y$

But $Y \subseteq Z \subseteq Z^+$ so

6. $Z = Y$

Proof of Theorem 5;iii. If **Cons** is proper, *i.e.* $\emptyset \in$ **Cons**, then, by the regularity of **Cons**, there is $X \in$ **Cons** such that $Y = X$ for any $Y \in$ **Cons**$\cap 2_X$. Hence, by the definition of $\mathbf{Lin}_{\mathbf{Cons}}$, $X \in \mathbf{Lin}_{\mathbf{Cons}}(\emptyset)$. Hence $\mathbf{Lin}_{\mathbf{Cons}}(\emptyset) \neq \emptyset$ so $\mathbf{Lin}_{\mathbf{Cons}}$ is proper. If, on the other hand, **Lin** is proper, then, by Remark 8, **Lin** is proper on $\emptyset$, *i.e.* **Lin**($\emptyset$) $\neq \emptyset$. It follows that there is X such that $X \in$ **Lin**($\emptyset$) and that $X \in$ **Lin**$(\emptyset) \cap 2_\emptyset$ so **Lin**$(\emptyset) \cap 2_\emptyset \neq \emptyset$. By the definition of $\mathbf{Cons}_{\mathbf{Lin}}$, $\emptyset \in \mathbf{Cons}_{\mathbf{Lin}}$ so $\mathbf{Cons}_{\mathbf{Lin}}$ is proper.

Proof of Theorem 5;iv. Suppose **Cons** is a (#)-saturated regular consistency property. To verify that $\mathbf{Lin}_{\mathbf{Cons}}$ is (#)-saturated assume that $X \in \mathbf{Lin}_{\mathbf{Cons}}(\emptyset)$, for any X. Hence, by the definition of $\mathbf{Lin}_{\mathbf{Cons}}$, $X \in$ **Cons** and $Y = X$ for any $Y \in$ **Cons**$\cap 2_X$. Hence, by the hypothesis, X is (#)-complete. To verify that $\mathbf{Cons}_{\mathbf{Lin}}$ is (#)-saturated, for any (#)-saturated regular Lindenbaum operator **Lin**, assume that $X \in \mathbf{Cons}_{\mathbf{Lin}}$ and $Y \doteq X$ for any $Y \in \mathbf{Cons}_{\mathbf{Lin}} \cap 2_X$. By the definition of $\mathbf{Cons}_{\mathbf{Lin}}$, there is X^+ such that $X^+ \in$ **Lin**$(\emptyset) \cap 2_X$. Hence, by taking Y to be X^+, $X^+ = X$ so $X \in$ **Lin**($\emptyset$). Hence, by the hypothesis, X is (#)-complete.
Q.E.D.

COROLLARY.
The definitions of $\mathbf{Lin}_{\mathbf{Cons}}$ in terms of **Cons** and of $\mathbf{Cons}_{\mathbf{Lin}}$ in terms of **Lin** establish a one-to-one correspondence between regular consistency properties and regular Lindenbaum operators.

The following remark provides a justification of our "translating" definitions of $\mathbf{Lin}_{\mathbf{Cons}}$ in terms of **Cons** and of $\mathbf{Cons}_{\mathbf{Lin}}$ in terms of **Lin**.

REMARK 9.

i. If **Cons** is a regular consistency property, then, for any X, the fact that $X \in$ **Cons** is equivalent to the fact that there is $Y \in 2_X$ such that $Y \in$ **Cons**$\cap 2_\emptyset$ and $Z = Y$ for any $Z \in$ **Cons**$\cap 2_Y$.

ii. If **Lin** is a regular Lindenbaum operator, then, for any X and Y, the fact that $Y \in$ **Lin** is equivalent to the fact that $Y \in 2_X$, **Lin**$(\emptyset) \cap 2_Y \neq \emptyset$ and $Z = Y$ for any $Z \in 2_Y$ such that **Lin**$(\emptyset) \cap 2_Z \neq \emptyset$.

8. MAXIMAL FAMILIES OF SETS.

8.1. Definition of a maximal family of sets. The idea of maximality can be categorized not only in terms of maximal extension mappings such as Lindenbaum operators **Lin** but also in terms of families of maximally extended sets.

The most common definition of such families makes use of the concept of provability **Pr** and it runs as follows.

DEFINITION (Dn-16).
$\mathbf{Cpl}_{\mathbf{Pr}} = \{X: \mathbf{Pr}(X) \neq S$ and $Y = X$ for any $Y \in 2_X$ such that $\mathbf{Pr}(Y) \neq S\}$

8.2. An axiomatization of the concept of a maximal family of sets. In order to separate a particular property of the maximal family $\mathbf{Cpl}_{\mathbf{Pr}}$ and study its relative strength independently of definition (Dn-16), the family $\mathbf{Cpl}_{\mathbf{Pr}}$ is denoted hereafter as **Cpl**. We adopt the following definition which focuses on some particular properties of maximality as a family of sets.

DEFINITION (Dn-17). Let $\mathbf{Cpl} \subseteq 2^S$.

i. **Cpl** is trivial iff $S \in \mathbf{Cpl}$.

ii. **Cpl** is proper iff $\mathbf{Cpl} \neq \emptyset$.

iii. **Cpl** is regular iff the fact that $X \in \mathbf{Cpl}$ and $Y \in \mathbf{Cpl} \cap 2_X$ implies that $X = Y$, for any X, Y.

iv. **Cpl** is a maximal family iff it is both non-trivial and regular.

v. **Cpl** is (#)-saturated iff the fact that $X \in \mathbf{Cpl}$ implies that X is (#)-complete, for any X.

THEOREM 6.

i. If **Lin** is a regular Lindenbaum operator and

$$\mathbf{Cpl}_{\mathbf{Lin}} = \{X: X \in \mathbf{Lin}(\emptyset)\},$$

then $\mathbf{Cpl}_{\mathbf{Lin}}$ is a maximal family. If **Cpl** is a maximal family and

$$\mathbf{Lin}_{\mathbf{Cpl}}(X) = \{Y \in 2_X: Y \in \mathbf{Cpl}\},$$

for any X, then $\mathbf{Lin}_{\mathbf{Cpl}}$ is a regular Lindenbaum operator.

ii. If **Lin** is a regular Lindenbaum operator, then $\mathbf{Lin}_{\mathbf{Cpl}_{\mathbf{Lin}}} = \mathbf{Lin}$.
If **Cpl** is a maximal family, then $\mathbf{Cpl}_{\mathbf{Lin}_{\mathbf{Cpl}}} = \mathbf{Cpl}$.

iii. If the operator **Lin** is proper, then $\mathbf{Cpl}_{\mathbf{Lin}}$ is proper.
If the family **Cpl** is proper, then $\mathbf{Lin}_{\mathbf{Cpl}}$ is proper.

iv. If the operator **Lin** is (#)-saturated, then $\mathbf{Cpl}_{\mathbf{Lin}}$ is (#)-saturated.
If the family **Cpl** is (#)-saturated, then $\mathbf{Lin}_{\mathbf{Cpl}}$ is (#)-saturated.

PROOF. Proof of Theorem 6;i. That $\mathbf{Cpl}_{\mathbf{Lin}}$ is a maximal family follows directly from the definition of $\mathbf{Cpl}_{\mathbf{Lin}}$ and the fact that **Lin** is proper, inclusive and regular. That $\mathbf{Lin}_{\mathbf{Cpl}}$ is a regular Lindenbaum operator can also be seen to follow immediately from the definition of $\mathbf{Lin}_{\mathbf{Cpl}}$ and the fact that **Cpl** is a maximal family.

Proof of Theorem 6;ii. Suppose that **Lin** is a regular Lindenbaum operator. Theorem 6;i gives at once that

$\mathbf{Cpl}_{\mathbf{Lin}}$ is a maximal family

and that

LinCpl is a regular Lindenbaum operator

Claim: $\mathbf{Lin}_{\mathbf{Cpl}_{\mathbf{Lin}}} \subseteq \mathbf{Lin}$. To show this inclusion, suppose that $Y \in \mathbf{Lin}_{\mathbf{Cpl}_{\mathbf{Lin}}}(X)$. By the definition of $\mathbf{Lin}_{\mathbf{Cpl}_{\mathbf{Lin}}}$ it follows that $X \subseteq Y$ and $Y \in \mathbf{Cpl}_{\mathbf{Lin}}$. Hence, by the definition of $\mathbf{Cpl}_{\mathbf{Lin}}$, $Y \in \mathbf{Lin}(\emptyset)$. **Lin** is inclusive so $Y \in \mathbf{Lin}(X)$. Claim: $\mathbf{Lin} \subseteq \mathbf{Lin}_{\mathbf{Cpl}_{\mathbf{Lin}}}$. Suppose that $Y \in \mathbf{Lin}(X)$. Since **Lin** is antimonotonic and extensive we may conclude that $X \subseteq Y$ and $Y \in$ **Lin**)Ø), and, by the definition of $\mathbf{Cpl}_{\mathbf{Lin}}$, $Y \in \mathbf{Cpl}_{\mathbf{Lin}}$. Now suppose that **Cpl** is a maximal family. Theorem 6;i gives at ones that

$\mathbf{Lin}_{\mathbf{Cpl}}$ is a regular Lindenbaum operator

and that

$\mathbf{Cpl}_{\mathbf{Lin}}$ is a maximal family

Claim: $\mathbf{Cpl}_{\mathbf{Lin}_{\mathbf{Cpl}}} \subseteq \mathbf{Cpl}$. If $X \in \mathbf{Cpl}_{\mathbf{Lin}_{\mathbf{Cpl}}}$, then, by the definition of $\mathbf{Cpl}_{\mathbf{Lin}_{\mathbf{Cpl}}}$, $X \in \mathbf{LinCpl}(\emptyset)$. Hence, by the definition of $\mathbf{Lin}_{\mathbf{Cpl}}$, $X \in \mathbf{Cpl}$. Claim: $\mathbf{Cpl} \subseteq \mathbf{Cpl}_{\mathbf{Lin}_{\mathbf{Cpl}}}$. Suppose that $X \in \mathbf{Cpl}$. Hence $X \in \mathbf{Lin}_{\mathbf{Cpl}}(\emptyset)$ and hence $X \in \mathbf{Cpl}_{\mathbf{Lin}_{\mathbf{Cpl}}}$.

Proof of Theorem 6;iii. If **Lin** is proper, then by Remark 8, it is proper on Ø, *i.e.* $\mathbf{Lin}(\emptyset) \neq \emptyset$ so, by the definition of $\mathbf{Cpl}_{\mathbf{Lin}}$, $\mathbf{Cpl} \neq \emptyset$, *i.e.* $\mathbf{Cpl}_{\mathbf{Lin}}$ is proper. If **Cpl** is proper, then $X \in \mathbf{Cpl}$ for some X and, by the definition of $\mathbf{Lin}_{\mathbf{Cpl}}$, $\mathbf{Lin}_{\mathbf{Cpl}}(\emptyset) \neq \emptyset$, *i.e.* $\mathbf{Lin}_{\mathbf{Cpl}}$ is proper.

Proof of Theorem 6;iv. If a regular Lindenbaum operator **Lin** is (#)-saturated and $X \in \mathbf{Cpl}_{\mathbf{Lin}}$, for any X, then $X \in \mathbf{Lin}(\emptyset)$ and so X is (#)-complete. Conversely, if a maximal family **Cpl** is (#)-saturated and $X \in \mathbf{Lin}_{\mathbf{Cpl}}(\emptyset)$, then $X \in \mathbf{Cpl}$ and so X is (#)-complete.

COROLLARY.

The definitions of $\mathbf{Cpl}_{\mathbf{Lin}}$ in terms of **Lin** and of $\mathbf{Lin}_{\mathbf{Cpl}}$ in terms of **Cpl** establish a one-to-one correspondence between regular consistency properties and regular Lindenbaum operators.

The justification of the "translating definitions of $\mathbf{Cpl}_{\mathbf{Lin}}$ in terms of **Lin** and of $\mathbf{Lin}_{\mathbf{Cpl}}$ in terms of **Cpl** is straightforward and is omitted.

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Combining Type Disciplines

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Introduction

Current research in the theory of programming languages has emphasized the importance of studying the properties of type disciplines with a sufficiently rich set of type constructors, in order to achieve greater flexibility in the use of type annotations as descriptions of the functional behaviour of programs. One of the many possible ways to meet this requirement is exemplified by the family of higher-order, strongly typed languages patterned after the Theory of Constructions [Coquand and Huet 1986], in which a type of a program can be considered as a *logical specification* of it. Another, orthogonal, research direction consists in investigating first order type constructors more directly related to the intuition of a type as a collection of values. From the latter viewpoint it is important to emphasize the connections between computational properties of expressions (e.g. strong normalizability, subject reduction or even strictness properties) and the possible forms of their typings. There are by now several examples in the literature which demonstrate the interest of this line of research: *intersection types* have been shown to characterize exactly the class of strongly normalizable λ-terms [Coppo Dezani 1980], [Leivant 1986]; *recursive types* in which the recursion variable occurs only positively can be assigned only to strongly normalizing terms [Mendler 1987]; there are interesting developments in the use of intersection types to detect strictness properties of functional programs [Coppo Ferrari 1992]. Recent research by John Reynolds [Reynolds 1988, 1989] has shown how to encode record types using intersection, and their interaction with general type recursion allows, to a certain extent, to formalize examples from object-oriented programming, like that of a class of points described by the type:

$$\text{Point} = \mu P.(\text{setX}:\text{Int} \to P \wedge \text{getX}:\mathbf{Int}).$$

Benjamin Pierce [Pierce 1990] has also devised the following example, showing that some real expressive power is gained when *union types* are added to intersection types. The IF•THEN•ELSE• operator is such that the type Neg-num ∨ Pos-num can be assigned to the term **n** = IF b THEN 1 ELSE -1. To test whether **n** is zero, using the function

Is-Zero : (Neg-num→False) ∧ (Zero→True) ∧ (Pos-num→False),

amounts to derive the typing

(Is-Zero **n**) : False.

Without using union types the best information about (Is-Zero **n**) is that it is just a boolean value.

The form of polymorphism described by universal and existential type quantification has been used in the description of free algebras [Böhm, Berarducci 1985] and abstract data types [Mitchell, Plotkin 1988], and this suggests looking at the interplay of polymorphism and the first-order type constructors. An area in which their interaction has proved to be a flexible tool is the analysis of object-oriented programming, especially when a notion of subtyping is added to the system and polymorphic type constructors include bounded universal quantification and record types as in the languages Fun [Cardelli, Wegner 1985] and Quest [Cardelli 1989].

In this perspective, the present paper studies some syntactical and semantical aspects of a type inference system which, in addition to the standard function type constructor, has intersection and union operators, type recursion and both universal and existential type quantifiers. This system is that introduced in [MacQueen, Plotkin and

Sethi 1986], where a semantical interpretation for the system is presented; we also follow that paper in not including subtyping rules nor bounded polymorphism, because it has been shown [Pierce 1991] that these features make the subtyping relation undecidable. Here we give two equivalent formulations of the type system, one in a natural deduction style and the other as a sequent calculus, and show the invariance of types with respect to a general notion of *parallel reduction* of terms, a property which fails for ordinary β-reduction of λ-terms. From the semantical point of view, we describe a new interpretation for the system, which does not require any restriction on the formation rules for types as those imposed in [MacQueen, Plotkin and Sethi 1986], and is based on the approximation properties of D_∞ λ-models [Scott 1972] of which the types are taken to be particular families of subsets.

1. The type assignment system.

In the present section we describe the syntax of the type inference system that we shall study in the sequel. The rules for the system are presented in two equivalent ways: the first one is that given by [MacQueen, Plotkin and Sethi 1986], and is patterned after a natural deduction system of the kind considered in [Prawitz 1965]. The second formulation is a sequent calculus in the style of [Gentzen 1969], for which a restricted form of cut elimination will be proved in the next section.

1.1 <u>Definition</u> (*Types*).
The set **T** of *types* is inductively defined by

- $t_0, t_1, \ldots \in T$ (V set of type variables, ranged over by s,t,...)
- $\omega \in T$ (type constant)
- $\sigma, \tau \in T \Rightarrow (\sigma \to \tau), (\sigma \wedge \tau), (\sigma \vee \tau) \in T$
- $t \in V,\ \tau \in T \Rightarrow \mu t.\tau,\ \forall t.\tau,\ \exists t.\tau \in T.$

Convention : we omit parentheses according to the precedence rule: "$\wedge$ and $\vee$ over $\to$".

A *typing statement* is an expression of the form $M:\sigma$, where M is a λ-term and σ a type; M is called the *subject* and σ the *predicate* of the typing statement. A *basic*

typing statement is a typing statement whose subject is a variable. A *basis* is a set of basic typing statements such that subjects are pairwise distinct. If B is a basis, FV(B) will denote the set of term and type variables which occur in B.
B, $x{:}\sigma$ will denote the basis $B \cup \{x{:}\sigma\}$ when B is a basis such that either $x \notin FV(B)$ or $x{:}\sigma \in B$.

A *statement* is an expression of the form $B \vdash M{:}\sigma$ (where B is a basis and $M{:}\sigma$ is a typing statement) that can be derived by the axioms and rules of Definition 1.2.

1.2 <u>Definition</u> *(Natural Deduction Formulation)*
The axioms and rules to derive statements are:

$$(Ax)\ \frac{}{B,\ x{:}\sigma \vdash x{:}\sigma} \qquad (\omega)\ \frac{}{B \vdash M{:}\omega}$$

$$(\to E)\ \frac{B \vdash M{:}\sigma\to\tau \quad B \vdash N{:}\sigma}{B \vdash MN{:}\tau} \qquad (\to I)\ \frac{B, x{:}\sigma \vdash M{:}\tau}{B \vdash \lambda x.M{:}\sigma\to\tau}$$

$$(\mu I)\ \frac{B \vdash M{:}\sigma[\mu t.\sigma/t]}{B \vdash M{:}\mu t.\sigma} \qquad (\mu E)\ \frac{B \vdash M{:}\mu t.\sigma}{B \vdash M{:}\sigma[\mu t.\sigma/t]}$$

$$(\forall I)\ \frac{B \vdash M{:}\sigma}{B \vdash M{:}\forall t.\sigma}\ \text{if } t\notin FV(B) \qquad (\forall E)\ \frac{B \vdash M{:}\forall t.\sigma}{B \vdash M{:}\sigma[\tau/t]}$$

$$(\exists I)\ \frac{B \vdash M{:}\sigma[\tau/t]}{B \vdash M{:}\exists t.\sigma} \qquad (\exists E)\ \frac{B, x{:}\sigma \vdash M{:}\tau \quad B \vdash N{:}\exists t.\sigma}{B \vdash M[N/x]{:}\tau}\ \text{if } t\notin FV(B)\cup FV(\tau)$$

$$(\wedge I)\ \frac{B \vdash M{:}\sigma \quad B \vdash M{:}\tau}{B \vdash M{:}\sigma\wedge\tau} \qquad (\wedge E)\ \frac{B \vdash M{:}\sigma\wedge\tau}{B \vdash M{:}\sigma} \quad \frac{B \vdash M{:}\sigma\wedge\tau}{B \vdash M{:}\tau}$$

$$(\vee I)\ \frac{B \vdash M{:}\sigma}{B \vdash M{:}\sigma\vee\tau} \quad \frac{B \vdash M{:}\tau}{B \vdash M{:}\sigma\vee\tau} \qquad (\vee E)\ \frac{B, x{:}\sigma \vdash M{:}\rho \quad B, x{:}\tau \vdash M{:}\rho \quad B \vdash N. \quad \tau}{B \vdash M[N/x]{:}\rho}$$

The admissibility of the following rules is straightforward.

(Weakening) $\dfrac{B \vdash M:\tau}{B, x:\sigma \vdash M:\tau}$ (Strengthening) $\dfrac{B, x:\sigma \vdash M:\rho}{B \vdash M:\rho}$ if $x \notin FV(M)$.

This presentation looks familiar and intuitive, and it shares with natural deduction systems the nice property of defining the "meaning" of type operators directly, through introduction-elimination rules.

However, if one is interested in the proof theoretical properties of the system it is useful to translate it in a sequent calculus style. To this aim we introduce a type assignment system whose rules are symmetrically introductions to the left and to the right of type constructors. Although we do this primarily for technical reasons, we think that this system is of interest in its own.

This calculus is not a pure sequent calculus since we still consider bases as *sets* of premises, in order both to keep the two systems as close as possible, and to avoid in the following proofs the boring treatment of structural rules. The "multiplicative" character of its rules is however a typical feature of Gentzen's original calculus that we preserve.

A ***sequent*** is an expression of the form $B :- M:\sigma$ that can be derived by the axioms and rules of Definition 1.3.
We write B, B' to mean $B \cup B'$, provided that this is a still a basis, i.e. x is the subject of the same basic typing statement whenever x is in FV(B) and FV(B').

1.3 Definition *(Sequent Calculus Formulation)*
The axioms and rules to derive sequents are:

$$\dfrac{}{B, x:\sigma :- x:\sigma}\ \omega \qquad \dfrac{}{B :- M:\omega}\ Ax$$

$$\dfrac{B, x:\tau :- M:\rho \quad B' :- N:\sigma}{B, B', y:\sigma\to\tau :- M[yN/x]:\rho}\ \to L \qquad \dfrac{B, x:\sigma :- M:\tau}{B :- \lambda x.M:\sigma\to\tau}\ \to R$$

$$\dfrac{B, x:\sigma[\mu t.\sigma] :- M:\tau}{B, x:\mu t.\sigma :- M:\tau}\ \mu L \qquad \dfrac{B :- M:\sigma[\mu t.\sigma]}{B :- M:\mu t.\sigma}\ \mu R$$

$$\dfrac{B, x:\sigma[\rho/t] :- M:\tau}{B, x:\forall t.\sigma :- M:\tau}\ \forall L \qquad \dfrac{B :- M:\tau}{B :- M:\forall t.\tau}\ \forall R \quad \text{if } t\notin FV(B)$$

$$\dfrac{B, x:\sigma :- M:\tau}{B, x:\exists t.\sigma :- M:\tau}\ \exists L \quad \text{if } t\notin FV(B)\cup FV(\tau) \qquad \dfrac{B :- M:\tau[\rho/t]}{B :- M:\exists t.\tau}\ \exists R$$

$$\frac{B, x:\sigma :- M:\rho}{B, x:\sigma\wedge\tau :- M:\rho}\ \wedge L \qquad \frac{B, x:\tau :- M:\rho}{B, x:\sigma\wedge\tau :- M:\rho}\ \wedge L \qquad \frac{B :- M:\sigma \quad B' :- M:\tau}{B, B' :- M:\sigma\wedge\tau}\ \wedge R$$

$$\frac{B, x:\sigma :- M:\rho \quad B', x:\tau :- M:\rho}{B, B', x:\sigma\vee\tau :- M:\rho}\ \vee L \qquad \frac{B :- M:\sigma}{B :- M:\sigma\vee\tau}\ \vee R \qquad \frac{B :- M:\tau}{B :- M:\sigma\vee\tau}\ \vee R$$

$$\frac{B, x:\sigma :- M:\tau \quad B' :- N:\sigma}{B, B' :- M[N/x]:\tau}\ \text{cut}$$

It is immediate to verify that rules (Weakening) and (Strengthening) are admissible also in the sequent formulation.

Recall that, by the notational convention just before this definition, the set of assumptions in the conclusion of each rule has to be actually a basis, that is each term variable must occur at most once. There is no loss of generality in this restriction. In fact if $B :- M:\sigma$ and $B' :- N:\tau$, using ($\wedge$L) we can always build B" and B''' such that $B'' :- M:\sigma$ and $B''' :- N:\tau$, and B", B''' is a basis. More precisely B" and B''' will contain $x:\rho\wedge\upsilon$ whenever $x:\rho\in B$ and $x:\upsilon\in B'$.

Notation

(i) If $\mathbf{D}$ is a derivation showing $B :- M:\sigma$ we write $\mathbf{D}: B :- M:\sigma$.

(ii) We shall denote an arbitrary rule by

$$\frac{B_i :- M:\sigma_i \ (i=0, 1 \text{ or } 2)}{B :- N:\tau}\ \text{rule}$$

to take into account that there are rules with zero, one or two premises.

The proof of the equivalence between the systems $\mathbf{N}$ and $\mathbf{S}$ is standard.

1.4 Lemma (*Substitution Lemma*)

$$B, x:\sigma \vdash M:\tau \text{ and } B \vdash N:\sigma \Rightarrow B \vdash M[N/x]:\tau.$$

Proof. This can be proved as usual for Curry type systems, by induction on the derivation of $B, x:\sigma \vdash M:\tau$. However in the present setting we can shorten the proof as follows:

$$(\vee E)\ \frac{B, x:\sigma \vdash M:\tau \quad B, x:\sigma \vdash M:\tau \quad (\vee I)\dfrac{B \vdash N:\sigma}{B \vdash N:\sigma\vee\sigma}}{B \vdash M[N/x]:\tau} \qquad \square$$

1.5 *Theorem* (*Equivalence between* $\vdash$ *and* :-)
For all bases B, $M \in \Lambda$ and types σ,

$$B \vdash M:\sigma \text{ iff } B :- M:\sigma.$$

Proof.
($\Rightarrow$) By induction on the derivation of $B \vdash M:\sigma$, distinguishing cases according to the last inference. We consider only the interesting cases, i.e. the elimination rules.

Case ($\rightarrow$E):

$$(\rightarrow E)\ \frac{B \vdash M:\sigma\rightarrow\tau \quad B \vdash N:\sigma}{B \vdash MN:\tau}$$

becomes

$$\cfrac{\cfrac{\cfrac{}{x:\tau :- x:\tau}\text{Ax} \quad \cfrac{\text{ind. hyp.}}{B :- N:\sigma}}{B, y:\sigma\rightarrow\tau :- yN:\tau}\rightarrow L \quad \cfrac{\text{ind. hyp.}}{B :- M:\sigma\rightarrow\tau}}{B :- MN:\tau}\text{cut}$$

Case ($\exists$E):

$$(\exists E)\ \frac{B, x:\sigma \vdash M:\rho \quad B \vdash \exists t.N:\sigma}{B \vdash M[N/x]:\rho}$$

becomes

$$\cfrac{\cfrac{\cfrac{\text{ind. hyp.}}{B, x:\sigma :- M:\rho}}{B, x:\exists t.\sigma :- M:\rho}\exists L \quad \cfrac{\text{ind. hyp.}}{B :- N:\exists t.\sigma}}{B :- M[N/x]:\rho}$$

since $t \notin FV(B) \cup FV(\tau)$.

Case (χE) where $\chi \in \{\mu, \forall, \wedge\}$:

$$(\chi E)\ \frac{B \vdash M:\tau}{B \vdash M:\sigma}$$

becomes

$$\cfrac{\cfrac{\cfrac{}{x:\sigma :- x:\sigma}\text{Ax}}{x:\tau :- x:\sigma}\chi L \quad \cfrac{\text{ind. hyp.}}{B :- M:\tau}}{B :- M:\sigma}\text{cut}$$

Case ($\vee$E):

$$(\vee E)\ \frac{B, x:\sigma \vdash M:\rho \quad B, x:\tau \vdash M:\rho \quad B \vdash N:\sigma\vee\tau}{B \vdash M[N/x]:\rho}$$

becomes

$$\dfrac{\dfrac{\dfrac{\text{ind. hyp.}}{B, x{:}\sigma :- M{:}\rho} \quad \dfrac{\text{ind. hyp.}}{B, x{:}\tau :- M{:}\rho}}{B, x{:}\sigma\vee\tau :- M{:}\rho}\vee L \quad \dfrac{\text{ind. hyp.}}{B :- N{:}\sigma\vee\tau}}{B :- M[N/x]{:}\rho}\text{cut}$$

($\Leftarrow$) Symmetrically, by induction on the derivation of $B{:-}\ M{:}\sigma$. The proof is immediate in almost all cases. In case of cut use the Substitution Lemma 1.4. The other non trivial cases are $\to$L, $\forall$L, $\exists$L and $\vee$L.

Case $\to$L:

$$\dfrac{B, x{:}\tau :- M{:}\rho \qquad B' :- N{:}\sigma}{B, B', y{:}\sigma\to\tau :- M[yN/x]{:}\rho}\to L$$

We can freely assume that $x \notin FV(B')$. Now by the induction hypothesis and the admissibility of weakening we have

$$B,\ B',\ y{:}\sigma\to\tau,\ x{:}\tau \vdash M{:}\rho \text{ and } B,\ B',\ y{:}\sigma\to\tau \vdash N{:}\sigma.$$

From this, by (Ax) and ($\to$E), we get $B,\ B',\ y{:}\sigma\to\tau \vdash yN{:}\sigma$, hence the thesis follows by the Substitution Lemma.

Case χL where $\chi\in\{\mu,\forall,\wedge\}$:

$$\dfrac{B,x{:}\sigma :- M{:}\tau}{B,x{:}\rho :- M{:}\tau}\chi L$$

We have that $B,x{:}\sigma \vdash M{:}\tau$ implies $B,x{:}\rho \vdash M{:}\tau$ since each occurrence of the axiom

$$(Ax)\ \dfrac{}{B,x{:}\sigma \vdash x{:}\sigma}$$

can be replaced by the following deduction

$$(\forall E)\ \dfrac{(Ax)\ \dfrac{}{B,x{:}\rho \vdash x{:}\rho}}{B,x{:}\rho \vdash x{:}\sigma}\ .$$

Case $\exists$L:

$$\dfrac{B, x{:}\sigma :- M{:}\rho}{B, x{:}\exists t.\sigma :- M{:}\rho}\exists L \quad \text{where } t\notin FV(B)\cup FV(\tau).$$

By the induction hypothesis $B,\ x{:}\sigma \vdash M{:}\rho$, hence, if y is new, $B,\ y{:}\sigma \vdash M[y/x]{:}\rho$, being deductions independent of the names of variables. By weakening $B,\ x{:}\exists t.\sigma, y{:}\sigma \vdash M[y/x]{:}\rho$ so we can conclude:

$$(\exists E)\ \frac{B, x{:}\exists t.\sigma, y{:}\sigma \vdash M[y/x]{:}\rho \quad B, x{:}\exists t.\sigma \vdash x{:}\exists t.\sigma}{B, x{:}\exists t.\sigma \vdash M{:}\rho} \text{ where } t \notin FV(B) \cup FV(\tau)$$

since $M[y/x][x/y] \equiv M$.

Case ∨L:

$$\frac{B, x{:}\sigma :\!- M{:}\rho \quad B', x{:}\tau :\!- M{:}\rho}{B, B', x{:}\sigma\vee\tau :\!- M{:}\rho} \ \vee L$$

By the induction hypothesis $B,\ x{:}\sigma \vdash M{:}\rho$, hence, if y is fresh, $B,\ B',\ x{:}\sigma\vee\tau,\ y{:}\sigma \vdash M[y/x]{:}\rho$, being deductions independent of the names of variables and using weakening. Similarly, $B,\ B',\ x{:}\sigma\vee\tau,\ y{:}\tau \vdash M[y/x]{:}\rho$; hence the thesis follows from:

$$(\vee E)\ \frac{B, x{:}\sigma\vee\tau, y{:}\sigma \vdash M[y/x]{:}\rho \quad B, x{:}\sigma\vee\tau, y{:}\tau \vdash M[y/x]{:}\rho \quad B, x{:}\sigma\vee\tau \vdash x{:}\sigma\vee\tau}{B, x{:}\sigma\vee\tau \vdash M{:}\rho}.$$

□

2. Invariance of types under parallel reduction of subjects.

The type assignment systems $\vdash$ and $:-$ are not invariant under β-reduction of subjects. The problem is that in rule (cut) we lose the correspondence between subterms and subdeductions; many occurrences of the same subterm correspond in fact to a unique subdeduction. For example one can deduce the type $(\sigma\to\sigma\to\tau)\wedge(\rho\to\rho\to\tau)\to(\xi\to\sigma\vee\rho)\to\xi\to\tau$ both for $\lambda xyz.x(yz)(yz)$ and for $\lambda xyz.x((\lambda u.u)yz)((\lambda u.u)yz)$, but this type cannot be deduced for $\lambda xyz.x(yz)((\lambda u.u)yz)$ or $\lambda xyz.x((\lambda u.u)yz)(yz)$. (Benjamin Pierce showed us this example.) Analogously the type $(\forall t.\sigma\to\sigma\to\tau)\to(\xi\to\exists t.\sigma)\to\xi\to\tau$ (where t does not occur in τ) can be deduced for the first two but not for the second two λ-terms. Then in this system types are preserved neither under β-reduction nor under β-expansion. Similar examples show that types are not preserved under η-reduction.

Invariance under β-reduction is a desirable property, since it gives us the feeling that the rules in the system are, in a sense, correct. But in the present case rules (∃E), (∨E)

and (cut) could hardly be considered uncorrect, since they are sound for the semantics introduced in section 3.
Instead it seems that the phenomenon we are about arises from a mismatch between the type system and the possibility of unbalanced reductions of the subject. In fact we prove the system :- to be invariant under parallel reduction, hence, by the equivalence of :- and $\vdash$, the same property is established for $\vdash$. More precisely types are invariant under β-reductions which are done simultaneously on all the occurrences of the same subterm which correspond to the same subdeduction.

Our proof is inspired by Gentzen's proof of the Haupsatz [Gentzen 1969]. Following his lines we introduce the notions of rank and degree of cuts. A similar but simpler proof is carried out for a system without μ, $\forall$ and $\exists$ in [Barbanera, Dezani and de' Liguoro 1992].
Usually the cut elimination proves the (weak) normal form theorem for λ-terms having types. This is not true here, since we allow recursive types and the universal type ω.

2.1 Definition (*Fathers and Generated Types*).
(i) Given an arbitrary rule

$$\frac{B_i :- M: \sigma_i}{B :- N: \tau} \text{ rule}$$

where i=1 or 2 according to Definition 1.3. we say that:
- the occurrence of σ_i in a premise is the *father* of the occurrence of τ iff $\sigma_i \equiv \tau$ (this is true in all left rules and in rule (cut))
- if $x: \upsilon \in B_i$ and $x: \xi \in B$ with $\upsilon \equiv \xi$ then the occurrence of υ is a *father* of the occurence of ξ.

(ii) Looking at the shapes of logical rules as shown in Figure 1 we say that:
- $\sigma \to \tau$ is the *type generated* by $\to$L and $\to$R from σ and τ
- $\sigma \wedge \tau$ is the *type generated* by $\wedge$R from σ and τ
- $\sigma \vee \tau$ is the *type generated* by $\vee$L from σ and τ
- σ is the *type generated* by χL and χ'R from τ.

Clearly a type either is generated or it has at least one father.

$$\frac{B, x{:}\tau :\!- M{:}\rho \quad B' :\!- N{:}\sigma}{B, B', y{:}\sigma\rightarrow\tau :\!- M[yN/x]{:}\rho} \rightarrow L \qquad \frac{B, x{:}\sigma :\!- M{:}\tau}{B :\!- \lambda x.M{:}\sigma\rightarrow\tau} \rightarrow R$$

$$\frac{B, x{:}\tau :\!- M{:}\rho}{B, x{:}\sigma :\!- M{:}\rho}\ \chi L\ \ (\chi\in\{\wedge,\forall,\exists,\mu\}) \qquad \frac{B :\!- M{:}\tau}{B :\!- M{:}\sigma}\ \chi' R\ (\chi'\in\{\vee,\forall,\exists,\mu\})$$

$$\frac{B :\!- M{:}\sigma \quad B' :\!- M{:}\tau}{B, B' :\!- M{:}\sigma\wedge\tau} \wedge R \qquad \frac{B, x{:}\sigma :\!- M{:}\rho \quad B', x{:}\tau :\!- M{:}\rho}{B, B', x{:}\sigma\vee\tau :\!- M{:}\rho} \vee L$$

Figure 1 - *Possible shapes of logical rules.*

2.2 Definition (*Degree of Type Occurrences*)

The *degree* of an occurrence of a type σ in a deduction $\mathfrak{D}$ is defined by induction on $\mathfrak{D}$:

- if σ occurs in the conclusion of (Ax) or (ω) the its degree is 0
- if σ occurs in the conclusion of a logical rule and is not the generated type, then its degree is the maximum of the degrees of its fathers
- if $\sigma\rightarrow\tau$ is the type generated by $\rightarrow$L or $\rightarrow$R then its degree is 0.5
- if $\sigma\wedge\tau$ ($\sigma\vee\tau$) is the type generated by $\wedge$R ($\vee$L) from σ and τ then its degree is the maximum of the degrees of σ and τ plus 1
- if σ is the type generated by χL ($\chi\in\{\mu,\forall,\exists,\wedge\}$) or by ($\chi'$R) ($\chi'\in\{\mu,\forall,\exists,\vee\}$) from τ then its degree is the degree of τ plus 1.

2.3 Definition (*Cut-type and Rank*)

Given a cut:

$$\frac{B, x{:}\sigma :\!- M{:}\tau \qquad B' :\!- N{:}\sigma}{B, B' :\!- M[N/x]{:}\tau} \text{cut}$$

we call B, $x{:}\sigma$:- $M{:}\tau$ the left premise, B' :- $N{:}\sigma$ the right premise and σ the *cut-type*.

The *left rank* of the cut is the largest number of consecutive statements in a path such that the lowest of these statements is the left premise of the cut and the basis of each of these statements contains $x{:}\sigma$.

The *right rank* of the cut is the largest number of consecutive statements in a path such that the lowest of these statements is the right premise of the cut and σ is the predicate of the succedent of each of these statements.

The *rank* of the cut is the sum of its left and right ranks.

2.4 Definition (*Ready Cuts and Ready Derivations*)

(i) A *cut* is *ready* iff it has rank 2;

(ii) A *derivation* is *ready* iff it contains only ready cuts.

If a cut is ready either its left and right premises are respectively the conclusions of the left and right introduction rule for the principal constructor of the cut-type, or one of its premises is an axiom or an instance of the rule ω. Figure 2 shows all possible shapes of ready cuts.

We introduce now a notion of parallel reduction. The parallelism comes out from the fact that in each reduction step more than a single redex is contracted.

In order to formalize this idea we define the notion of uniform set of redex occurrences in a term. Informally a set of redex occurrences in a term M is called uniform if, whenever it contains a redex occurrence, every other occurrence of the same redex is in the set as well.

As usual any λ-term M can be identified with its binary syntactical tree, where nodes are represented by the subset of $\{0,1\}^*$ which is the tree domain of M; if N is a subterm of M, and its syntactical subtree is rooted at $\alpha \in \{0,1\}^*$ in the tree of M, then we say that "N occurs at α in M", and denote this occurrence by $\langle\alpha,N\rangle$.

2.5 Definition (*Uniform and 1-uniform Sets of Redexes*)

Let $M \in \Lambda$, then:

(i) $\mathrm{Occ}(M) = \{\langle\alpha,N\rangle \mid N \text{ occurs at } \alpha \text{ in } M\}$;

(ii) $\mathrm{Red}(M) = \{\langle\alpha,(\lambda x.P)Q\rangle \in \mathrm{Occ}(M) \mid P,Q \in \Lambda,\ x \in \mathrm{Var}\}$;

(iii) $\mathcal{F} \subseteq \mathrm{Red}(M)$ is *uniform* iff

$$\langle\alpha,R\rangle \in \mathcal{F} \wedge \langle\beta,R\rangle \in \mathrm{Red}(M) \Rightarrow \langle\beta,R\rangle \in \mathcal{F};$$

(iv) $\mathcal{F} \subseteq \mathrm{Red}(M)$ is *1-uniform* iff $\mathcal{F}$ is uniform and $\{R \mid \exists\, \alpha.\ \langle\alpha,R\rangle \in \mathcal{F}\}$ is a singleton.

(a) $$\dfrac{\dfrac{B, z:\xi :- P:\tau \quad B' :- R:\upsilon}{B, B', x:\upsilon\to\xi :- P[xR/z]:\tau}\to L \qquad \dfrac{B'', y:\upsilon :- Q:\xi}{B'' :- \lambda y.Q:\upsilon\to\xi}\to R}{B, B', B'' :- P[xR/z][\lambda y.Q/x]:\tau}\ \text{cut}$$

(b) $$\dfrac{\dfrac{B, x:\xi :- P:\tau}{B, x:\upsilon :- P:\tau}\chi L \qquad \dfrac{B' :- Q:\xi}{B' :- Q:\upsilon}\chi R}{B, B' :- P[Q/x]:\tau}\ \text{cut}$$

where $\chi \in \{\mu, \exists, \forall\}$.

(c) $$\dfrac{\dfrac{B, x:\xi :- P:\tau}{B, x:\xi\wedge\upsilon :- P:\tau}\wedge L \qquad \dfrac{B' :- Q:\xi \quad B'' :- Q:\upsilon}{B', B'' :- Q:\xi\wedge\upsilon}\wedge R}{B, B', B'' :- P[Q/x]:\tau}\ \text{cut}$$

(d) $$\dfrac{\dfrac{B, x:\xi :- P:\tau \quad B'', x:\upsilon :- P:\tau}{B, B'', x:\xi\vee\upsilon :- P:\tau}\vee L \qquad \dfrac{B' :- Q:\xi}{B' :- Q:\xi\vee\upsilon}\vee R}{B, B', B'' :- P[Q/x]:\tau}\ \text{cut}$$

(e) $$\dfrac{\dfrac{B_i :- M:\rho (i=1 \text{ or } 2)}{B, x:\sigma :- M:\rho}\chi L \qquad \dfrac{}{B', y:\sigma :- y:\sigma}\text{Ax}}{B, B', y:\sigma :- M[y/x]:\rho}\ \text{cut}$$

(f) $$\dfrac{\dfrac{}{B, x:\sigma :- x:\sigma}\text{Ax} \qquad \dfrac{B_i :- M:\sigma_i (i=1 \text{ or } 2)}{B' :- M:\sigma}\chi R}{B, B' :- M:\sigma}\ \text{cut}$$

where χ is any type constructor, and the premises of rules χL and χR are one or two according to Definition 1.3.

(g) $$\dfrac{\dfrac{}{B, x:\omega :- x:\omega}\Theta \qquad \dfrac{}{B' :- M:\omega}\omega}{B, B' :- M:\omega}\ \text{cut}$$

where Θ is either Ax or ω.

Figure 2 - *Possible shapes of ready cuts.*

In the sequel we need the notions of residual and development, which are well known concepts of λ-calculus theory: we refer the reader to [Barendregt 1984, chapter 11] from which we take our notation.

2.6 *Definition* (*Parallel Reduction*)

The reduction relation $\Rightarrow_p$ over Λ^2 is defined by:

$$M \Rightarrow_p N \text{ iff } \exists \mathcal{F} \subseteq \text{Red}(M).\ \mathcal{F} \text{ is uniform and } (M,\mathcal{F}) \twoheadrightarrow_{cpl} N$$

where $(M,\mathcal{F}) \twoheadrightarrow_{cpl} N$ is the complete development of $(M,\mathcal{F})$ (see [Barendregt 1984, p.286]).

In the literature the idea of parallel reduction in the framework of the λ-calculus is usually formalized by the the Gross-Knuth Reduction as defined in [Böhm 1966] and [Knuth 1970] (cf. also [Barendregt 1984, 13.2.7]).

2.7 *Definition* (*Gross-Knuth Reduction*)

$$M \rightarrow_{gk} N \text{ iff } (M, \text{Red}(M)) \twoheadrightarrow_{cpl} N.$$

Actually this is a particular case of the relation $\Rightarrow_p$ since Red(M) is trivially uniform.

2.8 *Lemma*

Let $\mathcal{F} \subseteq \text{Red}(M)$ be uniform and $\mathcal{F} = \mathcal{F}_1 \cup .. \cup \mathcal{F}_m$ where each $\mathcal{F}_i$ is 1-uniform. If $(M,\mathcal{F}) \twoheadrightarrow_{cpl} N$, then there are $N_0,\dots,N_m$ and a permutation π of order m such that $M \equiv N_0$, $N \equiv N_m$, and for all $1 \leq i \leq m$,

$$(N_{i-1}, \mathcal{F}'_{\pi(i)}) \twoheadrightarrow_{cpl} N_i$$

where $\mathcal{F}'_{\pi(i)}$ is the set of residuals of redexes in $\mathcal{F}_{\pi(i)}$ with respect to the reduction $M \twoheadrightarrow N_{i-1}$, and $\mathcal{F}'_{\pi(i)}$ is 1-uniform.

Proof.

We introduce an ordering on the 1-uniform sets of redexes of M. Let $\mathcal{F} = \{\langle\alpha_h, R\rangle \mid h \in H\}$ and $\mathcal{F}' = \{\langle\alpha_k, R'\rangle \mid k \in K\}$ then

$$\mathcal{F} \leq \mathcal{F}' \text{ iff } \exists h \in H\ \forall k \in K.\ \alpha_k \leq_{lex} \alpha_h,$$

where $\leq_{lex}$ is the lexicographic order. This is a total ordering and we take π as the permutation s.t. for all i, $\mathcal{F}_{\pi(i)} < \mathcal{F}_{\pi(i+1)}$. If $i < j$ then $\mathcal{F}_{\pi(i)} < \mathcal{F}_{\pi(j)}$, so that all occurrences of $R' \in \mathcal{F}_{\pi(j)}$ either contain the same copies of redexes $R \in \mathcal{F}_{\pi(i)}$ by 1-uniformity, so that they are modified in the same way; or they remain unchanged, when reducing all occurrences of R. In any case their residuals are syntactically equal, so that the $\mathcal{F}'_{\pi(i)}$ are 1-uniform. Now the thesis follows from the uniqueness of the result of a complete development [Barendregt 1984, Theorem 11.2.25] since then the reduction $(M,\mathcal{F}) \twoheadrightarrow_{cpl} N$ can be rearranged as we need. $\square$

2.9 <u>*Definition*</u> *(Cuts generating $\mathcal{F}$ -redexes).*

Let $\mathcal{D}$ be a deduction whose conclusion subject is M; fix a uniform $\mathcal{F} \subseteq$ Red(M); finally let

$$\frac{B, x{:}\sigma :\text{-}\ P{:}\tau \qquad B' :\text{-}\ \lambda y.Q{:}\sigma}{B, B' :\text{-}\ P[\lambda y.Q/x]{:}\tau}\ \text{cut}$$

be a cut in $\mathcal{D}$. We say that the above cut *generates* ***$\mathcal{F}$-redexes of*** M iff P has at least a subterm of the shape xR, such that all occurrences of (λy.Q')R' in M belong to $\mathcal{F}$, where (λy.Q')R' is an arbitrary substitution instance of (λy.Q)R.

2.10 <u>*Remark*</u> We have to consider the substitution instance (λx.Q')R' instead of (λy.Q)R itself because of the possibility that other cuts, under the given one in the current deduction, substitute variables inside (λy.Q)R.

2.11 <u>*Definition*</u> (*Cut Degree*)

Let $\mathcal{D}$ be a deduction whose conclusion subject is M; fix a uniform $\mathcal{F} \subseteq$ Red(M). Moreover let

$$\frac{B, x{:}\sigma :\text{-}\ M{:}\tau \qquad B' :\text{-}\ N{:}\sigma}{B, B' :\text{-}\ M[N/x]{:}\tau}\ \text{cut}$$

be a cut inference in $\mathcal{D}$. Then the *degree* of this cut in $\mathcal{D}$ relative to $\mathcal{F}$ is defined by cases:

- if this cut does not generates $\mathcal{F}$-redexes then its degree is 0
- otherwise its degree is the sum of the degrees of the occurences of its cut-type in the left and right premises of the cut.

2.12 <u>*Remark*</u> . Our definition of degree is not the classical one, which simply expresses the height of the synctactical tree of the type expression. The distinctive features of our cut degree are:

(i) the degree is 0 if the cut does not generates $\mathcal{F}$-redexes

(ii) the degree is 1 if the cut-type is an arrow

(iii) otherwise the degree is a function of the degrees of the occurences of its cut-type, i.e. of the number of inferences of the forms χL and χR which have been used in building these occurrences.

Condition (i) has to be so because we will eliminate only cuts generating $\mathcal{F}$-redexes. Moreover when eliminating a cut generating an $\mathcal{F}$-redex, whose cut-type is of the form $\sigma \to \tau$, the redexes generated by the new cuts cannot be in $\mathcal{F}$; hence their degree relative to $\mathcal{F}$ will be 0. Indeed if the contraction of a redex in $\mathcal{F}$ produces new redexes, these will be never contracted in the development of $\mathcal{F}$.

Condition (ii) is useful since we can eliminate ready cuts whose cut-type is an arrow only when all cuts generating the same $\mathcal{F}$-redex have as premises the conclusions of $\to$L and $\to$R (i.e. the cut has the shape of Figure 2(a)). When $\mathcal{F}$ is 1-uniform this implies that all cuts generating $\mathcal{F}$-redexes of shapes (b), (c), (d) or (e) have been eliminated.

Condition (iii) takes into account that when we shall eliminate a ready cut whose premises are conclusions of say $\forall$L and $\forall$R, the heigth of the cut-type of the newly generated cut may well be greater than that of the original cut-type, despite of the intuition that the derivation has been simplified. Note that in this third case the degree of the cut is always greater than 2 .

2.13 <u>*Definition*</u> (*Degree of Deductions and Non-increasing Deductions*).

(i) Let $\mathcal{D}$ be a deduction whose conclusion subject is M; fix a uniform $\mathcal{F} \subseteq$ Red(M).

The *degree* $d(\mathcal{D},\mathcal{F})$ of $\mathcal{D}$ relative to $\mathcal{F}$ is the highest degree of a cut in $\mathcal{D}$.

(ii) Let $\mathcal{D}$: B :- M:σ and $\mathcal{D}'$: B' :- M:σ : then $\mathcal{D}'$ is *non-increasing* with respect to $\mathcal{D}$ iff $d(\mathcal{D}',\mathcal{F}) \leq d(\mathcal{D},\mathcal{F})$ for all uniform $\mathcal{F} \subseteq$ Red(M).

We prove that we can transform deductions by adding assumptions, by changing the names of (free) term variables and by eliminating the cuts of the shapes (f) and (g) (of Figure 2) preserving readiness and non increasing the cut degrees.

We say that a deduction $\mathcal{D}'$ is *similar* to a deduction $\mathcal{D}$ iff $\mathcal{D}'$ can be obtained from $\mathcal{D}$ by adding (basic) typing statements to the bases and renaming term variables. This means that $\mathcal{D}'$ and $\mathcal{D}$ have the same deduction tree and differ only for the bases and the names of term variables.

2.14 *Lemma*

(i) If $\mathfrak{D}$: B :- M:σ and B' $\supseteq$ B then there is $\mathfrak{D}'$: B' :- M:σ such that $\mathfrak{D}'$ is similar to $\mathfrak{D}$.

(ii) If $\mathfrak{D}$: B, x:τ :- M:σ and y is fresh then there is $\mathfrak{D}'$: B, y:τ :- M[y/x]:σ such that $\mathfrak{D}'$ is similar to $\mathfrak{D}$.

(iii) All occurrences of ready cuts of the shapes (f) and (g) (of Figure 2) can be eliminated preserving readiness and obtaining a non increasing deduction of the same statement.

Proof.

(i). Easy by adding the premises of B'-B to each statement (possibly renaming some variables not in B).

(ii). Immediate, since deductions are independent of the names of (free) term variables.

(iii). Let us consider a cut of the shape (f) or (g):

$$\dfrac{\dfrac{}{B,\, x{:}\sigma :- x{:}\sigma}\,Ax \qquad \dfrac{B_i :- M{:}\sigma_i\,(i{=}\,1 \text{ or } 2)}{B' :- M{:}\sigma}\,\chi R}{B,\, B' :- M{:}\sigma}\ \text{cut}$$

$$\dfrac{\dfrac{}{B,\, x{:}\omega :- x{:}\omega}\,\Theta \qquad \dfrac{}{B' :- M{:}\omega}\,\omega}{B,\, B' :- M{:}\omega}\ \text{cut}$$

where Θ is either Ax or ω.

In both cases (i) implies that we can replace B' :- M:σ by B, B' :- M:σ obtaining a deduction which satisfies the given conditions. □

The following lemma (proved in the Appendix) claims that every deduction can be transformed into a ready deduction of the same statement. This can be accomplished without increasing the complexity of the proof. Formally

2.15 *Lemma (Rank Lemma)*

For any derivation $\mathfrak{D}$: B :- M:σ there exists a ready derivation $\mathfrak{D}'$: B :- M:σ such that $\mathfrak{D}'$ is *non-increasing* with respect to $\mathfrak{D}$.

2.16 *Theorem* (*Invariance of Types under Parallel Reduction*)

(i) $B :- M:\rho$ and $M \Rightarrow_p N$ imply $B :- N:\rho$.

(ii) $B \vdash M:\rho$ and $M \Rightarrow_p N$ imply $B \vdash N:\rho$.

Proof.

(i). If $M \Rightarrow_p N$, then for some uniform $\mathcal{F} \subseteq Red(M)$, $(M,\mathcal{F}) \twoheadrightarrow_{cpl} N$. By Lemma 2.8 this reduction can be splitted into a finite number of steps, whose $\mathcal{F}$'s are 1-uniform. Hence it suffices to prove the thesis when $\mathcal{F}$ is 1-uniform. On the other hand by Lemmas 2.14(iii) and 2.15, we can assume that the given $\mathfrak{D}:B:-M:\rho$ is ready and does not contain cuts of shapes (f) and (g). Hence we proceed by induction on $d(\mathfrak{D},\mathcal{F})$.

If $d(\mathfrak{D},\mathcal{F}) = 0$ and $\mathfrak{D}$ is ready, then no cut generates $\mathcal{F}$–redexes, hence $\mathcal{F}$–redexes may occur only as subterms of terms of type Σ: in this case clearly $B :- N:\rho$ is derivable introducing by rule ω the reducta of those subtems of M.

If $d(\mathfrak{D},\mathcal{F}) = 1$ then all cuts which generate $\mathcal{F}$ redexes have an arrow type as cut-type; hence, by the readiness of $\mathfrak{D}$, they have the shape

$$\dfrac{\dfrac{B, z:\xi :- P:\tau \quad B' :- R:\upsilon}{B, B', x:\upsilon\to\xi :- P[xR/z]:\tau}\to L \qquad \dfrac{B'', y:\upsilon :- Q:\xi}{B'' :- \lambda y.Q:\upsilon\to\xi}\to R}{B, B', B'' :- P[xR/z][\lambda y.Q/x]:\tau}\,cut$$

Then all these cuts can be replaced as follows:

$$\dfrac{B, z:\xi :- P:\tau \qquad \dfrac{B'', y:\upsilon :- Q:\xi \quad B' :- R:\upsilon}{B', B'' :- Q[R/y]:\xi}\,cut}{B, B', B'' : P[Q[R/y]/z]:\tau}\,cut$$

and in the rest of $\mathfrak{D}$ the subject parts have to be reduced (by $\Rightarrow_p$) in order to match $P[Q[R/y]/z]$. Eventually we get a deduction of degree 0.

If $d(\mathfrak{D},\mathcal{F}) > 1$ let k be the number of cuts with degree $d(\mathfrak{D},\mathcal{F})$. They are all of the shape of Figure 2(b),(c),or (d). In fact all cuts of shape (e) have degree 0.
We can replace the cuts of the shape of Figure 2(b) by:

$$\frac{B, x{:}\xi :- P{:}\tau \qquad B' :- Q{:}\xi}{B, B' :- P[Q/x]{:}\tau} \text{ cut}$$

We can replace the cuts of the shape of Figure 2(c) or 2(d) by:

$$\frac{B, x{:}\xi :- P{:}\tau \qquad B', B'' :- Q{:}\xi}{B, B', B'' :- P[Q/x]{:}\tau} \text{ cut}$$

where we can add statements in the bases of the premises by **2.14(i)**.

We can apply to the so obtained deduction the Rank Lemma. In this way we obtain a ready deduction of lower degree, so the induction hypothesis applies.
(ii) Immediate from (i).

□

2.17 _Corollary_ (*Invariance under Gross-Knuth Reduction*)
(i) $B \vdash M{:}\sigma,\ M \rightarrow_{gk} N \Rightarrow B \vdash N{:}\sigma;$
(ii) $B \vdash M{:}\sigma,\ M \rightarrow_{\beta} N \Rightarrow \exists L.\ N \twoheadrightarrow_{\beta} L \ \&\ B \vdash L{:}\sigma.$

Proof.
(i) Immediate from 2.16(ii).
(ii) is a consequence of (i) and of the cofinality of $\rightarrow_{gk}$ with respect to $\rightarrow_{\beta}$, proved in [Barendregt 1984, 13.2.11]. □

Note that the above proof cannot be transformed into a proof of normalization of λ-terms (which does not hold since we allow recursive types). In fact when we eliminate a cut whose cut-type is an arrow we obtain two cuts whose cut-types have in general bigger degrees relative to the newly generated redex occurrences.

3. Semantics.

In the present section we describe the construction of a model for our type system, based on a technique introduced in [Coppo 1985] that permits to interpret type recursion even when type constructors are not monotonic with respect to the (semantical) relation of subtyping induced by the interpretation of types as sets of values. This technique differs from that used by [MacQueen, Plotkin and Sethi 1986] in exploiting just the approximation properties of the domain in which untyped terms are interpreted, without any appeal to a complete metric space structure that can be defined on the collection of (denotations of) types. This approach to type interpretations has some advantages over that followed by [MacQueen, Plotkin and Sethi 1986], in not requiring any restriction on the formation rules for types and also in offering a powerful proof technique for many semantical properties (see [Cardone and Coppo 1991] for some examples) of the type system.

Starting from the standard model D_∞ of the λ-calculus, obtained as the inverse limit of complete lattices [Scott 1972], each type will be interpreted as a subset of D satisfying constraints derived from the order-theoretical nature of this model. For example, as every type can be assigned to the combinator $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$, and the interpretation of Ω in every continuous λ-model D is the least element $\perp$ (cp. [Barendregt 1984]), we require that each of the subsets of D that we will consider as candidates for the interpretation of types should contain $\perp$. Furthermore, if $A \subseteq D$ is such a subset and $\Delta \subseteq A$ is a directed set, then also $\sqcup\Delta \in A$: this condition yields the correctness of the typing of the fixed-point combinator Y as a term of type $(\sigma \to \sigma) \to \sigma$, for any type σ.

The explicit construction of the domain D (see [Barendregt 1984] for a detailed description) starts from the two-points lattice and at every step, given the lattice D_n, D_{n+1} is defined to be the lattice of *continuous functions* from D_n to itself, with an embedding $\phi_n: D_n \to D_{n+1}$ and a projection $\psi_n: D_{n+1} \to D_n$. The inverse limit of the diagram of projections $\langle D_n, \psi_n \rangle_{n \in N}$ satisfies the recursive domain equation $D \cong [D \to D]$, with isomorphisms Φ: $D \to [D \to D]$ and Ψ: $[D \to D] \to D$: this induces an operation of application over D defined by

$$d \cdot e =_{Def} \Phi(d)(e), \text{ for } d, e \in D.$$

The construction of D endows it with an approximation structure which is orthogonal to its order structure, in the form of a denumerable sequence of continuous mappings

$$(.)_n: D \to D,$$

which provide inside D an image of each D_n. In particular, the following properties hold (see [Barendregt 1984], 18.2.8 and 18.2.9):

3.1 Proposition (*Properties of Approximation Mappings over D_∞*)
The family $\{(.)_n: D \to D\}_{n \in N}$ satisfies the following conditions: For all $d \in D$ and $n, m, k \in N$:

A.1. If $d = \top$ then $d_0 = \top$, otherwise $d_0 = \bot$.

A.2. $(d_n)_m = (d_m)_n = d_{\min(m,n)}$.

A.3. $d = \bigsqcup \{ d_n \mid n \in N \}$.

If f, $d \in D$, then:

F.1. If $n \leq k$, then $f_{n+1} \cdot d_k = f_{n+1} \cdot d_n$.

F.2. If $n \leq k$ then $(f_{k+1} \cdot d_n)_n = f_{n+1} \cdot d_n$.

F.3. $f_{n+1} \cdot d_n = f_{n+1} \cdot d = (f \cdot d_n)_n$.

F.4. $f = f_{n+1}$ if and only if $\forall d \in D.\ f \cdot d = (f \cdot d_n)_n$.

A consequence of this Proposition is that for $d \in [D \to D]$, $d \cdot e = \bigsqcup_{n \in N} d_{n+1} \cdot e_n$. Observe also that each $D_n = \{d_n \mid d \in D\}$ is a finite lattice.

We shall require each subset A of D which interprets a type to be closed under each of these mappings, in the sense that $d \in A$ implies that $d_n \in A$ for all $n \in N$. We obtain in this way a collection of subsets of D that is closed under a wide range of type constructors, and also offers some technical advantages over *ideals* that have been often used in the semantics of type inference systems (see, for example, [MacQueen, Plotkin and Sethi 1986]).

3.2 Definition (*Profinite Subsets*)
(i) The class P of *profinite subsets* of D is defined as follows. $A \in P$ whenever A is a subset of D such that:

- $\bot \in A$ and $\top \notin A$;
- A is *complete*: if $< d^{(i)} >_{i \in N}$ is an increasing chain in D such that $\forall i \in N.\ d^{(i)} \in A$, then $\bigsqcup_i d^{(i)} \in A$;
- A is *closed under approximations*: $d \in A$ implies $d_n \in A$, for all $n \in N$.

(ii) For profinite subset A, B of D, the relation $A \twoheadrightarrow_n B$ is defined to hold if and only if $A \subseteq B$ and $d_n \in A$ whenever $d \in B$.

Observe the close connection between this class of subsets of D and the class of relations used in [Amadio 1991] and [Cardone 1991] in order to build models for strongly typed, polymorphic languages with type recursion and subtyping.

Obviously the intersection of an arbitrary family X of profinite subsets of D is profinite, so the structure $\langle \mathbf{P}, \subseteq\rangle$ is a complete lattice. We shall denote the infinitary lattice operations by **inf** and **sup** when convenient. The following Proposition collects the main properties of this lattice, where $A_n = \{d_n \mid d \in A\}$.

3.3 *Proposition* (*Properties of Profinite Subsets*)
For $A \in \mathbf{P}$ and any $n \in \mathbf{N}$:
(i) $A_n \subseteq A$;
(ii) $A_n \in \mathbf{P}$;
(iii) $A_n \twoheadrightarrow_n A_{n+1}$
(iv) For $X \subseteq \mathbf{P}$, $(\bigcup_{A\in X} A_n) \in \mathbf{P}$;
(v) For any family $X \subseteq \mathbf{P}$ we have

$$(\mathbf{sup}\ X)_n = \bigcup_{A\in X} A_n.$$

Proof. (i) Immediate, as A is closed under approximations.
(ii) First we show that A_n is complete: if $<d^{(i)}>_{i\in\omega}$ is a chain such that $d^{(i)} \in A_n$ for any $i\in \mathbf{N}$, each $d^{(i)}$ belongs to A by (i), so $\sqcup_i d^{(i)} \in A$ because A is complete. But now observe that $(\sqcup_i d^{(i)})_n \in A_n$ and this yields $\sqcup_i d^{(i)} \in A_n$ by the continuity of $(.)_n$ and the fact that $d^{(i)} \in D_n$ for all $i\in \mathbf{N}$.
Furthermore, to show that A_n is closed under approximations, let $d \in A_n$. There are two cases:
- $m>n$: $d_m = d$ (because $d\in D_n$), so $d_m \in A_n$.
- $m\leq n$: As $d\in A$ (by (i)), $d_m \in A$, and $d_m = (d_m)_n \in A_n$

concluding the proof of (ii).
(iii) To show that $A_n \twoheadrightarrow_n A_{n+1}$ assume $d_n \in A_n$. This implies $d_n \in A$ by (*i*) so $d_n \in A_{n+1}$ because $(d_n)_{n+1} = d_n$. If $d\in A_{n+1}$ then $d \in A$ (by (i)) , so $d_n \in A_n$.
(iv) The interesting part concerns completeness. If $\{d^{(i)}\}_{i\in N}$ is an increasing chain such that $\{d^{(i)}\}_{i\in N} \subseteq \bigcup_{A\in X} A_n$, then it follows from the finiteness of each D_n that there is

an index j such that $d^{(j)} = \bigsqcup_{i\in N} d^{(i)}$, so $\bigsqcup_{i\in N} d^{(i)} \in \bigcup_{A\in X} A_n$ and this shows that $(\bigcup_{A\in X} A_n) \in \mathbf{P}$.

(v) It is easily seen that the following properties hold:

- $\bigcup_{A\in X} A_n \subseteq \bigcup_{A\in X} A_{n+1}$
- $d \in \bigcup_{A\in X} A_{n+1} \Rightarrow d_n \in \bigcup_{A\in X} A_n$

yielding that the set $U = \{d \mid \forall n\in \omega.\ d_n \in \bigcup_{A\in X} A_n\}$ is a profinite subset (using (iv)) such that $\bigcup_{A\in X} A_n \subseteq U$ for each $n\in\omega$. If $S\in \mathbf{P}$ is such that $S \supseteq \bigcup X$ then also $S \supseteq U$ because

$$\begin{aligned} d\in U &\Rightarrow \forall n\in \mathbf{N}.\ d_n \in \bigcup_{A\in X} A_n \\ &\Rightarrow \forall n\in \mathbf{N}.\ d_n \in \bigcup X \text{ (as } A_n \subseteq A \text{ by } (iii)) \\ &\Rightarrow \forall n\in \mathbf{N}.\ d_n \in S \\ &\Rightarrow d = \bigsqcup_{n\in N} d_n \in S. \end{aligned}$$

So $\mathbf{sup}\ X = U$ and $(\mathbf{sup}\ X)_n = \bigcup_{A\in X} A_n$. □

3.4 *Definition*

If A, B are profinite subsets of D, we introduce the following notations:

- $A \to B =_{Def} \{d \mid d\in D,\ \forall a \in A.\ d\cdot a\in B\}$
- $A_n \to^{n+1} B_n =_{Def} \{d \mid d\in D_{n+1},\ \forall a \in A_n.\ d\cdot a \in B_n\}$.

Then $A\to B$ and $A_n \to^{n+1} B_n$ are also profinite. We consider only the case of $A_n \to^{n+1} B_n$. If $a\in A_n$ and $<d^{(i)}>_{i\in N}$ is an increasing chain such that $<d^{(i)}>_{i\in N} \subseteq A_n \to^{n+1} B_n$, $<d^{(i)}\cdot a>_{i\in N} \subseteq B_n$ which is complete by Proposition 3.3(ii), so:

$$\bigsqcup_i (d^{(i)}\cdot a) = (\bigsqcup_i d^{(i)})\cdot a \in B_n$$

yielding the completeness of $A_n \to^{n+1} B_n$.

In order to show that this relation is also closed under approximations, take $d \in A_n \to^{n+1} B_n$. For any $k \in \mathbf{N}$ and $a \in A_n$, observe that $a_{k-1} \in A_n$ being A_n a profinite subset.

So $(d\cdot a_{k-1}) \in B_n$ and also $d_k\cdot a = d_k\cdot a_{k-1} = (d\cdot a_{k-1})_{k-1} \in B_n$.

We shall define the interpretation of a type by means of a construction which involves denumerably many approximations, matching the approximation structure of the λ-model in which untyped terms are interpreted.

3.5 _Definition_ (*Approximate Interpretations of Types*).
For all $n \in \mathbf{N}$, any type σ and any environment η, i.e. any function mapping type variables to profinite subsets of D, define $\mathfrak{I}^n [\![\sigma]\!]\eta$ (*the n-th approximation of the interpretation of σ in the environment η*) as follows, by induction on n and the complexity of the type σ:

$\mathfrak{I}^0[\![\sigma]\!]\eta = \{\perp\}$
$\mathfrak{I}^{n+1}[\![\omega]\!]\eta = D_{n+1}$
$\mathfrak{I}^{n+1}[\![t]\!]\eta = (\eta(t))_{n+1}$
$\mathfrak{I}^{n+1}[\![\sigma_1 \wedge \sigma_2]\!]\eta = \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta \cap \mathfrak{I}^{n+1}[\![\sigma_2]\!]\eta$
$\mathfrak{I}^{n+1}[\![\sigma_1 \vee \sigma_2]\!]\eta = \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta \cup \mathfrak{I}^{n+1}[\![\sigma_2]\!]\eta$
$\mathfrak{I}^{n+1}[\![\sigma_1 \to \sigma_2]\!]\eta = \mathfrak{I}^{n}[\![\sigma_1]\!]\eta \to^{n+1} \mathfrak{I}^{n}[\![\sigma_2]\!]\eta$
$\mathfrak{I}^{n+1}[\![\mu t.\ \sigma_1]\!]\eta = \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t := \mathfrak{I}^{n}[\![\mu t.\sigma_1]\!]\eta]$
$\mathfrak{I}^{n+1}[\![\forall t.\sigma_1]\!]\eta = \bigcap_{A \in \mathbf{P}} \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t:=A]$
$\mathfrak{I}^{n+1}[\![\exists t.\sigma_1]\!]\eta = \bigcup_{A \in \mathbf{P}} \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t:=A]$. □

By a simple inductive argument we prove that each $\mathfrak{I}^n[\![\sigma]\!]\eta$ is a profinite subset of D_n. We shall make frequent use in what follows of the fact that for any type σ and any $n \in \mathbf{N}$, $\mathfrak{I}^n[\![\sigma]\!]\eta$ depends only on the values $(\eta(t))_n$ of the type variables occurring in σ, as can be easily seen from the definition of the approximate type interpretations.

3.6 _Lemma_
For all types σ, for all $n \in \mathbf{N}$, all type environments η, $A \in \mathbf{P}$ and all type variables t:
(a) $\mathfrak{I}^{n+1}[\![\sigma]\!]\eta[t:=A_n] \subseteq \mathfrak{I}^{n+1}[\![\sigma]\!]\eta[t:=A_{n+1}]$;
(b) If $d \in \mathfrak{I}^{n+1}[\![\sigma]\!]\eta[t:=A_{n+1}]$ and $d \notin \mathfrak{I}^{n+1}[\![\sigma]\!]\eta[t:=A_n]$, then $d \in A_{n+1}$.

Proof.
By induction on the complexity of the type σ, the result being obvious when σ is a type variable or ω.

(**a**) If $d \in \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t:=A_n, s:=B]$ for some $B \in \mathbf{P}$, then by induction hypothesis we also have that $d \in \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t:=A_{n+1}, s:=B]$, so $d \in \mathfrak{I}^{n+1}[\![\exists s.\sigma_1]\!]\eta[t:=A_{n+1}]$.
(**b**) If $d \in \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t:=A_{n+1} s:=B']$ for some $B' \in \mathbf{P}$ and $d \notin \mathfrak{I}^{n+1}[\![\exists s.\sigma_1]\!]\eta[t:=A_n]$ then, in particular, $d \notin \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t:=A_n, s:=B']$, so the induction hypothesis applies and yields $d \in A_{n+1}$. □

3.7 <u>Lemma</u>

For all types σ, $n>0$ and any type environment η:

If, for all type environments η',

(i) $\mathfrak{I}^n[\![\sigma]\!]\eta' \Rightarrow_n \mathfrak{I}^{n+1}[\![\sigma]\!]\eta'$ and

(ii) $\mathfrak{I}^{n-1}[\![\mu t.\sigma]\!]\eta' \Rightarrow_{n-1} \mathfrak{I}^n[\![\mu t.\sigma]\!]\eta'$,

then $\mathfrak{I}^n[\![\sigma]\!]\eta[t:=\mathfrak{I}^{n-1}[\![\mu t.\sigma]\!]\eta] \Rightarrow_n \mathfrak{I}^{n+1}[\![\sigma]\!]\eta[t:=\mathfrak{I}^n[\![\mu t.\sigma]\!]\eta]$.

Proof.

Let $d \in \mathfrak{I}^n[\![\sigma]\!]\eta[t:=\mathfrak{I}^{n-1}[\![\mu t.\sigma]\!]\eta]$: observe that $\mathfrak{I}^{n-1}[\![\mu t.\sigma]\!]\eta = (\mathfrak{I}^n[\![\mu t.\sigma]\!]\eta)_{n-1}$ according to assumption (ii), so we can apply Lemma 3.6(**a**) in order to show that $d \in \mathfrak{I}^n[\![\sigma]\!]\eta[t:=\mathfrak{I}^n[\![\mu t.\sigma]\!]\eta]$ and from (i) it follows that

$$d \in \mathfrak{I}^{n+1}[\![\sigma]\!]\eta[t:=\mathfrak{I}^n[\![\mu t.\sigma]\!]\eta].$$

If $d \in \mathfrak{I}^{n+1}[\![\sigma]\!]\eta[t:=\mathfrak{I}^n[\![\mu t.\sigma]\!]\eta]$, then by (i) $d_n \in \mathfrak{I}^n[\![\sigma]\!]\eta[t:=\mathfrak{I}^n[\![\mu t.\sigma]\!]\eta]$ and by Lemma 3.6(**b**) $d_n \in \mathfrak{I}^n[\![\mu t.\sigma]\!]\eta = \mathfrak{I}^n[\![\sigma]\!]\eta[t:=\mathfrak{I}^{n-1}[\![\mu t.\sigma]\!]\eta]$. □

3.8 <u>Lemma</u>

For any $n \in \mathbf{N}$, all types σ and any environment η :

$$\mathfrak{I}^n[\![\sigma]\!]\eta \Rightarrow_n \mathfrak{I}^{n+1}[\![\sigma]\!]\eta.$$

Proof.

By induction on n. The basis is obvious, and the induction step is proved by induction on the complexity of the type σ.

• $\sigma \equiv t$, a type variable.

$\mathfrak{I}^n[\![t]\!]\eta = (\eta(t))_n$ and the result follows from Proposition 3.3(iii).

• $\sigma \equiv \omega$: it follows from Proposition 3.3(iii) that $D_n \Rightarrow_n D_{n+1}$.

• $\sigma \equiv \sigma_1 \rightarrow \sigma_2$:

In order to prove that

$$\mathfrak{I}^{n-1}[\![\sigma_1]\!]\eta \rightarrow^n \mathfrak{I}^{n-1}[\![\sigma_2]\!]\eta \subseteq \mathfrak{I}^n[\![\sigma_1]\!]\eta \rightarrow^{n+1} \mathfrak{I}^n[\![\sigma_2]\!]\eta$$

assume that $d \in \mathfrak{I}^{n-1}[\![\sigma_1]\!]\eta \rightarrow^n \mathfrak{I}^{n-1}[\![\sigma_2]\!]\eta$ and $a \in \mathfrak{I}^n[\![\sigma_1]\!]\eta$. By induction hypothesis $a_{n-1} \in \mathfrak{I}^{n-1}[\![\sigma_1]\!]\eta$, so

• $\sigma \equiv \sigma_1 \wedge \sigma_2$:

(**a**) $\Im^{n+1}[\![\sigma_1 \wedge \sigma_2]\!]\eta[t:=A_n] = \Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_n] \cap \Im^{n+1}[\![\sigma_2]\!]\eta[t:=A_n]$ which is included in $\Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_{n+1}] \cap \Im^{n+1}[\![\sigma_2]\!]\eta[t:=A_{n+1}] = \Im^{n+1}[\![\sigma_1 \wedge \sigma_2]\!]\eta[t:=A_{n+1}]$ by induction hypothesis.

(**b**) if $d \in \Im^{n+1}[\![\sigma_1 \wedge \sigma_2]\!]\eta[t:=A_{n+1}] = \Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_{n+1}] \cap \Im^{n+1}[\![\sigma_2]\!]\eta[t:=A_{n+1}]$ but $d \notin \Im^{n+1}[\![\sigma_1 \wedge \sigma_2]\!]\eta[t:=A_n] = \Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_n] \cap \Im^{n+1}[\![\sigma_2]\!]\eta[t:=A_n]$ then, for $i=1$ or $i=2$, $d \in \Im^{n+1}[\![\sigma_i]\!]\eta[t:=A_{n+1}]$ and $d \notin \Im^{n+1}[\![\sigma_i]\!]\eta[t:=A_n]$. Using the induction hypothesis we have that $d \in A_{n+1}$.

• $\sigma \equiv \sigma_1 \vee \sigma_2$: dual to the preceding case.

• $\sigma \equiv \sigma_1 \rightarrow \sigma_2$:

$$\Im^{n+1}[\![\sigma_1 \rightarrow \sigma_2]\!]\eta[t:=A_{n+1}] =$$
$$\Im^{n}[\![\sigma_1]\!]\eta[t:=A_{n+1}] \rightarrow^{n+1} \Im^{n}[\![\sigma_2]\!]\eta[t:=A_{n+1}] =$$
$$\Im^{n}[\![\sigma_1]\!]\eta[t:=A_n] \rightarrow^{n+1} \Im^{n}[\![\sigma_2]\!]\eta[t:=A_n]$$
$$\Im^{n+1}[\![\sigma_1 \rightarrow \sigma_2]\!]\eta[t:=A_n],$$

hence (**a**) holds, while (**b**) is vacuously true.

• $\sigma \equiv \mu s.\sigma_1$:

$$\Im^{n+1}[\![\mu s.\ \sigma_1]\!]\eta[t:=A_{n+1}] =$$
$$\Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_{n+1}, s:=\Im^{n}[\![\mu s.\sigma_1]\!]\eta[t:=A_{n+1}]] =$$
$$\Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_{n+1}, s:=\Im^{n}[\![\mu s.\sigma_1]\!]\eta[t:=A_n]]$$

and

$$\Im^{n+1}[\![\mu s.\ \sigma_1]\!]\eta[t:=A_n] =$$
$$\Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_n, s:=\Im^{n}[\![\mu s.\sigma_1]\!]\eta[t:=A_n]].$$

(**a**) Follows from the induction hypothesis.

(**b**) If $d \in \Im^{n+1}[\![\mu s.\ \sigma_1]\!]\eta[t:=A_{n+1}]$ but $d \notin \Im^{n+1}[\![\mu s.\ \sigma_1]\!]\eta[t:=A_n]$, then the induction hypothesis shows that $d \in A_{n+1}$.

• $\sigma \equiv \forall s.\sigma_1$:

(**a**) For any $B \in P$, $\Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_n, s:=B] \subseteq \Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_{n+1}, s:=B]$ by the induction hypothesis. Therefore $d \in \Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_n, s:=B]$ for all $B \in \mathbf{P}$ and it is also true that for all $B \in \mathbf{P}$ $d \in \Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_{n+1}, s:=B]$.

(**b**) Assume that $d \in \Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_{n+1}, s:=B]$ for all $B \in \mathbf{P}$, but there is some $B' \in \mathbf{P}$ such that $d \notin \Im^{n+1}[\![\sigma_1]\!]\eta[t:=A_n, s:=B']$. Then, by induction hypothesis we have that $d \in A_{n+1}$.

• $\sigma \equiv \exists s.\sigma_1$:

$$d\cdot a_{n-1} = d\cdot a \in \mathfrak{I}^{n-1}[\![\sigma_2]\!]\eta \subseteq \mathfrak{I}^{n}[\![\sigma_2]\!]\eta$$

using the induction hypothesis again, and Proposition 3.1.

If $d\in \mathfrak{I}^{n}[\![\sigma_1]\!]\eta \rightarrow^{n+1} \mathfrak{I}^{n}[\![\sigma_2]\!]\eta$ and $a\in \mathfrak{I}^{n-1}[\![\sigma_1]\!]\eta$, $a\in \mathfrak{I}^{n}[\![\sigma_1]\!]\eta$ by induction hypothesis, so $d\cdot a \in \mathfrak{I}^{n}[\![\sigma_2]\!]\eta$ and by induction hypothesis $(d\cdot a)_{n-1}\in \mathfrak{I}^{n-1}[\![\sigma_2]\!]\eta$. The result follows by observing that, using Proposition 3.1, $(d\cdot a)_{n-1} = d_n\cdot a$, so

$$d_n \in \mathfrak{I}^{n-1}[\![\sigma_1]\!]\eta \rightarrow^{n} \mathfrak{I}^{n-1}[\![\sigma_2]\!]\eta.$$

• $\sigma\equiv\sigma_1\wedge\sigma_2$:

$\mathfrak{I}^{n}[\![\sigma_1\wedge\sigma_2]\!]\eta = \mathfrak{I}^{n}[\![\sigma_1]\!]\eta \cap \mathfrak{I}^{n}[\![\sigma_2]\!]\eta \Rrightarrow_n \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta \cap \mathfrak{I}^{n+1}[\![\sigma_2]\!]\eta = \mathfrak{I}^{n+1}[\![\sigma_1\wedge\sigma_2]\!]\eta$,

using the induction hypothesis.

• $\sigma\equiv\sigma_1\vee\sigma_2$:

$\mathfrak{I}^{n}[\![\sigma_1\vee\sigma_2]\!]\eta = \mathfrak{I}^{n}[\![\sigma_1]\!]\eta \cup \mathfrak{I}^{n}[\![\sigma_2]\!]\eta \Rrightarrow_n \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta \cup \mathfrak{I}^{n+1}[\![\sigma_2]\!]\eta = \mathfrak{I}^{n+1}[\![\sigma_1\vee\sigma_2]\!]\eta$,

using the induction hypothesis.

• $\sigma\equiv\mu t.\sigma_1$:

The induction hypotheses make possible the application of Lemma 3.7 :

$\mathfrak{I}^{n}[\![\mu t.\sigma_1]\!]\eta =$

$\mathfrak{I}^{n}[\![\sigma_1]\!]\eta[t:=\mathfrak{I}^{n-1}[\![\mu t.\sigma_1]\!]\eta] \Rrightarrow_n \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t:=\mathfrak{I}^{n}[\![\mu t.\sigma_1]\!]\eta] =$

$\mathfrak{I}^{n+1}[\![\mu t.\sigma_1]\!]\eta$.

• $\sigma\equiv\forall t.\sigma_1$:

Assume that $d\in \mathfrak{I}^{n}[\![\forall t.\sigma_1]\!]\eta$ and $A\in \mathbf{P}$. Then, by induction hypothesis, $d\in \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t:=A]$ and therefore $d\in \mathfrak{I}^{n+1}[\![\forall t.\sigma_1]\!]\eta$, as $A\in \mathbf{P}$ was arbitrary. Conversely, if $d\in \mathfrak{I}^{n+1}[\![\forall t.\sigma_1]\!]\eta$ and $A\in \mathbf{P}$, by induction hypothesis we have $d_n\in \mathfrak{I}^{n}[\![\sigma_1]\!]\eta[t:=A]$, and finally that $d_n\in \mathfrak{I}^{n}[\![\forall t.\sigma_1]\!]\eta$.

• $\sigma\equiv\exists t.\sigma_1$:

Assume that $d\in \mathfrak{I}^{n}[\![\exists t.\sigma_1]\!]\eta$. Then, for some $A\in \mathbf{P}$, $d\in \mathfrak{I}^{n}[\![\sigma_1]\!]\eta[t:=A]$ and by induction hypothesis, $d\in \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t:=A]$; therefore $d\in \mathfrak{I}^{n+1}[\![\exists t.\sigma_1]\!]\eta$. Conversely, if $d\in \mathfrak{I}^{n+1}[\![\sigma_1]\!]\eta[t:=A]$ for some $A\in \mathbf{P}$, by induction hypothesis we have that $d_n\in \mathfrak{I}^{n}[\![\sigma_1]\!]\eta[t:=A]$, and finally that $d_n\in \mathfrak{I}^{n}[\![\exists t.\sigma_1]\!]\eta$. □

The preceding Lemma justifies the definition of the interpretation of a type by cumulating the information given by its approximate interpretations.

3.9 <u>Definition</u> (*Type Interpretations*)

For every type σ and type environment η,

$$\Im[\![\sigma]\!]\eta =_{Def} \{d \in D \mid \forall n \in N.\ d_n \in \Im^n[\![\sigma]\!]\eta\}.$$

3.10 *Proposition*

(i) $\forall n \in N.\ \Im^n[\![\sigma]\!]\eta \subseteq \Im[\![\sigma]\!]\eta$.

(ii) For any type σ and environment η, $\Im[\![\sigma]\!]\eta$ is a profinite subset of D.

Proof.

(i) Easy, using Lemma 3.8.

(ii) Let $<d^{(i)}>_{i \in N} \subseteq \Im[\![\sigma]\!]\eta$ be an increasing chain. We have for all $n \in N$ that $<(d^{(i)})_n>_{i \in N} \subseteq \Im^n[\![\sigma]\!]\eta$, so $(\sqcup_i d^{(i)})_n \in \Im^n[\![\sigma]\!]\eta$ by the continuity of the mappings $(.)_n$, yielding the completeness of $\Im[\![\sigma]\!]\eta$. If $d \in \Im[\![\sigma]\!]\eta$, then $d_n \in \Im^n[\![\sigma]\!]\eta \subseteq \Im[\![\sigma]\!]\eta$ by (i). □

This Proposition entails that, for any type σ and environment η,

$$(\Im[\![\sigma]\!]\eta)_n = \Im^n[\![\sigma]\!]\eta,$$

for all $n \in N$.

3.11 *Lemma*

For any type σ and environment η:

$$\Im[\![\mu t.\ \sigma]\!]\eta = \Im[\![\sigma]\!]\eta[t := \Im[\![\mu t.\sigma]\!]\eta].$$

Proof.

$(\subseteq)$ $\quad d \in \Im[\![\mu t.\ \sigma]\!]\eta \Rightarrow \forall n \in N.\ d_n \in \Im^n[\![\sigma]\!]\eta[t := \Im^{n-1}[\![\mu t.\sigma]\!]\eta]$

(by Definition 3.5)

$\Rightarrow \forall n \in N.\ d_n \in \Im^n[\![\sigma]\!]\eta[t := \Im^n[\![\mu t.\sigma]\!]\eta]$

(by Lemma 3.6(**a**))

$\Rightarrow \forall n \in N.\ d_n \in \Im^n[\![\sigma]\!]\eta[t := \Im[\![\mu t.\sigma]\!]\eta]$

$\Rightarrow d \in \Im[\![\sigma]\!]\eta[t := \Im[\![\mu t.\sigma]\!]\eta]$.

$(\supseteq)$ $d \in \Im[\![\sigma]\!]\eta[t := \Im[\![\mu t.\sigma]\!]\eta] \Rightarrow$

$\Rightarrow \forall n \in N.\ d_n \in \Im^n[\![\sigma]\!]\eta[t := \Im[\![\mu t.\sigma]\!]\eta]$

$\Rightarrow \forall n \in N.\ d_n \in \Im^n[\![\sigma]\!]\eta[t := \Im^n[\![\mu t.\sigma]\!]\eta]$

$\Rightarrow \forall n \in N.\ d_n \in \Im^n[\![\mu t.\sigma]\!]\eta$

(by Lemma 3.6(**b**))

$\Rightarrow d \in \Im[\![\mu t.\sigma]\!]\eta$. □

3.12 *Lemma*

For any pair of types σ,τ, any type variable t and any environment η,

$$\mathfrak{I}[\![\sigma[\tau/t]]\!]\eta = \mathfrak{I}[\![\sigma]\!]\eta[t:=\mathfrak{I}[\![\tau]\!]\eta].$$

Proof.

We show by induction on n that :

$$\forall n\in \mathbf{N}.\mathfrak{I}^n[\![\sigma[\tau/t]]\!]\eta = \mathfrak{I}^n[\![\sigma]\!]\eta[t:=\mathfrak{I}[\![\tau]\!]\eta] .$$

The basis is obvious, and the induction step is proved by induction on the complexity of σ, the only interesting case being that in which $\sigma \equiv \mu s.\sigma_1$, where s is not free in τ. Then $(\mu s.\sigma_1)[\tau/t] \equiv \mu s.(\sigma_1[\tau/t]) \equiv \mu s.\, \tilde{\sigma}_1$, where $\tilde{\sigma}_1 \equiv \sigma_1[\tau/t]$.

$$\mathfrak{I}^n[\![\ \mu s.\tilde{\sigma}_1\ ]\!]\eta = \mathfrak{I}^n[\![\sigma_1[\tau/t]]\!]\eta[s:=\mathfrak{I}^{n-1}[\![\ \mu s.\ \tilde{\sigma}_1\ ]\!]\eta] =$$

$$= \mathfrak{I}^n[\![\sigma_1]\!]\eta[t:=\mathfrak{I}[\![\tau]\!]\eta,\ s:=\mathfrak{I}^{n-1}[\![\mu s.\ \tilde{\sigma}_1]\!]\eta]$$

by induction hypothesis and because $s \notin FV(\tau)$

$$= \mathfrak{I}^n[\![\sigma_1]\!]\eta[t:= \mathfrak{I}[\![\tau]\!]\eta,$$

$$s:= \mathfrak{I}^{n-1}[\![\mu s.\sigma_1]\!]\eta[t:=\mathfrak{I}[\![\tau]\!]\eta]\ ]$$

by induction hypothesis on n applied to $(\mu s.\sigma_1)[\tau/t]$

$$= \mathfrak{I}^n[\![\mu s.\ \sigma_1]\!]\eta[t := \mathfrak{I}[\![\tau]\!]\eta] . \quad \square$$

3.13 *Theorem* (*Properties of Type Interpretations*)

The following conditions are satisfied, for any type environment η:

(1) $\mathfrak{I}[\![\ t\ ]\!]\eta = (\eta(t))$;

(2) $\mathfrak{I}[\![\ \omega\ ]\!]\eta = D$;

(3) $\mathfrak{I}[\![\ \sigma \to \tau\ ]\!]\eta = \mathfrak{I}[\![\ \sigma\ ]\!]\eta \to \mathfrak{I}[\![\ \tau\ ]\!]\eta$;

(4) $\mathfrak{I}[\![\ \sigma \wedge \tau\ ]\!]\eta = \mathfrak{I}[\![\ \sigma\ ]\!]\eta \cap \mathfrak{I}[\![\ \tau\ ]\!]\eta$;

(5) $\mathfrak{I}[\![\ \sigma \vee \tau\ ]\!]\eta = \mathfrak{I}[\![\ \sigma\ ]\!]\eta \cup \mathfrak{I}[\![\ \tau\ ]\!]\eta$;

(6) $\mathfrak{I}[\![\ \mu t.\ \sigma\ ]\!]\eta = \mathfrak{I}[\![\sigma[\mu t.\ \sigma/t]]\!]\eta$;

(7) $\mathfrak{I}[\![\forall t.\sigma\ ]\!]\eta = \inf\{\mathfrak{I}[\![\ \sigma\ ]\!]\eta[t:=A] \mid A \in \mathbf{P}\}$;

(8) $\mathfrak{I}[\![\exists t.\sigma\ ]\!]\eta = \sup\{\mathfrak{I}[\![\ \sigma\ ]\!]\eta[t:=A] \mid A \in \mathbf{P}\}$.

Proof.

(1) $d \in \mathfrak{I}[\![\ t\ ]\!]\eta$ iff, for all $n\in N$, $d_n \in \mathfrak{I}^n[\![\ t\ ]\!]\eta = (\eta(t))_n$ iff $d \in (\eta(t))$.

(2) Analogous to the preceding case.

(3) Assume $d \in \mathfrak{I}[\![\sigma \to \tau]\!]\eta$, so for all n

$$d_n\in \mathfrak{I}^{n-1}[\![\ \sigma\ ]\!]\eta \to^n \mathfrak{I}^{n-1}[\![\ \tau\ ]\!]\eta.$$

If $a \in \mathfrak{I}[\![\sigma]\!]\eta$, then $a_{n-1} \in \mathfrak{I}^{n-1}[\![\sigma]\!]\eta$ and

$$d_n \cdot a_{n-1} \in \mathfrak{I}^{n-1}[\![\tau]\!]\eta \subseteq \mathfrak{I}[\![\tau]\!]\eta ,$$

uniformly for all $n \in \mathbf{N}$. But $\mathfrak{I}[\![\tau]\!]\eta$ is complete so

$$d \cdot a = \bigsqcup_n (d_{n+1} \cdot a_n) \in \mathfrak{I}[\![\tau]\!]\eta,$$

showing that :

$$\mathfrak{I}[\![\sigma \to \tau]\!]\eta \subseteq \mathfrak{I}[\![\sigma]\!]\eta \to \mathfrak{I}[\![\tau]\!]\eta.$$

Conversely, if $d \in \mathfrak{I}[\![\sigma]\!]\eta \to \mathfrak{I}[\![\tau]\!]\eta$ and $a \in \mathfrak{I}^{n-1}[\![\sigma]\!]\eta \subseteq \mathfrak{I}[\![\sigma]\!]\eta$, we have that $d \cdot a \in \mathfrak{I}[\![\tau]\!]\eta$ and $(d \cdot a)_{n-1} \in \mathfrak{I}^{n-1}[\![\tau]\!]\eta$, but now it is sufficient to observe that $(d \cdot a)_{n-1} = d_n \cdot a$ to conclude that

$$d_n \in \mathfrak{I}^{n-1}[\![\sigma]\!]\eta \to^n \mathfrak{I}^{n-1}[\![\tau]\!]\eta.$$

Then $d_n \in \mathfrak{I}^n[\![\sigma \to \tau]\!]\eta$ for all $n \in \mathbf{N}$, yielding the conclusion.

(4) If $d \in \mathfrak{I}[\![\sigma \wedge \tau]\!]\eta$, then for any $n \in \mathbf{N}$ $d_n \in \mathfrak{I}^n[\![\sigma \wedge \tau]\!]\eta = \mathfrak{I}^n[\![\sigma]\!]\eta \cap \mathfrak{I}^n[\![\tau]\!]\eta \subseteq \mathfrak{I}[\![\sigma]\!]\eta \cap \mathfrak{I}[\![\tau]\!]\eta$. Conversely, given $d \in \mathfrak{I}[\![\sigma]\!]\eta \cap \mathfrak{I}[\![\tau]\!]\eta$, for every $n \in \mathbf{N}$ $d_n \in \mathfrak{I}^n[\![\sigma]\!]\eta \cap \mathfrak{I}^n[\![\tau]\!]\eta = \mathfrak{I}^n[\![\sigma \wedge \tau]\!]\eta$, therefore $d \in \mathfrak{I}[\![\sigma \wedge \tau]\!]\eta$.

(5) If $d \in \mathfrak{I}[\![\sigma \vee \tau]\!]\eta$, then for any $n \in \mathbf{N}$ $d_n \in \mathfrak{I}^n[\![\sigma \vee \tau]\!]\eta = \mathfrak{I}^n[\![\sigma]\!]\eta \cup \mathfrak{I}^n[\![\tau]\!]\eta \subseteq \mathfrak{I}[\![\sigma]\!]\eta \cup \mathfrak{I}[\![\tau]\!]\eta$. Conversely, given $d \in \mathfrak{I}[\![\sigma]\!]\eta \cup \mathfrak{I}[\![\tau]\!]\eta$, for every $n \in \mathbf{N}$ $d_n \in \mathfrak{I}^n[\![\sigma]\!]\eta$ or $d_n \in \mathfrak{I}^n[\![\tau]\!]\eta$, so $d_n \in \mathfrak{I}^n[\![\sigma]\!]\eta \cup \mathfrak{I}^n[\![\tau]\!]\eta$ and finally $d \in \mathfrak{I}[\![\sigma \wedge \tau]\!]\eta$..

(6) From Lemmas 3.11, 3.12.

(7) $$\begin{aligned} d \in \mathfrak{I}[\![\forall t.\sigma]\!]\eta &\Leftrightarrow \forall n \in \mathbf{N}.\, d_n \in \mathfrak{I}^n[\![\forall t.\sigma]\!]\eta = \textstyle\bigcap_{A \in \mathbf{P}} \mathfrak{I}^n[\![\sigma]\!]\eta[t:=A] \\ &\Leftrightarrow \forall n \in \mathbf{N}.\forall A \in \mathbf{P}.\ d_n \in \mathfrak{I}^n[\![\sigma]\!]\eta[t:=A] \\ &\Leftrightarrow \forall n \in \mathbf{N}.\forall A \in \mathbf{P}.\ d_n \in \mathfrak{I}[\![\sigma]\!]\eta[t:=A] \\ &\Leftrightarrow \forall A \in \mathbf{P}.\ d \in \mathfrak{I}[\![\sigma]\!]\eta[t:=A] \\ &\Leftrightarrow d \in \textstyle\bigcap_{A \in \mathbf{P}} \mathfrak{I}[\![\sigma]\!]\eta[t:=A]. \end{aligned}$$

(8) $(\sup\{\mathfrak{I}[\![\sigma]\!]\eta[t:=A] \mid A \in \mathbf{P}\})_n = \bigcup_{A \in \mathbf{P}} \mathfrak{I}^n[\![\sigma]\!]\eta[t:=A] = \mathfrak{I}^n[\![\exists t.\sigma]\!]\eta$ by Proposition 3.3(v) and the remark following Proposition 3.10(ii). The result follows immediately by observing that for any profinite subset A of D, $d \in A$ if and only if $d_n \in A_n$, for every $n \in \mathbf{N}$. □

The soundness of the typing rules of our type inference system can be shown quite naturally, given the set-theoretic flavor of the type interpretations we have defined and their closeness to the intuitions behind the rules for type constructors.

In the sequel, by a *term environment* we mean any function mapping term variables to elements of D. The interpretation of the set of λ-terms in the λ-model D is defined as usual, by induction on the structure of terms, by the clauses:

- $[\![x]\!]\rho = \rho(x)$;
- $[\![MN]\!]\rho = [\![M]\!]\rho \cdot [\![N]\!]\rho$
- $[\![\lambda x.M]\!]\rho = \Psi(f)$, where $f(d) = [\![M]\!]\rho[x:=d]$,

observing that every such an f is a continuous function from D to itself.

3.15 *Definition* (*Satisfaction*)
Given a type environment η and a term environment ρ, the pair $\langle\eta,\rho\rangle$ *satisfies* a typing statement $M:\sigma$, written $\langle\eta,\rho\rangle \models M:\sigma$, if $[\![M]\!]\rho \in \mathfrak{I}[\![\sigma]\!]\eta$. Given a basis B, $\langle\eta,\rho\rangle \models B$ if $\langle\eta,\rho\rangle \models x:\tau$ for all basic typing statements $x:\tau \in B$. We write $\langle\eta,\rho\rangle : B \models M:\sigma$ when $\langle\eta,\rho\rangle \models B$ implies $\langle\eta,\rho\rangle \models M:\sigma$, and $B \models M:\sigma$ when $\langle\eta,\rho\rangle : B \models M:\sigma$ for all η and ρ.

3.16 *Theorem* (*Soundness*)
For any basis B, any λ-term M and any type σ:

$$\text{if } B \vdash M:\sigma \text{ then } B \models M:\sigma.$$

Proof.
By a straightforward induction on the derivation of the statement $B \vdash M:\sigma$. We only outline the proof in the case in which the last rule applied in the derivation is:

$$(\exists E)\quad \frac{B, x:\sigma \vdash M:\tau \qquad B \vdash N:\exists t.\sigma}{B \vdash M[N/x]:\tau}$$

where $t \notin FV(B) \cup FV(\tau)$.
Assume that $\langle\eta,\rho\rangle \models B$, and observe that $[\![M[N/x]]\!]\rho = [\![M]\!]\rho[x:=[\![N]\!]\rho]$, for all λ-terms M, N. By induction hypothesis $[\![N]\!]\rho \in \mathfrak{I}[\![\exists t.\sigma]\!]\eta$, and therefore $([\![N]\!]\rho)_n \in \mathfrak{I}^n[\![\exists t.\sigma]\!]\eta = \bigcup_{A\in \mathbf{P}} \mathfrak{I}^n[\![\sigma]\!]\eta[t:=A]$, so there exists $C \in \mathbf{P}$ such that $([\![N]\!]\rho)_n \in \mathfrak{I}^n[\![\sigma]\!]\eta[t:=C]$. Now, $\langle\eta[t:=C], \rho[x:=([\![N]\!]\rho)_n]\rangle \models B, x:\sigma$, and this entails that $[\![M]\!]\rho[x:=([\![N]\!]\rho)_n] \in \mathfrak{I}[\![\tau]\!]\eta$, because $t \notin FV(\tau)$. This argument works uniformly for all $n \in \mathbf{N}$, and we can conclude that

$$[\![M]\!]\rho[x:=[\![N]\!]\rho] = \bigsqcup_{n\in\mathbf{N}} [\![M]\!]\rho[x:=([\![N]\!]\rho)_n] \in \mathfrak{I}[\![\tau]\!]\eta,$$

using the continuity properties of the interpretation mapping for terms. □

Acknowledgement

The authors are grateful to Franco Barbanera for his assistance in editing the derivations in the proof of the Rank Lemma.

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Appendix: Proof of the Rank Lemma.

It suffices to show that, if $\mathfrak{D}$: B :- M:σ is a derivation ending with one occurrence of a cut of rank r>2, in which all other cuts are ready, then $\mathfrak{D}$ can be transformed into a derivation $\mathfrak{D}'$: B :- M:σ containing some occurrences of the original cut with lower rank and possibly some new cuts of rank 2, while preserving the readiness of all other cuts. Concerning the non-increasing property of $\mathfrak{D}'$ wrt $\mathfrak{D}$, it is established by routine calculations: we shall actually perform just in the first case we treat and in some other relevant ones. Note that these calculations are parametric wrt $\mathcal{F}$, hence it is actually an arbitrary 1-uniform set of redex occurrences.

By Lemma 2.14(iii) we can assume that $\mathcal{D}$ does not contains ready cuts of the shape (f) or (g).

We split the proof into two parts: in the first one we lower the right rank of the considered cut, which is supposed to be > 1; in the second part we assume the right rank to be 1 and we lower the left rank supposed to be > 1. Since all transformations we do leave unchanged the left rank in part 1 and the right rank in part 2, this establishes the thesis.

Part 1: right rank > 1.

We distinguish various cases according to the shape of the last rule above the right premise of the cut. Note that Ax and ω cannot occur because of right rank being >1. For the same reason cases of rules of introducing type constructors to the right are impossible.

Case $\rightarrow$L:

$$\dfrac{B, x{:}\sigma :\!- M{:}\rho \quad \dfrac{B', u{:}\tau :\!- N{:}\sigma \quad B'' :\!- P{:}\xi}{B', B'', v{:}\xi\rightarrow\tau :\!- N[vP/u]{:}\sigma}\rightarrow L}{B, B', B'', v{:}\xi\rightarrow\tau :\!- M[N[vP/u]/x]{:}\rho}\text{cut} .$$

We can freely assume that $u \notin FV(B, B'')$ and $u \notin FV(M)$; hence we transform it into

$$\dfrac{\dfrac{B, x{:}\sigma :\!- M{:}\rho \quad B', u{:}\tau :\!- N{:}\sigma}{B, B', u{:}\tau :\!- M[N/x]{:}\rho}\text{cut} . \quad B'' :\!- P{:}\xi}{B, B', B'', v{:}\xi\rightarrow\tau :\!- M[N/x][vP/u]{:}\rho}\rightarrow L$$

Note that the new cuts in the figure above are legal since $u \notin FV(B, B'')$ and B, B', B" is a basis. Finally $u \notin FV(M)$ implies that $M[N[vP/u]/x] \equiv M[N/x][vP/u]$.

To verify the non-increasing property, suppose that the cut inference in the given derivation $\mathcal{D}$ is generating an $\mathcal{F}$-redex (otherwise the thesis is trivially true); then the corresponding cut in the transformed derivation $\mathcal{D}'$ is generating an $\mathcal{F}$-redex too. We have immediately that the degre of occurrences of the cut-type do not change.

Case χL $(\chi \in \{\mu, \forall, \exists, \wedge, \vee\})$:

$$\dfrac{B, x:\sigma :- M:\rho \qquad \dfrac{B_i, y:\tau :- N:\sigma \ (i=1 \text{ or } 2)}{B', y:\tau' :- N:\sigma}\,\chi L}{B, B', y:\tau' :- M[N/x]:\rho}$$

if $y \notin FV(B)$, we transform it into

$$\dfrac{\dfrac{B, x:\sigma :- M:\rho \quad B_i, y:\tau :- N:\sigma}{B, B_i, y:\tau :- M[N/x]:\rho}\,\text{cut } (i=1 \text{ or } 2)}{B, B', y:\tau' :- M[N/x]:\rho}\,\chi L \quad .$$

Otherwise $B = B'', y:\tau'$ for some B'', since $B, B', y:\tau'$ is a basis. Let w be any fresh variable; define $N' \equiv N[w/y]$, so that by 2.14(ii) there exists a derivation of $B', w:\tau :- N':\sigma$ similar to the given subderivation of $B', y:\tau :- N:\sigma$. Hence we make the following transformation

$$\dfrac{\dfrac{\dfrac{B'', y:\tau', x:\sigma :- M:\rho \quad B_i, w:\tau :- N':\sigma}{B_i, B'', y:\tau', w:\tau :- M[N'/x]:\rho}\,\text{cut } (i=1 \text{ or } 2)}{B', B'', y:\tau', w:\tau' :- M[N'/x]:\rho}\,\chi L \qquad \dfrac{}{y:\tau' :- y:\tau'}\,Ax}{B', B'', y:\tau' :- M[N'/x][y/w]:\rho}\,\text{cut}$$

where $M[N'/x][y/w] \equiv M[N/x]$ since $w \notin FV(M)$. Note that the last cut whose right permise is an axiom is ready and surely it doesn't create any redex, hence its degree is 0. The given transformations are sound; in fact B, B_i is a basis since B, B' is a basis and B_i differs from B only for the type of y and $y \notin FV(B'')$ by assumption.

In case $(\exists L)$ it has to be true that $t \notin FV(B, B') \cup FV(\rho)$: since the original instance of the $(\exists L)$ was correct by hypothesis we know that $t \notin FV(B') \cup FV(\sigma)$, hence it can always be choosen such that $t \notin FV(B) \cup FV(\rho)$, so that the restriction is met.

Case cut: in this case, by the hypothesis, the upper cut is ready. We must distinguish subcases according to the shapes of Figure 2.

We can assume that $x \notin FV(B, B', B'')$ and $z, y, u \notin FV(B''')$.

(a)

$$\dfrac{B''', x:\sigma :- M:\rho \qquad \dfrac{\dfrac{B, z:\xi :- N:\sigma \quad B' :- R:\upsilon}{B, B', u:\upsilon\to\xi :- N[uR/z]:\sigma}\,\to L \qquad \dfrac{B'', y:\upsilon :- Q:\xi}{B'' :- \lambda y.Q:\upsilon\to\xi}\,\to R}{B, B', B'' :- N[uR/z][\lambda y.Q/u]:\sigma}\,\text{cut}}{B, B', B'', B''' :- M[N[uR/z][\lambda y.Q/u]/x]:\rho}\,\text{cut}$$

can be replaced by:

$$\dfrac{\dfrac{\dfrac{B''', x:\sigma :- M:\rho \quad B, z:\xi :- N:\sigma}{B, B''', z:\xi :- M[N/x]:\rho}\,\text{cut} \quad B' :- R:\upsilon}{B, B', B''', u:\upsilon\to\xi :- M[N/x][uR/z]:\rho}\to L \quad \dfrac{B'', y:\upsilon :- Q:\xi}{\lambda y.Q:\upsilon\to\xi}\to R}{B, B', B'', B''' :- M[N/x][uR/z][\lambda y.Q/u]:\rho}\,\text{cut} \; .$$

Notice that $M[N[uR/z][\lambda y.Q/u]/x] \equiv M[N/x][uR/z][\lambda y.Q/u]$ since we can assume $x \notin FV(N)$ and $u, z \notin FV(M)$.

(b), (c) and (d):

$$\dfrac{B''', x:\sigma :- M:\rho \quad \dfrac{\dfrac{B_i, y:\xi_i :- N:\sigma\ (i=1 \text{ or } 2)}{B, y:\upsilon :- N:\sigma}\chi L \quad \dfrac{B_j :- Q:\xi_j\ (j=1 \text{ or } 2)}{B' :- Q:\upsilon}\chi R}{B, B' :- N[Q/y]:\sigma}\,\text{cut}}{B, B', B'' :- M[N[Q/y]/x] : \rho}\,\text{cut}$$

where $\chi \in \{\mu, \forall, \exists, \wedge, \vee\}$, can be replaced by:

$$\dfrac{\dfrac{\dfrac{B''', x:\sigma :- M:\rho \quad B_i, y:\xi_i :- N:\sigma}{B_i, B''', y:\xi_i :- M[N/x] : \rho}\,\text{cut}\ (i=1 \text{ or } 2)}{B, B''', y:\upsilon :- M[N/x] : \rho}\chi L \quad \dfrac{B_j :- Q:\xi_j}{B' :- Q:\upsilon}\chi R\ (j=1 \text{ or } 2)}{B, B', B'' :- M[N/x][Q/y] : \rho}\,\text{cut} \; .$$

Notice that $M[N[Q/y]/x] \equiv M[N/x][Q/y]$ since we can assume $y \notin FV(M)$.

(e) we are given a figure of the shape

$$\dfrac{B'', x:\sigma :- M:\rho \quad \dfrac{\dfrac{B_i :- N:\sigma\ (i=1 \text{ or } 2)}{B, y:\tau :- N:\sigma}\chi L \quad \dfrac{}{B', z:\tau :- z:\tau}\text{Ax}}{B, B', z:\tau :- N[z/y]:\sigma}\,\text{cut}}{B, B', B'', z:\tau :- M[N[z/y]/x]:\rho}\,\text{cut}$$

where $z \notin FV(B)$

or of the shape

$$\dfrac{B'', x:\sigma :- M:\rho \quad \dfrac{\dfrac{B_i :- N:\sigma\ (i=1 \text{ or } 2)}{B, z:\tau, y:\tau :- N:\sigma}\chi L \quad \dfrac{}{B', z:\tau :- z:\tau}\text{Ax}}{B, B', z:\tau :- N[z/y]:\sigma}\,\text{cut}}{B, B', B'', z:\tau :- M[N[z/y]/x]:\rho}\,\text{cut} \; .$$

In both cases we can assume that $y \notin FV(B'')$ and $y \notin FV(M)$.

We transform them into

$$\cfrac{\cfrac{B'', x:\sigma :- M:\rho \quad B_i[z/y] :- N[z/y]:\sigma}{B'', B_i[z/y] :- M[N[z/y]/x]:\rho}\text{cut (i= 1 or 2)}}{B'', B, y:\tau :- M[N[z/y]/x]:\rho}\chi L$$

and into

$$\cfrac{\cfrac{\cfrac{B'', x:\sigma :- M:\rho \quad B_i :- N:\sigma}{B'', B_i :- M[N/x]:\rho}\text{cut (i= 1 or 2)}}{B'', B, y:\tau :- M[N/x]:\rho}\chi L \quad \cfrac{}{B', z:\tau :- z:\tau}\text{Ax}}{B, B', B'' :- M[N/x][z/y]:\rho}\text{cut}$$

respectively.
Notice that there is a deduction of $B_i[z/y]$, $B' :- N[z/y]:\sigma$ similar to the given deduction of $B_i :- N:\sigma$ by Lemma 2.14 (i) and (ii). Moreover by the assumption $y \notin FV(M)$: $M[N[z/y]/x] \equiv M[N/x][z/y]$.

Part 2: right rank =1 and left rank >1.
In this case the deduction will have the shape:

$$\cfrac{\cfrac{B_i :- N_i:\sigma \text{ (i= 1 or 2)}}{B :- M:\rho}\text{(rule)} \quad B' :- N:\sigma}{B, B' :- M[N/x]:\rho}\text{cut} .$$

By the applicability of the cut, we must have $x:\sigma \in B$; therefore if i=1 we must have $x:\sigma \in B_1$, because of the hypothesis about the rank. If i = 2 then either $x:\sigma \in B_1$ or $x:\sigma \in B_2$, or both: just the third case will be considered, since the treatment of the first two is similar and simpler

We distinguish various cases according to the last rule above the left premise of the cut. Note that cases Ax and ω are impossible because left rank > 1.
Case $\rightarrow$L: we are given either a figure of the shape

$$\cfrac{\cfrac{B, x:\sigma, u:\tau :- M:\rho \quad B', x:\sigma :- P:\xi}{B, B', v:\xi\rightarrow\tau, x:\sigma :- M[vP/u]:\rho}\rightarrow L \quad B'' :- N:\sigma}{B, B', B'', v:\xi\rightarrow\tau :- M[vP/u][N/x]:\rho}\text{cut}$$

or of the shape

$$\cfrac{\cfrac{B, x:\sigma\rightarrow\tau, u:\sigma :- M:\rho \quad B', x:\sigma\rightarrow\tau :- P:\sigma}{B, B', x:\sigma\rightarrow\tau :- M[vP/u]:\rho}\rightarrow L \quad B'' :- N:\sigma}{B, B', B'' :- M[vP/u][N/x]:\rho}\text{cut} .$$

In both cases we can assume that $u \notin FV(B', B'')$ and $u \notin FV(N)$.

We transform them into

$$\dfrac{\dfrac{B, u{:}\tau, x{:}\sigma :- M{:}\rho \quad B'' :- N{:}\sigma}{B, B'', u{:}\tau :- M[N/x]{:}\rho}\,\text{cut} \quad \dfrac{B', x{:}\sigma :- P{:}\xi \quad B'' :- N{:}\sigma}{B', B'' :- P[N/x]{:}\xi}\,\text{cut}}{B, B', B'', v{:}\xi\to\tau :- M[N/x][vP[N/x]/u]{:}\rho}\,\to L$$

and into

$$\dfrac{\dfrac{\dfrac{B, u{:}\sigma, x{:}\sigma\to\tau :- M{:}\rho \quad B'' :- N{:}\sigma\to\tau}{B, B'', u{:}\sigma :- M[N/x]{:}\rho}\,\text{cut} \quad \dfrac{B', x{:}\sigma\to\tau :- P{:}\sigma \quad B'' :- N{:}\sigma\to\tau}{B', B'' :- P[N/x]{:}\sigma}\,\text{cut}}{B, B', B'', z{:}\sigma\to\tau :- M[N/x][zP[N/x]/u]{:}\rho}\,\to L \quad B'' :- N{:}\sigma\to\tau}{B, B', B'', z{:}\sigma\to\tau :- M[N/x][zP[N/x]/u][N/z]{:}\rho}\,\text{cut}$$

respectively, where z is a fresh variable.

Note that $u \notin FV(N)$ and z fresh imply $M[N/x][vP[N/x]/u] \equiv M[vP/u][N/x]$ and $M[N/x][zP[N/x]/u][N/z] \equiv M[xP/u][N/x]$.

Case $\to R$:

$$\dfrac{\dfrac{B, x{:}\sigma, y{:}\tau :- M{:}\rho}{B, x{:}\sigma :- \lambda y.M{:}\tau\to\rho}\,\to R \quad B' :- N{:}\sigma}{B, B' :- (\lambda y.M)[N/x]{:}\tau\to\rho}\,\text{cut}\ .$$

Clearly we can assume $y \notin FV(N)$ and $y \notin FV(B')$. The derivation transforms into

$$\dfrac{\dfrac{B, y{:}\tau, x{:}\sigma :- M{:}\rho \quad B' :- N{:}\sigma}{B, B', y{:}\tau :- M[N/x]{:}\rho}\,\text{cut}}{B, B' :- \lambda y.M[N/x]{:}\tau\to\rho}\,\to R$$

and, by the fact that $y \notin FV(N)$, $(\lambda y.M)[N/x] \equiv \lambda y.M[N/x]$.

Case χL ($\chi \in \{\mu, \forall, \exists, \wedge, \vee\}$):

$$\dfrac{\dfrac{B_i, y{:}\tau, x{:}\sigma :- M{:}\rho}{B, y{:}\tau', x{:}\sigma :- M{:}\rho}\,\chi L \quad B' :- N{:}\sigma}{B, B', y{:}\tau' :- M[N/x]{:}\rho}\,\text{cut}\ .$$

If $y \notin FV(B')$, this derivation is transformed into

$$\dfrac{\dfrac{B_i, y{:}\tau, x{:}\sigma :- M{:}\rho \quad B' :- N{:}\sigma}{B_i, B', y{:}\tau :- M[N/x]{:}\rho}\,\text{cut}\,(i=1 \text{ or } 2)}{B, B', y{:}\tau' :- M[N/x]{:}\rho}\,\chi L$$

Otherwise we use the same renaming technique of the case χL in the first part of this proof.

Case χR $(\chi \in \{\mu, \forall, \exists, \wedge, \vee\})$:

$$\dfrac{\dfrac{B_i, x:\sigma :- M:\rho \ (i=1 \text{ or } 2)}{B, x:\sigma :- M:\tau}\chi R \quad B' :- N:\sigma}{B, B' :- M[N/x]:\tau}\text{cut}$$

becomes

$$\dfrac{\dfrac{B_i, x:\sigma :- M:\rho \quad B' :- N:\sigma}{B_i, B' :- M[N/x]:\rho}\text{cut}\ (i=1\text{ or }2)}{B, B' :- M[N/x]:\tau}\chi R$$

Case cut: in this case by hypothesis the upper cut is ready. We must distinguish subcases according to the shapes of Figure 2. There is no loss of generality in assuming that $y \notin FV(N)$ and $y \notin FV(B''')$.

(a)

$$\dfrac{\dfrac{\dfrac{B, x:\sigma, z:\xi :- M:\rho \quad B', x:\sigma :- R:\upsilon}{B, B', x:\sigma, y:\upsilon\to\xi :- M[yR/z]:\rho}\to L \quad \dfrac{B'', x:\sigma, u:\upsilon :- Q:\xi}{B'', x:\sigma :- \lambda u.Q:\upsilon\to\xi}\to R}{B, B', B'', x:\sigma :- M[yR/z][\lambda u.Q/y]:\rho}\text{cut} \quad B''' :- N:\sigma}{B, B', B'', B''' :- M[yR/z][\lambda u.Q/y][N/x]:\rho}\text{cut}$$

is replaced by:

$$\dfrac{\dfrac{\dfrac{B,x:\sigma,z:\xi :- M:\rho \quad B''' :- N:\sigma}{B, B''', z:\xi :- M[N/x]:\rho}\text{cut} \quad \dfrac{B', x:\sigma :- R:\upsilon \quad B''' :- N:\sigma}{B', B''' :- R[N/x]:\upsilon}\text{cut}}{B, B', B''', y:\upsilon\to\xi :- M[N/x][yR[N/x]/z]:\rho}\to L \quad \dfrac{\dfrac{B'', x:\sigma, u:\upsilon :- Q:\xi \quad B''' :- N:\sigma}{B'', B''', u:\upsilon :- Q[N/x]:\xi}\text{cut}}{B'', B''' :- \lambda u.Q[N/x]:\upsilon\to\xi}\to R}{B, B', B'', B''' :- M[N/x][yR[N/x]/z][\lambda u.Q[N/x]/y]:\rho}\text{cut}$$

Notice that $M[yR/z][\lambda u.Q/y][N/x] \equiv M[N/x][yR[N/x]/z][\lambda u.Q[N/x]/y]$.

(b), (c) and (d):

$$\dfrac{\dfrac{\dfrac{B_i, x:\sigma, y:\xi_i :- M:\rho\ (i=1\text{ or }2)}{B, x:\sigma, y:\upsilon :- M:\rho}\chi L \quad \dfrac{B_j, x:\sigma :- P:\xi_j\ (j=1\text{ or }2)}{B', x:\sigma :- P:\upsilon}\chi R}{B, B', x:\sigma :- M[P/y]:\rho}\text{cut} \quad B'' :- N:\sigma}{B, B', B'' :- M[P/y][N/x]:\rho}\text{cut}$$

where $\chi \in \{\mu, \forall, \exists, \wedge, \vee\}$, can be replaced by

$$\dfrac{\dfrac{\dfrac{B_i, x:\sigma, y:\xi_i :- M:\rho \quad B'' :- N:\sigma}{B_i, B'', y:\xi_i :- M[N/x]:\rho}\,\text{cut } (i=1 \text{ or } 2)}{B, B'', y:\upsilon :- M[N/x]:\rho}\,\chi L \qquad \dfrac{\dfrac{B_j, x:\sigma :- P:\xi_j \quad B'' :- N:\sigma}{B_j, B'' :- P[N/x]:\xi_j}\,\text{cut } (j=1 \text{ or } 2)}{B', B'' :- P[N/x]:\upsilon}\,\chi R}{B, B', B'' :- M[N/x][P[N/x]/y]:\rho}\,\text{cut.}$$

Notice that $M[P/y][N/x] \equiv M[N/x][P[N/x]/y]$.

Finally in case (e) we are given a figure of the shape

$$\dfrac{\dfrac{\dfrac{B_i, x:\sigma :- M:\rho \ (i=1 \text{ or } 2)}{B, y:\tau, x:\sigma :- M:\rho}\,\chi L \qquad \dfrac{}{B', z:\tau :- z:\tau}\,\text{Ax}}{B, B', z:\tau, x:\sigma :- M[z/y]:\rho}\,\text{cut} \qquad B''' :- N:\sigma}{B, B', B''', z:\tau :- M[z/y][N/x]:\rho}\,\text{cut}$$

or of the shape

$$\dfrac{\dfrac{\dfrac{B_i, x:\sigma :- M:\rho \ (i=1 \text{ or } 2)}{B, y:\sigma, x:\sigma :- M:\rho}\,\chi L \qquad \dfrac{}{B', x:\sigma :- x:\sigma}\,\text{Ax}}{B, B', x:\sigma :- M[x/y]:\rho}\,\text{cut} \qquad B''' :- N:\sigma}{B, B', B''' :- M[x/y][N/x]:\rho}\,\text{cut .}$$

We transform them into

$$\dfrac{\dfrac{B_i[z/y], x:\sigma :- M[z/y]:\rho \qquad B'' :- N:\sigma}{B_i[z/y], B'' :- M[z/y][N/x]:\rho}\,\text{cut}}{B, B'', y:\tau :- M[z/y][N/x]:\rho}\,\chi L$$

and into

$$\dfrac{\dfrac{\dfrac{B_i, x:\sigma :- M:\rho \qquad B'' :- N:\sigma}{B_i, B'' :- M[N/x]:\rho}\,\text{cut}}{B, B'', y:\sigma :- M[N/x]:\rho}\,\chi L \qquad B'' :- N:\sigma}{B, B', B'' :- M[N/x][N/y]:\rho}\,\text{cut}$$

respectively.

Notice that in the second case the newly generated cut is ready, since by hypothesis the right rank of the current cut was 1. Moreover, the degree of this new cut is the degree of the lower cut of the original figure.

Notice that in all transformations the degrees of cuts do not increase, and the degrees of newly generated cuts is either the same as that of the old cuts or 0. □

COMPACIDAD Y COMPACTIFICACIÓN EN TEORÍA DE MODELOS

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INTRODUCCIÓN

La finalidad de este artículo es doble: primero, dar una caracterización de la compacidad de una lógica abstracta (model - theoretic logic) en términos de la convergencia de familias de estructuras via ultrafiltros definidos en esas familias, donde los límites correspondientes de cada familia generalizan debilmente los ultraproductos de la familia; y segundo, construir una compactificación (topológica) de cada espacio de estructuras, la cual permitirá definir una nueva semántica para la lógica original que sea (logicamente) compacta.

Los espacios de estructuras con la topología elemental definida por la lógica subyacente son espacios cero-dimensionales, ie. admiten una base de clopens (abiertos-cerrados), y la compactificación construida es tal que preserva la cero-dimensionalidad de estos espacios. En este caso, la compacidad topológica y la compacidad lógica coinciden.

El método seguido es puramente topológico y permite aplicarlo a cualquier espacio cero-dimensional, inclusive cuando la colección de puntos que subyace al espacio es una clase propia, ie. un espacio "grande", que es el caso de los espacios de estructuras que nos interesa; en este caso, los abiertos de la topología son en general clases propias. Sin embargo, restringiremos nuestro estudio al caso en que la topología es "pequeña", ie. cuando la colección de abiertos puede ser parametrizada por un conjunto, o lo que es lo mismo, tratándose de espacios de estructuras, cuando la lógica es "pequeña", ie. la colección de enunciados o fórmulas cerradas correspondiente a cada tipo de similaridad es un conjunto, ya que en este caso la teoría de conjuntos que permite nuestras construcciones puede ser debidamente fundamentada. En particular, la compacidad de un espacio grande con una topología pequeña involucra solo cubrimientos pequeños de abiertos.

CARACTERIZACIÓN DE LA COMPACIDAD

Sea $L \geq L_{\omega\omega}$ una lógica abstracta regular (cf. [E] pag. 31), y para cada tipo de similaridad τ sean St^{τ} la colección de estructuras de tipo τ y L^{τ} la colección de enunciados de L de tipo τ.

Supondremos que L es una lógica pequeña (small logic), ie. para cada tipo τ, L^τ es un conjunto.

La topología elemental sobre St^τ es cero-dimensional y está dada por la siguiente base de clopens: $\{\,\mathrm{Mod}(\varphi)/\varphi \in L^\tau\}$, la cual es una colección pequeña de clases en St^τ (ie. una colección de clases parametrizada por un conjunto) cerrada para intersecciones finitas y complementos.

Esta topología admite una estructura uniforme que la genera (cf. [K] cap. 6) dada por la siguiente base de uniformidad: $\{\mathcal{U}_\Phi/\Phi$ es un subconjunto finito de $L^\tau\}$, donde para cada $\Phi, \mathcal{U}_\Phi = \{(\mathcal{A},\mathcal{B}) \in St^\tau \times St^\tau/\mathcal{A} \equiv_\Phi \mathcal{B}\}$, siendo $\equiv_\Phi$ la relación de equivalencia elemental con respecto a la colección Φ de enunciados (cf. [Ca]).

Es fácil ver que la base de uniformidad dada tiene la siguiente propiedad para cualquier tipo τ(aquí $\mathcal{U}_\varphi$ abrevia $\mathcal{U}_{\{\varphi\}}$): para cada $\varphi, \mathcal{U}_\varphi[\mathcal{A}] = \{\mathcal{B} \in St^\tau/(\mathcal{A},\mathcal{B}) \in \mathcal{U}_\varphi\} =$
$$\begin{cases} \mathrm{Mod}(\varphi), \text{ si } \mathcal{A} \models \varphi \\ \mathrm{Mod}(\neg\varphi), \text{ si } \mathcal{A} \not\models \varphi, \end{cases}$$
(los $\mathcal{U}_\varphi[\mathcal{A}]$ son el correspondiente a las "bolas" de un espacio seudométrico). Esta propiedad garantiza que el espacio uniforme resultante es totalmente acotado (totally bounded) o precompacto (ie. dado φ, el conjunto de las bolas $\mathcal{U}_\varphi[\mathcal{A}]$ que cubren St^τ es finito), y que esta base de uniformidad genera la topología elemental del espacio.

En la teoria general de espacios uniformes se tiene la siguiente caracterización, válida también para espacios grandes con bases pequeñas de uniformidad (cf. [K] pag. 198):

$$\text{COMPACIDAD} = \text{COMPLETITUD DE CAUCHY} + \text{ACOTACION TOTAL},$$

donde la completitud de Cauchy es definida en términos de la convergencia de toda red de Cauchy (las redes son definidas sobre conjuntos). Por lo tanto, para los espacios de estructuras St^τ son equivalentes *Compacidad* y *Completitud de Cauchy.*

A continuación daremos una caracterización de la compacidad topológica de cada espacio St^τ, ie., dada la cero-dimensionalidad del espacio, de la compacidad lógica de cada L^τ, en términos de una versión abstracta del Teorema de Ultraproductos de Łos. Ésta será equivalente a la completitud de Cauchy de esos espacios, mostrando así que el teorema de Łos es un teorema de completitud topológica.

DEFINICIÓN 1.

1.1 Sea $\{\mathcal{A}_i\}_{i\in I}$ una familia de estructuras en St^τ y U un ultrafiltro sobre I. Definimos $\lim_U \mathcal{A}_i$ como la colección de las estructuras $\mathcal{A} \in St^\tau$ tal que para todo $\varphi \in L^\tau$ existe $X \in U$ tal que para todo $i \in X, (\mathcal{A},\mathcal{A}_i) \in \mathcal{U}_\varphi$; o equivalentemente, si para todo $\varphi, \{i \in I/(\mathcal{A},\mathcal{A}_i) \in \mathcal{U}_\varphi\} \in U$.

1.2 Sea $(D, \leq)$ un "conjunto" dirigido. Una *red* (net) en St^τ es cualquier familia de estructuras $\{\mathcal{A}_i\}_{i\in D}$.

1.3 Una red $\{\mathcal{A}_i\}_{i\in D}$ es de *Cauchy* si para todo $\varphi \in L^\tau$ existe $k \in D$ tal que para todo $i, j \geq k, (\mathcal{A}_i, \mathcal{A}_j) \in \mathcal{U}_\varphi$.

1.4 Sea $\{\mathcal{A}_i\}_{i\in D}$ una red en St^τ. Definimos $\lim_i \mathcal{A}_i$ como la colección de las estructuras $\mathcal{A} \in St^\tau$ tal que para todo $\varphi \in L^\tau$ existe $k \in D$ tal que para todo $i \geq k, (\mathcal{A}, \mathcal{A}_i) \in \mathcal{U}_\varphi$.

1.5 Un ultrafiltro U sobre un conjunto dirigido D es llamado *libre* (free ultrafilter) si contiene todos los subconjuntos $Y_k = \{i \in D/i \geq k\}$ con $k \in D$. (La noción de ultrafiltro libre sobre un conjunto dirigido generaliza la de ultrafiltro no-principal en ω; observese que la colección $\{Y_k\}_{k\in D}$ tiene la propiedad de intersección finita).

OBSERVACIÓN 1. Debido a la cero-dimensionalidad de los espacios St^τ, la colección $\lim_U \mathcal{A}_i$ puede ser definida de la siguiente manera (cf. [F-M-S] pag. 223): $\mathcal{A} \in \lim_U \mathcal{A}_i$ si y solo si para todo $\varphi \in L^\tau$,

$$\mathcal{A} \in \text{Mod}(\varphi) \Leftrightarrow \{i \in I/\mathcal{A}_i \in \text{Mod}(\varphi)\} \in U.$$

Así, toda estructura $\mathcal{A} \in \lim_U \mathcal{A}_i$ satisface la propiedad fundamental, dada por el teorema de Łos, que el ultraproducto $\Pi_U \mathcal{A}_i$ satisface en el caso $L = L_{\omega\omega}$.

Se sigue de la observación anterior que $\lim_U \mathcal{A}_i = \bigcap\{\text{Mod}(\varphi)/\{i \in I/\mathcal{A}_i \in \text{Mod}(\varphi)\} \in U\}$.

El siguiente lema es esencial para la demostración de nuestra caracterización de la compacidad y es debido fundamentalmente a D. Mundici y A. M. Sette (cf. [M-S-C] pag. 7). La importancia que él tiene para la interpretación de la convergencia en los espacios de estructuras sugiere darle un nombre, y aquí lo llamaremos "Lema de Convergencia".

LEMA DE CONVERGENCIA. Si $\{\mathcal{A}_i\}_{i\in I}$ es una red de Cauchy en St^τ y U es un ultrafiltro libre sobre D, entonces, $\lim_i \mathcal{A}_i = \lim_U \mathcal{A}_i$. En particular, $\lim_U \mathcal{A}_i$ no depende del ultrafiltro libre U.

DEMOSTRACIÓN.

a) Sean $\mathcal{A} \in \lim_i \mathcal{A}_i$ y $\varphi \in L^\tau$, entonces existe $k \in D$ tal que para todo $i \geq k, (\mathcal{A}, \mathcal{A}_i) \in \mathcal{U}_\varphi$, ie., $Y_k \subseteq \{i \in I/(\mathcal{A}, \mathcal{A}_i) \in \mathcal{U}_\varphi\}$, luego, como U es libre, $\{i \in I/(\mathcal{A}, \mathcal{A}_i) \in \mathcal{U}_\varphi\} \in U$, por lo tanto, $\mathcal{A} \in \lim_U \mathcal{A}_i$.

b) Sean $\mathcal{A} \in \lim_U \mathcal{A}_i$ y $\varphi \in L^\tau$, entonces existe $X_\varphi \in U$ tal que para todo $i \in X_\varphi, (\mathcal{A}, \mathcal{A}_i) \in \mathcal{U}_\varphi$. Por otro lado, como la red $\{\mathcal{A}_i\}_{i\in D}$ es de Cauchy, existe $k_\varphi \in D$ tal que para todo $i, j \geq k_\varphi, (\mathcal{A}_i, \mathcal{A}_j) \in \mathcal{U}_\varphi$.

Consideremos $Z = X_\varphi \cap Y_{k_\varphi}$, entonces $Z \in U$ por ser U libre, en particular, $Z \neq \phi$. Sea $k \in Z$ cualquiera.

AFIRMACIÓN. Para todo $i \geq k, (\mathcal{A}, \mathcal{A}_i) \in \mathcal{U}_\varphi$.

En efecto, si $i \geq k$, entonces, como $k \in X_\varphi$ tenemos que $(\mathcal{A}, \mathcal{A}_k) \in \mathcal{U}_\varphi$. Además, como $i, k \geq k_\varphi$ tenemos que $(\mathcal{A}_k, \mathcal{A}_i) \in \mathcal{U}_\varphi$, luego, $(\mathcal{A}, \mathcal{A}_i) \in \mathcal{U}_\varphi \circ \mathcal{U}_\varphi \subseteq \mathcal{U}_\varphi$ (pues cada $\mathcal{U}_\varphi$ está definido en términos de una relación de equivalencia).

Ésto prueba que $\mathcal{A} \in \lim_i \mathcal{A}_i$ ■

PROPOSICIÓN 1. Para cada tipo τ son equivalentes:

i) Para toda familia $\{\mathcal{A}_i\}_{i \in I}$ en St^τ y todo ultrafiltro U sobre $I, \lim_U \mathcal{A}_i \neq \phi$.
ii) Para toda red de Cauchy $\{\mathcal{A}_i\}_{i \in D}$ en St^τ y todo ultrafiltro libre U sobre $D, \lim_U \mathcal{A}_i \neq \phi$.
iii) El espacio St^τ es completo, ie. para toda red de Cauchy $\{\mathcal{A}_i\}_{i \in D}, \lim_i \mathcal{A}_i \neq \phi$.
iv) El espacio St^τ es compacto.

DEMOSTRACIÓN.

(i $\rightarrow$ ii): Trivial.
(ii $\rightarrow$ iii): Es consecuencia inmediata del lema de convergencia.
(iii $\rightarrow$ iv): Es inmediato por ser St^τ un espacio uniforme totalmente acotado.
(iv $\rightarrow$ i): Sea $\{\mathcal{A}_i\}_{i \in I}$ una familia en St^τ y U un ultrafiltro sobre I. Por la observación (1), $\lim_U \mathcal{A}_i$ se expresa como la intersección de la siguiente familia de cerrados con la propiedad de intersección finita: $\{\text{Mod}(\varphi)/\{i \in I/\mathcal{A}_i \in \text{Mod}(\varphi)\} \in U\}$. Por lo tanto, como St^τ es compacto, $\lim_U \mathcal{A}_i \neq \phi$. ■

COROLARIO. Sea $L \geq L_{\omega\omega}$ una lógica abstracta regular pequeña, entonces son equivalentes:

i) L es compacta.
ii) Para todo tipo τ, toda familia $\{\mathcal{A}_i\}_{i \in I}$ en St^τ y todo ultrafiltro U sobre $I, \lim_U \mathcal{A}_i \neq \phi$. ■

La afirmación (ii) del corolario anterior es una versión abstracta y una generalización para lógicas compactas del teorema de ultraproductos de Łos. Ella siempre es válida en $L_{\omega\omega}$ pues en este caso $\Pi_U \mathcal{A}_i \in \lim_U \mathcal{A}_i$. Para otras lógicas compactas diferentes de $L_{\omega\omega}$ se puede plantear aquí el interesante problema de la "construcción", para cada familia $\{\mathcal{A}_i\}_{i \in I}$ y cada ultrafiltro U sobre I, de una estructura $\mathcal{A} \in \lim_U \mathcal{A}_i$.

COMPACTIFICACIÓN DE St^τ

Para cada τ definimos $CSt^\tau = \{(K, U)/K \subseteq St^\tau$ es un "conjunto" y U es un ultrafiltro sobre $K\}$, y para cada $\varphi \in L^\tau$ definimos $\text{Mod}^*(\varphi) = \{(K, U) \in CSt^\tau/\{\mathcal{A} \in K/\mathcal{A} \models \varphi\} \in U\} = \{(K, U) \in CSt^\tau/K \cap \text{Mod}(\varphi) \in U\}$.

$\text{Mod}^*(\varphi)$ es la colección de "Modelos Generalizados" de φ. De este punto de vista podemos definir la "verdad" (truth) de φ en (K, U) como

$$(K,U) \Vdash \varphi \Leftrightarrow (K,U) \in \text{Mod}^*(\varphi) \Leftrightarrow \{\mathcal{A} \in K/\mathcal{A} \models \varphi\} \in U \Leftrightarrow K \cap \text{Mod}(\varphi) \in U,$$

por lo tanto, (K, U) se comporta como un ultraproducto de K módulo U. De hecho, si $L = L_{\omega\omega}$, entonces para todo $\varphi, (K,U) \Vdash \varphi \Leftrightarrow \Pi_U K \models \varphi$, ie. $(K,U) \equiv \Pi_U K$ ($\equiv$ denota la relación de equivalencia elemental de L).

La colección $\{\text{Mod}^*(\varphi)/\varphi \in L^\tau\}$ tiene las siguientes propiedades:

i) $\text{Mod}^*(\varphi) \cap \text{Mod}^*(\psi) = \text{Mod}^*(\varphi \wedge \psi)$
ii) $\text{Mod}^*(\varphi)^c = \text{Mod}^*(\neg\varphi)$
iii) $(St^\tau)^* = CSt^\tau$.

Por lo tanto, es una base de clopens para una topología pequeña cero-dimensional en CSt^τ, que es cerrada para intersecciones finitas y complementos.

Consideremos la aplicación $h : St^\tau \to CSt^\tau$ dada por $h(\mathcal{A}) = (\{\mathcal{A}\}, \{\{\mathcal{A}\}\})$.

Puede probarse facilmente lo siguiente:

a) h es inyectiva.

b) para todo $\mathcal{A} \in St^\tau$ y todo $\varphi \in L^\tau : h(\mathcal{A}) \Vdash \varphi \Leftrightarrow \mathcal{A} \models \varphi$, por lo tanto, la semántica de CSt^τ extiende la semántica de St^τ (es importante observar que el lenguaje L^τ no ha cambiado al extender la semántica).

c) $\text{Mod}^*(\varphi) \cap h[St^\tau] = h[\text{Mod}(\varphi)]$, lo que implica que h es un homeomorfismo de St^τ sobre $h[St^\tau]$, ie. St^τ puede ser considerado como un subespacio de CSt^τ.

En lo que sigue identificaremos $h[St^\tau]$ con St^τ y $h(\mathcal{A})$ con $\mathcal{A}$.

PROPOSICIÓN 2. St^τ es un subespacio denso de CSt^τ.

DEMOSTRACIÓN. Sea $\varphi \in L^\tau$ tal que $\text{Mod}^*(\varphi) \neq \phi$, entonces existe $(K,U) \in \text{Mod}^*(\varphi)$, ie. $\{\mathcal{A} \in K/\mathcal{A} \models \varphi\} \in U$, en particular, como U es un filtro propio, existe $\mathcal{A} \in K$ tal que $\mathcal{A} \models \varphi$, ie. (por (b)) $\mathcal{A} \Vdash \varphi$, luego, $\mathcal{A} \in \text{Mod}^*(\varphi)$ ■

La *compacidad lógica* de L^τ con respecto a la semántica extendida CSt^τ puede ser formulada de la siguiente manera: sea $\sum \subseteq L^\tau$ tal que todo subconjunto finito de $\sum$ tiene modelo generalizado (ie. en CSt^τ), entonces $\sum$ tiene modelo generalizado.

La cero-dimensionalidad de CSt^τ implica que la compacidad lógica de L^τ es equivalente a la compacidad topológica de CSt^τ.

PROPOSICIÓN 3. CSt^τ es compacto.

DEMOSTRACIÓN. Sea $\{\text{Mod}^*(\varphi_i)\}_{i\in I}$ una colección (pequeña, ie. I es un conjunto) de cerrados básicos de CSt^τ con la propiedad de intersección finita.

Sea $J = \mathcal{P}_\omega(I)$ (partes finitas de I) y para cada $\Delta \in J$ "elegimos" $(K_\Delta, U_\Delta) \in \bigcap_{i\in\Delta} \text{Mod}^*(\varphi_i)$.

Aqui está siendo usado un axioma de elección para clases pequeñas.

Consideremos $K = \bigcap_{\Delta\in J} K_\Delta$ el cual es un conjunto. Construiremos un ultrafiltro W sobre K de tal modo que $(K, W) \in \bigcap_{i\in I} \text{Mod}^*(\varphi_i)$.

AFIRMACIÓN 1. $\{K \cap \text{Mod}(\varphi_i)\}_{i\in I}$ es una colección de subconjuntos de K que tiene la propiedad de intersección finita.

En efecto, sea $\Delta \in J$, entonces, como $(K_\Delta, U_\Delta) \in \bigcap_{i\in\Delta} \text{Mod}^*(\varphi_i)$ tenemos que para cada $i \in \Delta, K_\Delta \cap \text{Mod}(\varphi_i) \in U_\Delta$, luego, $K_\Delta \cap \text{Mod}(\bigwedge_{i\in\Delta} \varphi_i) \in U_\Delta$, en particular, existe $\mathcal{A} \in K_\Delta \subseteq K$ tal que $\mathcal{A} \models \bigwedge_{i\in\Delta} \varphi_i$.

Sea W un ultrafiltro sobre K que contiene la familia $\{K \cap \text{Mod}(\varphi_i)\}_{i\in I}$.

AFIRMACIÓN 2. $(K, W) \in \bigcap_{i\in I} \text{Mod}^*(\varphi_i)$.

En efecto, sea $i \in I$, entonces $K \cap \text{Mod}(\varphi_i) \in W$, ie. por definición, $(K, W) \in \text{Mod}^*(\varphi_i)$.

En consecuencia, CSt^τ es compacto ■

PROPOSICIÓN 4. Si St^τ es compacto, entonces para todo $(K, U) \in CSt^\tau$ existe $\mathcal{A} \in St^\tau$ tal que $\mathcal{A} \equiv (K, U)$, ie. para todo $\varphi \in L^\tau, \mathcal{A} \models \varphi \Leftrightarrow (K, U) \Vdash\varphi$. Además, podemos escoger $\mathcal{A} \in \overline{K}$, donde $\overline{K}$ es la clausura de K en St^τ.

DEMOSTRACIÓN. Sea $(K, U) \in St^\tau$ y consideremos la siguiente familia de cerrados de $\overline{K}$: $\{\overline{K} \cap \text{Mod}(\varphi)/(K, U) \Vdash\varphi\}$.

AFIRMACIÓN. La familia dada tiene la propiedad de intersección finita.

En efecto, sean $\varphi_1, ..., \varphi_n$ tales que $(K, U) \Vdash\varphi_i, i = 1, ..., n$, entonces $K \cap \text{Mod}(\varphi_1) \in U, ..., K \cap \text{Mod}(\varphi_n) \in U$, luego, $(\overline{K} \cap \text{Mod}(\varphi_1)) \cap ... \cap (\overline{K} \cap \text{Mod}(\varphi_n)) \in U$, en particular es $\neq \phi$.

En consecuencia, como St^τ es compacto, también $\overline{K}$ es compacto, por lo tanto, existe $\mathcal{A} \in \bigcap\{\overline{K} \cap \text{Mod}(\varphi)/(K, U) \Vdash\varphi\} = \overline{K} \cap \bigcap\{\text{Mod}(\varphi)/(K, U) \Vdash\varphi\}$, ie. existe $\mathcal{A} \in \overline{K}$ tal que para todo $\varphi \in L^\tau, (K, U) \Vdash\varphi \Rightarrow \mathcal{A} \models \varphi$, luego, considerando las negaciones, tenemos $(K, U) \Vdash\varphi \Leftrightarrow \mathcal{A} \models \varphi$ ■

Hemos conseguido entonces compactificar la lógica L extendiendo la semántica. Más aún, hemos obtenido una compactificación topológica del espacio St^{τ}. Sin embargo, si St^{τ} es compacto, CSt^{τ} no se reduce a St^{τ} pues este último no es Hausdorff (ver proposición 8, definición 2 y observación 4 más adelante).

UNA APLICACIÓN

En el parágrafo anterior se ha demostrado que toda lógica (regular y pequeña) admite una semántica compacta que extiende la semántica usual.

La compacidad (lógica) de la nueva semántica, al igual que la compacidad de la lógica elemental, es importante por sus consecuencias, no solo porque permite garantizar la existencia de ciertos "modelos" satisfaciendo determinadas propiedades, sino por su poder en el análisis de la expresabilidad de las teorías matemáticas.

A continuación veremos un pequeño ejemplo de este análisis.

Sea κ un cardinal infinito y consideremos la lógica infinitaria $L = L_{\kappa\kappa}(Q_\kappa)$ donde Q_κ es el cuantificador cardinal cuya interpretación en una estructura $\mathcal{A} = < A, ... >$ es la siguiente: $\mathcal{A} \models (Q_\kappa x)\varphi(x) \Leftrightarrow |\{a \in A/\mathcal{A} \models \varphi[a]\}| \geq \kappa$ (Q_κ es denotado a veces por Q_α siendo α el ordinal tal que $\kappa = \aleph_\alpha$). L permite conjunciones, disyunciones y cuantificaciones universales y existenciales de longitud $< \kappa$ (cf. [B-S] cap. 13 y 14).

En L puede ser expresado el hecho que un conjunto A tenga $|A| < \kappa$; en efecto, $|A| < \kappa \Leftrightarrow A \models \neg(Q_\kappa x)(x = x)$. Por otro lado, para cualquier $\lambda < \kappa$, puede ser expresado también el hecho que $|A| \geq \lambda$ de la siguiente manera: sea $\exists^{\geq\lambda}$ el enunciado $(\exists x_0)...(\exists x_\eta)... \bigwedge_{\alpha<\beta<\lambda} (x_\alpha \neq x_\beta)$ con $\eta < \lambda$, entonces, se verifica que $|A| \geq \lambda \Leftrightarrow A \models \exists^{\geq\lambda}$.

Puede observarse facilmente que si A es un conjunto (ie. $A \in St^{\phi}$, donde ϕ es el tipo de similaridad vacío, o sea, el único símbolo de relación permitido en las estructuras de St^{ϕ} es la igualdad $=$), entonces, $|A| \geq \lambda$ para todo $\lambda < \kappa$ implica $|A| \geq \kappa$, lo que está obviamente en contradición con $|A| < \kappa$. Sin embargo, mostraremos que el conjunto de enunciados $\sum = \{\exists^{\geq\lambda}/\lambda < \kappa\} \cup \{\neg(Q_\kappa x)(x = x)\}$ es no-contradictorio del punto de vista semántico, ie. existe algún modelo generalizado de $\sum$. De hecho, la existencia queda garantizada por la compacidad de CSt^{ϕ} pues todo subconjunto finito de $\sum$ tiene modelo en St^{ϕ}. En lo que sigue construiremos un modelo concreto de $\sum$.

Consideremos el espacio St^{ϕ} y para cada $\lambda < \kappa$ escojemos $A_\lambda \in St^{\phi}$ tal que $|A_\lambda| = \lambda$. Sea $K = \{A_\lambda/\lambda < \kappa\}$ (se observa que $|K| = \kappa$). Obviamente, para cualquier ultrafiltro U sobre K tenemos que $(K, U) \Vdash \neg(Q_\kappa x)(x = x)$ pues $\{A \in K/A \models \neg(Q_\kappa x)(x = x)\} = \{A \in K/|A| < \kappa\} = K \in U$.

Construiremos ahora un ultrafiltro U sobre K tal que para todo $\lambda < \kappa, (K, U) \Vdash \exists^{\geq\lambda}$.

Para cada $\lambda < \kappa$, sea $M_\lambda = \{A \in K/|A| \geq \lambda\}$, entonces, la familia $\{M_\lambda\}_{\lambda<\kappa}$ tiene la propiedad de intersección finita, pues si $\lambda_1, ..., \lambda_n < \kappa$, tenemos que $A \in M_{\lambda_1} \cap ... \cap M_{\lambda_n} \Leftrightarrow |A| \geq$ máx $(\lambda_1, ..., \lambda_n)$, y como máx $(\lambda_1, ..., \lambda_n) < \kappa$, existe $A \in M_{\lambda_1} \cap ... \cap M_{\lambda_n}$.

Es más, si κ es regular, ie. para toda familia de cardinales $\{\alpha_\eta\}_{\eta<\lambda}$ con $\lambda < \kappa$ y cada $\alpha_\eta < \kappa$ se tiene que $\sup_{\eta<\lambda} \alpha_\eta < \kappa$, entonces, la familia $\{M_\lambda\}_{\lambda<\kappa}$ tiene la propiedad de κ-intersección, ie. $\bigcap_{\eta<\lambda} M_{\alpha_\eta} \neq \phi$ para toda familia de cardinales $\{\alpha_\eta\}_{\eta<\lambda}$ como la descrita antes. En este caso, el filtro generado por la familia es κ-completo.

Sea U un ultrafiltro que contiene la familia $\{M_\lambda\}_{\lambda<\kappa}$ (de hecho U es no-principal), entonces para todo $\lambda < \kappa, (K, U) \Vdash \exists^{\geq\lambda}$ pues $\{\mathcal{A} \in K/\mathcal{A} \models \exists^{\geq\lambda}\} = \{\mathcal{A} \in K/|\mathcal{A}| \geq \lambda\} = M_\lambda \in U$.

Luego, (K, U) es el modelo generalizado que hace compatibles los conceptos de "tener cardinalidad $\geq \lambda$ para todo $\lambda < \kappa$" y "tener cardinalidad $< \kappa$". En el caso particular de ser $\kappa = \aleph_0$, son compatibles los conceptos de "ser arbitrariamente grande" y "ser finito".

DISCUSIÓN SOBRE LA COMPACIDAD DE CSt^τ

Ya hemos visto que la colección $\{\,\mathrm{Mod}^*(\varphi)/\varphi \in L^\tau\}$ es una base de clopens para el espacio CSt^τ, por lo tanto, podemos adaptar la definición (1.1), en su versión cero-dimensional, al espacio CSt^τ, y definir para una familia $\{(K_i, U_i)\}_{i\in I}$ y un ultrafiltro U sobre $I : (M, W) \in \lim_U(K_i, U_i)$ si y solo si para todo $\varphi \in L^\tau$,

$$(M, W) \in \mathrm{Mod}^*(\varphi) \Leftrightarrow \{i \in I/(K_i, U_i) \in \mathrm{Mod}^*(\varphi)\} \in U,$$

ie.

$$\{\mathcal{A} \in M/\mathcal{A} \models \varphi\} \in W \Leftrightarrow \{i \in I/\{\mathcal{A} \in K_i/\mathcal{A} \models \varphi\} \in U_i\} \in U,$$

o aún, escrita en términos de las familias $M = \{\mathcal{A}_j\}_{j\in J}, K_i = \{\mathcal{A}_{ij}\}_{j\in J_i}$, con ultrafiltros W en J y U_i en J_i respectivamente, tenemos

$$\{j \in J/\mathcal{A}_j \models \varphi\} \in W \overset{(*)}{\Leftrightarrow} \{i \in I/\{j \in J_i/\mathcal{A}_{ij} \models \varphi\} \in U_i\} \in U.$$

Ahora, la proposición (1), adaptada al caso del espacio CSt^τ, junto con la proposición (3), tienen como consecuencia para cualquier lógica L que, para cada familia $\{(K_i, U_i)\}_{i\in I}$ en CSt^τ y cada ultrafiltro U sobre $I, \lim_U(K_i, U_i) \neq \phi$, ie. existe $(M, W) \in CSt^\tau$ satisfaciendo la equivalencia $(*)$. Ésto puede ser considerado como el correspondiente Teorema de Łos para los espacios CSt^τ. El problema de la construcción del par (M, W) para cualquier lógica L queda aquí sugerido.

PROPOSICIÓN 5. Si L es compacta, entonces, para todo τ, toda familia $\{(K_i, U_i)\}_{i\in I}$ en CSt^τ, con $K_i = \{\mathcal{A}_{ij}\}_{j\in J_i}$ y U_i ultrafiltro sobre J_i, existe $\mathcal{A} \in St^\tau$ tal que para todo $\varphi \in L^\tau$:

$$\mathcal{A} \models \varphi \Leftrightarrow \{i \in I/\{j \in J_i/\mathcal{A}_{ij} \models \varphi\} \in U_i\} \in U.$$

DEMOSTRACIÓN. Es consecuencia inmediata de la discusión anterior y de la proposición (4) ∎

En $L_{\omega\omega}$ esta estructura $\mathcal{A}$ puede ser realizada explicitamente como el ultraproducto $\Pi_U(\Pi_{U_i}\mathcal{A}_{ij})$.

OBSERVACIÓN 2. Si $(K,U) \in CSt^\tau$, entonces considerando K como una familia en CSt^τ cuyos elementos están parametrizados por ellos mismos, es fácil probar que $\lim_U K = cl\{(K,U)\}$ (cl denota la clausura en CSt^τ). En particular, $(K,U) \in \lim_U K$, lo que refuerza la interpretación de (K,U) como un ultraproducto generalizado de K módulo U.

PROPIEDAD DE EXTENSIÓN

Sean $< X,\mathcal{B} >$ y $< Y,\mathcal{C} >$ dos espacios cero-dimensionales con $\mathcal{B}$ y $\mathcal{C}$ bases de clopens de X e Y respectivamente, cerradas para intersecciones finitas y complementos. Sea $f : X \to Y$ una función, decimos que f es s-continua (strongly continuous) si para todo $W \in \mathcal{C}, f^{-1}[W] \in \mathcal{B}$. Obviamente toda función s-continua es continua. También, decimos que f es s-abierta si para todo $V \in \mathcal{B}, f[V] \in \mathcal{C}$.

Son ejemplos de funciones s-continuas los siguientes:

1. h: $St^\tau \to CSt^\tau$ es s-continua ya que para todo $\varphi \in L^\tau$ tenemos $h^{-1}[\text{Mod}^*(\varphi)] = \text{Mod}^*(\varphi) \cap St^\tau = \text{Mod}(\varphi)$.

2. Denotando con $St^\tau(L)$ el espacio St^τ con la topología elemental dada por L tenemos que, si $L_1 \leq L_2$ (ie. para cada $\varphi \in L_1^\tau$ existe $\psi \in L_2^\tau$ tal que $\text{Mod}_{L_2}(\psi) = \text{Mod}_{L_1}(\varphi)$) entonces, la identidad $I : St^\tau(L_2) \to St^\tau(L_1)$ es s-continua. En particular, si $L_1 \leq L_2$ y $L_2 \leq L_1$ entonces la identidad I mencionada es un homeomorfismo que preserva las bases. Un tal homeomorfismo, ie. una función biyectiva s-continua y s-abierta, será llamado "s-homeomorfismo".

PROPOSICIÓN 6. Sean $< X,\mathcal{B} >$ un espacio cero-dimensional *compacto* con $\mathcal{B}$ una base de clopens cerrada para intersecciones finitas y complementos, y $f : St^\tau \to X$ una función s-continua (St^τ es considerado con la base dada anteriormente),entonces existe $F : CSt^\tau \to X$ s-continua tal que $F\lceil St^\tau = f$.

DEMOSTRACIÓN. Sea $(K,U) \in CSt^\tau$ y consideremos la colección $M = \{V \in \mathcal{B}/ K \cap f^{-1}[V] \in U\}$.

AFIRMACIÓN 1. M es una colección de cerrados de X con la propiedad de intersección finita.

En efecto, obviamente M consta de cerrados pues $\mathcal{B}$ es una base de clopens. Ahora, si $V_1, ..., V_n \in M$, entonces, $K \cap f^{-1}[V_1], ..., K \cap f^{-1}[V_n] \in U$, luego, $K \cap f^{-1}[V_1 \cap ... \cap V_n] = (K \cap f^{-1}[V_1]) \cap ... \cap (K \cap f^{-1}[V_n]) \in U$, en particular, $f^{-1}[V_1 \cap ... \cap V_n] \neq \phi$, ie. $V_1 \cap ... \cap V_n \neq \phi$.

En consecuencia, como X es compacto, para cada $(K, U) \in CSt^{\tau}$ tenemos que $\bigcap\{V \in \mathcal{B}/K \cap f^{-1}[V] \in U\} \neq \phi$.

Definimos $F(K, U) \in \bigcap\{V \in \mathcal{B}/K \cap f^{-1}[V] \in U\}$ en forma arbitraria con la única siguiente restricción: si $(K, U) = (\{\mathcal{A}\}, \{\{\mathcal{A}\}\})$ con $\mathcal{A} \in St^{\tau}$ se observa que $\bigcap\{V \in \mathcal{B}/K \cap f^{-1}[V] \in U\} = \bigcap\{V \in \mathcal{B}/f(\mathcal{A}) \in V\} = \overline{\{f(\mathcal{A})\}}$, luego, exigiendo que $F(\{\mathcal{A}\}, \{\{\mathcal{A}\}\}) = f(\mathcal{A})$ tenemos que $F \lceil St^{\tau} = f$.

AFIRMACIÓN 2. Para todo $V \in \mathcal{B}, F^{-1}[V] = (f^{-1}[V])^*$.

Observemos primero que como f es s-continua, para cada $V \in \mathcal{B}$ existe $\varphi \in L^{\tau}$ tal que $f^{-1}[V] = \text{Mod}(\varphi)$, luego, $(f^{-1}[V])^*$ significa $\text{Mod}^*(\varphi)$.

Sea $(K, U) \in F^{-1}[V]$, entonces, $F(K, U) \in V$, por otro lado, $F(K, U) \in \bigcap\{W \in \mathcal{B}/K \cap f^{-1}[W] \in U\}$. Supongamos que $(K, U) \notin (f^{-1}[V])^* = \text{Mod}^*(\varphi)$, entonces, $(K, U) \in \text{Mod}^*(\neg\varphi)$, ie. $K \cap \text{Mod}(\neg\varphi) \in U$, pero $\text{Mod}(\neg\varphi) = f^{-1}[V^c]$ y $V^c \in \mathcal{B}$, luego, $K \cap f^{-1}[V^c] = K \cap \text{Mod}(\neg\varphi) \in U$, ie. para $W = V^c$ tenemos que $(K, U) \in W$, una contradicción, por lo tanto, $(K, U) \in (f^{-1}[V])^*$.

Sea $(K, U) \in (f^{-1}[V])^* = \text{Mod}^*(\varphi)$, entonces, $K \cap \text{Mod}(\varphi) \in U$, ie. $K \cap f^{-1}[V] \in U$, luego, como $F(K, U) \in \bigcap\{W \in \mathcal{B}/K \cap f^{-1}[W] \in U\}$ tenemos que $F(K, U) \in V$, ie. $(K, U) \in F^{-1}[V]$.

En consecuencia, F es s-continua ∎

COROLARIO. Sean $L_1 \leq L_2$ con L_1 compacta, entonces, para todo tipo $\tau, St^{\tau}(L_1)$ es un *retracto* s-continuo de $CSt^{\tau}(L_2)$, ie. existe $F : CSt^{\tau}(L_2) \to St^{\tau}(L_1)$ s-continuo tal que $F\lceil St^{\tau}(L_2) = I$ (la identidad). F es llamado también un retracto.

DEMOSTRACIÓN. Consideremos la identidad $I : St^{\tau}(L_2) \to St^{\tau}(L_1)$ que, como ya vimos, es s-continua. Entonces, como $St^{\tau}(L_1)$ es compacto cero-dimensional, por la propiedad de extensión (proposición 6) existe $F : CSt^{\tau}(L_2) \to St^{\tau}(L_1)$ s-continua tal que $F\lceil St^{\tau}(L_2) = I$, ie. $St^{\tau}(L_1)$ es trivialmente un retracto s-continuo de $CSt^{\tau}(L_2)$ ∎

OBSERVACIÓN 3. La s-continuidad de la F construida en el corolario anterior se puede expresar, siguiendo la afirmación (2) de la proposición (6), de la siguiente manera: para todo $\varphi \in L_1^{\tau}$ existe $\psi \in L_2^{\tau}$ tal que $F^{-1}[\text{Mod}_{L_1}(\varphi)] = \text{Mod}^*_{L_2}(\psi)$.

A continuación analizaremos la relación estructural íntima que existe entre los espacios CSt^{τ} y St^{τ} en el caso de ser L una lógica compacta.

PROPOSICIÓN 7. Sea L una lógica compacta y para cualquier τ consideremos el retracto $F :< CSt^{\tau}, \mathcal{B} > \to < St^{\tau}, \mathcal{C} >$, dado por el corolario anterior, donde $\mathcal{B}$ y $\mathcal{C}$ son las bases elementales de clopens respectivas. Entonces:

a) La aplicación $G : \mathcal{B} \to \mathcal{C}$ definida por $G(M) = F[M]$ es una biyección.

b) Sea $\{M_i\}_{i\in I} \subseteq \mathcal{B}$, si $\beta\{M_i\}$ denota una combinación booleana arbitraria de los M_i (incluyendo uniones e intersecciones infinitas) entonces, $F[\beta\{M_i\}] = \beta\{F[M_i]\}$ (en particular, G es un isomorfismo de álgebras de Boole que preserva algunas uniones e intersecciones infinitas).

c) $F[cl\{(K,U)\}] = \overline{\{F(K,U)\}}$ (cl denota la clausura en CSt^τ y la barra en St^τ).

d) F es una aplicación propia s-abierta, ie. es s-abierta, cerrada y si $K \subseteq St^\tau$ es compacto, entonces $F^{-1}[K]$ es compacto en CSt^τ.

e) CSt^τ y St^τ tienen la topología inducida y co-inducida por F respectivamente.

DEMOSTRACIÓN. Consideremos $\mathcal{B} = \{\operatorname{Mod}^*(\varphi)/\varphi \in L^\tau\}$ y $\mathcal{C} = \{\operatorname{Mod}(\varphi)/\varphi \in L^\tau\}$.

a) G está definida de la siguiente manera: si $M \in \mathcal{B}$, entonces $M = \operatorname{Mod}^*(\varphi)$ con $\varphi \in L^\tau$, luego $G(M) = F[\operatorname{Mod}^*(\varphi)]$, pero, por la afirmación (2) de la proposición (6), $\operatorname{Mod}^*(\varphi) = F^{-1}[\operatorname{Mod}(\varphi)]$, entonces, como F es sobre, $F[\operatorname{Mod}^*(\varphi)] = \operatorname{Mod}(\varphi)$, ie. $G(\operatorname{Mod}^*(\varphi)) = \operatorname{Mod}(\varphi)$.

Sean $\varphi, \psi \in L^\tau$ tales que $G(\operatorname{Mod}^*(\varphi)) = G(\operatorname{Mod}^*(\psi))$, entonces, $\operatorname{Mod}(\varphi) = \operatorname{Mod}(\psi)$, luego, $\operatorname{Mod}^*(\varphi) = \operatorname{Mod}^*(\psi)$, ie. G es inyectiva. Por lo tanto, G es una biyección ya que es obviamente sobre.

b) Basta probar que F conmuta con los complementos y las intersecciones arbitrarias. El caso de los complementos es trivial pues las mismas bases $\mathcal{B}$ y $\mathcal{C}$ son cerradas con respecto a esa operación. Por la misma razón F conmuta con cualquier intersección finita.

Veamos el caso de una intersección arbitraria: sea $\{\operatorname{Mod}^*(\varphi_i)\}_{i\in I} \subseteq \mathcal{B}$, entonces $\bigcap\limits_{i\in I} \operatorname{Mod}^*(\varphi_i) = \bigcap\limits_{i\in I} F^{-1}[\operatorname{Mod}(\varphi_i)] = F^{-1}[\bigcap\limits_{i\in I} \operatorname{Mod}(\varphi_i)]$, luego, $F[\bigcap\limits_{i\in I} \operatorname{Mod}^*(\varphi_i)] = \bigcap\limits_{i\in I} \operatorname{Mod}(\varphi_i)$ por ser F sobre, ie. $F[\bigcap\limits_{i\in I} \operatorname{Mod}^*(\varphi_i)] = \bigcap\limits_{i\in I} F[\operatorname{Mod}^*(\varphi_i)]$.

c) Es consecuencia inmediata de (b) pues la clausura cl es una intersección de cerrados básicos.

d) Es inmediato que F es s-abierta pues en (a) fue verificado que para el caso de los abiertos básicos, $F[\operatorname{Mod}^*(\varphi)] = \operatorname{Mod}(\varphi)$.

El hecho de ser F cerrada es consecuencia de (b) ya que todo cerrado es intersección de cerrados básicos, ie. de abiertos básicos pues los espacios considerados son cero-dimensionales.

Sea $K \subseteq St^{\tau}$ compacto y supongamos que $F^{-1}[K] \subseteq \bigcup_{i\in I} \mathrm{Mod}^*(\varphi_i)$, entonces, $K = F[F^{-1}[K]] \subseteq F[\bigcup_{i\in I} \mathrm{Mod}^*(\varphi_i)] = \bigcup_{i\in I} F[\mathrm{Mod}^*(\varphi_i)] = \bigcup_{i\in I} \mathrm{Mod}(\varphi_i)$, luego, como K es compacto, existen $i_1, ..., i_n \in I$ tales que $K \subseteq \bigcup_{k=1}^{n} \mathrm{Mod}(\varphi_{i_k})$; por lo tanto, $F^{-1}[K] \subseteq \bigcup_{k=1}^{n} F^{-1}[\mathrm{Mod}(\varphi_{i_k})] = \bigcup_{k=1}^{n} \mathrm{Mod}^*(\varphi_{i_k})$, ie. $F^{-1}[K]$ es compacto.

e) Basta probar que CSt^{τ} y St^{τ} tienen como base la inducida y co-inducida por F respectivamente.

En efecto, como para todo $\varphi \in L^{\tau}$ tenemos que $F^{-1}[\mathrm{Mod}(\varphi)] = \mathrm{Mod}^*(\varphi)$, entonces: $\mathcal{B} = \{\mathrm{Mod}^*(\varphi)/\varphi \in L^{\tau}\} = \{F^{-1}[\mathrm{Mod}(\varphi)]/\varphi \in L^{\tau}$ siendo que $\mathrm{Mod}(\varphi) \in \mathcal{C}\} = \{F^{-1}[M]/M \in \mathcal{C}\} =$ BASE INDUCIDA POR F, y $\mathcal{C} = \{\mathrm{Mod}(\varphi)/\varphi \in L^{\tau}\} = \{\mathrm{Mod}(\varphi)/F^{-1}[\mathrm{Mod}(\varphi)] = \mathrm{Mod}^*(\varphi) \in \mathcal{B}\} = \{M/F^{-1}[M] \in \mathcal{B}\} =$ BASE CO-INDUCIDA POR F ∎

La parte (e) de la proposición anterior dice que St^{τ} es un cociente de CSt^{τ} (siempre en el caso compacto), y el siguiente corolario da el cociente explicitamente.

COROLARIO. Definiendo en CSt^{τ} la relación de equivalencia determinada por F, ie. $(K,U) \sim_F (M,W) \Leftrightarrow F(K,U) = F(M,W)$, tenemos que el espacio cociente CSt^{τ}/F es s-homeomorfo a St^{τ}.

DEMOSTRACIÓN. Denotando con $[K,U]$ la clase de equivalencia de (K,U) en CSt^{τ}, podemos definir la aplicación $\tilde{F} : CSt^{\tau}/F \to St^{\tau}$ por $\tilde{F}([K,U]) = F(K,U)$ y obviamente es una biyección.

Finalmente, a partir de las relaciones encontradas en la proposición anterior, puede probarse facilmente que para todo $\varphi \in L^{\tau}$:

$$\tilde{F}^{-1}[\mathrm{Mod}(\varphi)] = \pi[\mathrm{Mod}^*(\varphi)]$$

y

$$\tilde{F}[\pi[\mathrm{Mod}^*(\varphi)]] = \mathrm{Mod}(\varphi)$$

donde $\pi : CSt^{\tau} \to CSt^{\tau}/F$ es la proyección canónica, lo que demuestra que $\tilde{F}$ es continua e abierta, es más, $\tilde{F}$ es un s-homeomorfismo ∎

Desde el punto de vista lógico podemos obtener aún una mejor descripción de la relación entre CSt^{τ} y St^{τ} en el caso compacto. Primero observemos que la propiedad: $F^{-1}[\text{Mod}(\varphi)] = \text{Mod}^*(\varphi)$ para todo φ, dada en la afirmación (2) de la proposición (6), expresa lo siguiente: para todo $\varphi \in L^{\tau}, (K,U) \in \text{Mod}^*(\varphi) \Leftrightarrow F(K,U) \in \text{Mod}(\varphi)$; y dado que F es sobre tenemos que: para todo $\varphi \in L^{\tau}$

$$(K,U) \in \text{Mod}^*(\varphi) \Leftrightarrow F(K,U) \in F[\text{Mod}^*(\varphi)].$$

Consideremos estructuras topológicas $< X, \mathcal{B} >$ y $< Y, \mathcal{C} >$ siendo $\mathcal{B}$ y $\mathcal{C}$ bases de abiertos, y una aplicación $f : X \to Y$ sobreyectiva, s-continua y s-abierta tal que para todo $a \in X$ y para todo $A \in \mathcal{B}$:

$$a \in A \overset{(\Delta)}{\Leftrightarrow} f(a) \in f[A].$$

Se observa facilmente que la condición (Δ) exigida, es equivalente a: para todo $A \in \mathcal{B}, A = f^{-1}[f[A]]$, ie. cada $A \in \mathcal{B}$ es *saturado con respecto a* f. Una función f satisfaciendo las condiciones dadas en el párrafo anterior será llamada "q-homeomorfismo" (ie. quasi-homeomorfismo, ya que f sería inyectiva si y solo si para todo $B \subseteq X, B = f^{-1}[f[B]]$) (cf. [C] parag. 1.4.4).

El retracto $F : CSt^{\tau} \to St^{\tau}$ es un q-homeomorfismo. Es más, se puede demostrar facilmente, adaptando la demostración de la proposición (7), que todo q-homeomorfismo cumple las propiedades enunciadas en dicha proposición.

Para todo par de cardinales $\kappa \geq \lambda \geq \omega$, sea $L^{-}_{\kappa\lambda}$ el lenguaje formal que tiene como símbolos primitivos: variables $x_0, x_1, ..., x_{\alpha}, ..., \alpha < \kappa$, para elementos de X (respectivamente Y), $V_0, V_1, ..., V_{\alpha}, ..., \alpha < \kappa$, para elementos de $\mathcal{B}$ (respectivamente $\mathcal{C}$), el símbolo de pertenencia "$\in$" que relaciona elementos de X con elementos de $\mathcal{B}$ (ie. $x_i \in V_j$), y el símbolo de igualdad "$=$" entre elementos de $\mathcal{B}$ (ie. $V_i = V_j$). Este lenguaje permite conjunciones y disyunciones infinitas de longitud $< \kappa$, y cuantificación universal y existencial infinita de longitud $< \lambda$, pero no permite la igualdad entre elementos de X.

En este lenguaje podemos expresar, por ejemplo, el hecho de ser $\mathcal{B}$ base de una topología, así como el hecho de ser X un espacio con base enumerable, o de ser un espacio normal o regular. Sin embargo, la propiedad de ser Hausdorff no es expresable porque involucra la igualdad entre elementos de X.

En la siguiente proposición $\vec{x}$ y $\vec{V}$ denotan secuencias finitas o infinitas de variables.

PROPOSICIÓN 8. Sea $f :< X, \mathcal{B} > \to < Y, \mathcal{C} >$ un q-homeomorfismo. Si $\varphi(\vec{x}, \vec{V})$ es una fórmula de $L^{-}_{\kappa\lambda}$, entonces para toda secuencia $\vec{a}$ de elementos de X y $\vec{A}$ de elementos de $\mathcal{B}$ tenemos:

$$< X, \mathcal{B} > \models \varphi[\vec{a}, \vec{A}] \Leftrightarrow < Y, \mathcal{C} > \models \varphi[f(\vec{a}), f[\vec{A}]];$$

en particular, $< X, \mathcal{B} > \equiv_{L^{-}_{\kappa\lambda}} < Y, \mathcal{C} >$.

DEMOSTRACIÓN. La demostración es por inducción sobre la complejidad de la fórmula φ.

Caso 1. Si φ es atómica de la forma $x_i \in V_j$:
$< X, \mathcal{B} > \models (x_i \in V_j)[a, A] \Leftrightarrow a \in A \Leftrightarrow$ (por la condición (Δ))$f(a) \in f[A] \Leftrightarrow$ $< Y, \mathcal{C} > \models (x_i \in V_j)[f(a), f[A]]$.

Caso 2. Si φ es atómica de la forma $V_i = V_j$:
$< X, \mathcal{B} > \models (V_i = V_j)[A, B] \Leftrightarrow A = B \Leftrightarrow$ (por la proposición (7a)) $f[A] = f[B] \Leftrightarrow < Y, \mathcal{C} > \models (V_i = V_j)[f[A], f[B]]$.

Caso 3. Si φ es $\neg\psi$ o $\bigwedge_{i\in I}\varphi_i$ con $|I| < \kappa$: es trivial.

Caso 4. Si $\varphi(\vec{x}, \vec{V})$ es $(\exists\vec{z})\psi(\vec{z}, \vec{x}, \vec{V})$:
$< X, \mathcal{B} > \models (\exists\vec{z})\psi[\vec{a}, \vec{A}] \Leftrightarrow$ existe $\vec{b} \in X$ tal que $< X, \mathcal{B} > \models \psi[\vec{b}, \vec{a}, \vec{A}] \Leftrightarrow$ (por hipótesis inductiva) existe $\vec{b} \in X$ tal que $< Y, \mathcal{C} > \models \psi[f(\vec{b}), f(\vec{a}), f[\vec{A}]] \Leftrightarrow$ (por ser f sobre) existe $\vec{c}\ (= f(\vec{b})) \in Y$ tal que $< Y, \mathcal{C} > \models \psi[\vec{c}, f(\vec{a}), f[\vec{A}]] \Leftrightarrow$ $< Y, \mathcal{C} > \models (\exists\vec{z})\psi[f(\vec{a}), f[\vec{A}]]$.

Caso 5. Si $\varphi(\vec{x}, \vec{V})$ es $(\exists\vec{W})\psi(\vec{x}, \vec{V}, \vec{W})$:
$< X, \mathcal{B} > \models (\exists\vec{W})\psi[\vec{a}, \vec{A}] \Leftrightarrow$ existe $\vec{B} \in \mathcal{B}$ tal que $< X, \mathcal{B} > \models \psi[\vec{a}, \vec{A}, \vec{B}] \Leftrightarrow$ (por hipótesis inductiva) existe $\vec{B} \in \mathcal{B}$ tal que $< Y, \mathcal{C} > \models \psi[f(\vec{a}), f[\vec{A}], f[\vec{B}]] \Leftrightarrow$ (por la proposición (7a)) existe $\vec{C}(= f[\vec{B}]) \in \mathcal{C}$ tal que $< Y, \mathcal{C} > \models$ $\psi[f(\vec{a}), f[\vec{A}], \vec{C}] \Leftrightarrow < Y, \mathcal{C} > \models (\exists\vec{W})\psi[f(\vec{a}), f[\vec{A}]]$ ■

Una consecuencia inmediata de la proposición anterior es que, si $f : X \to Y$ es un q-homeomorfismo, entonces, X es compacto (normal, regular) si y solo si Y es compacto (normal, regular).

EL COCIENTE MÓDULO EQUIVALENCIA ELEMENTAL

Un ejemplo genérico de q-homeomorfismos se da en la siguiente proposición.

PROPOSICIÓN 9. Sea X un espacio topológico. Definimos sobre X la siguiente relación de equivalencia: $x \equiv y \Leftrightarrow \overline{\{x\}} = \overline{\{y\}}$, donde la barra denota la clausura en X. Entonces, si $X/\equiv$ es el espacio cociente respectivo (llamado "espacio de Kolmogoroff de X"), la proyección canónica $\pi : X \to X/\equiv$ es un q-homeomorfismo.

DEMOSTRACIÓN. Obviamente, considerando los espacios con sus topologías totales como bases, π es sobre y s-continua. Si probamos que todo abierto A de X es saturado con respecto a π, ie. $\pi^{-1}[\pi[A]] = A$, será inmediato que π es s-abierta y además satisface la condición (Δ).

Basta probar que $\pi^{-1}[\pi[A]] \subseteq A$. Sea $x \in \pi^{-1}[\pi[A]]$, entonces, $\pi(x) \in \pi[A]$, luego existe $z \in A$ tal que $\pi(x) = \pi(z)$, ie. $\overline{\{x\}} = \overline{\{z\}}$. Si $x \notin A$, entonces, $x \in A^c$ (que es cerrado), luego, $\overline{\{z\}} = \overline{\{x\}} \subseteq \overline{A^c} = A^c$, ie. $z \in A^c$, lo cual es una contradicción, por lo tanto, $x \in A$ ∎

En teoría de modelos tenemos que, si $\mathcal{A}, \mathcal{B} \in St^\tau(L)$ entonces, con respecto a la topología elemental, $\overline{\{\mathcal{A}\}} = \overline{\{\mathcal{B}\}} \Leftrightarrow \mathcal{A} \equiv_L \mathcal{B}$, por lo tanto, la proyección $\pi : St^\tau(L) \to St^\tau(L)/\equiv_L$ es un q-homeomorfismo; es más, es un q-homeomorfismo con respecto a las bases elementales. En particular, $St^\tau(L)$ es compacto si y solo si $St^\tau(L)/\equiv_L$ es compacto.

Análoga descripción puede hacerse con respecto al espacio $CSt^\tau(L)$. Aquí, $(K,U) \equiv_L (M,W)$ significa que para todo $\varphi \in L^\tau, (K,U) \Vdash \varphi \Leftrightarrow (M,W) \Vdash \varphi$. Luego, la proyección $\pi : CSt^\tau(L) \to CSt^\tau(L)/\equiv_L$ es un q-homeomorfismo (con respecto a las bases elementales). En este caso, ambos espacios son compactos.

Por ejemplo, en general tenemos que, si $\mathcal{A} \in K \subseteq St^\tau(L)$ y $U_\mathcal{A}$ es el ultrafiltro principal sobre K generado por $\mathcal{A}$, ie. $U_\mathcal{A} = \{M \subseteq K / \mathcal{A} \in M\}$, entonces $\mathcal{A} \equiv_L (K, U_\mathcal{A})$ (recordar que $\mathcal{A}$ está siendo identificado con el par $(\{\mathcal{A}\}, \{\{\mathcal{A}\}\})$). En efecto, si $\varphi \in L^\tau$, entonces, $(K, U_\mathcal{A}) \Vdash \varphi \Leftrightarrow \{\mathcal{B} \in K / \mathcal{B} \models \varphi\} \in U_\mathcal{A} \Leftrightarrow \mathcal{A} \in \{\mathcal{B} \in K / \mathcal{B} \models \varphi\} \Leftrightarrow \mathcal{A} \models \varphi$. Por lo tanto, dado que todo ultrafiltro sobre un conjunto finito es principal, en $CSt^\tau(L) \backslash St^\tau(L)$ solo interesan, en esencia, los pares (K,U) con K infinito y U no-principal sobre K.

Esta última observación motiva la siguiente modificación en nuestra construcción de una compactificación de los espacios de estructuras St^τ.

DEFINICIÓN 2. Sea $\mathcal{F}(St^\tau) = \{(K,U)/K \subseteq St^\tau$ es un "conjunto" infinito y U un ultrafiltro no-principal sobre $K\}$.

Definimos, $C'St^\tau = St^\tau \cup \mathcal{F}(St^\tau)$ y para cada $\varphi \in L^\tau$, $\text{Mod}'(\varphi) = \text{Mod}(\varphi) \cup \{(K,U) \in \mathcal{F}(St^\tau)/K \cap \text{Mod}(\varphi) \in U\}$.

El nuevo espacio $C'St^\tau$ es una compactificación reducida de St^τ y, aunque con propiedades análogas a la compactificación CSt^τ, tiene la siguiente importante ventaja.

OBSERVACIÓN 4. La propiedad que define los pares (K,U) en $\text{Mod}'(\varphi)$, ie. "$K \cap \text{Mod}(\varphi) \in U$", es equivalente a "existe $A \in U$ con $A \subseteq \text{Mod}(\varphi)$", por lo tanto, para cada $\varphi \in L^\tau$, $\text{Mod}'(\varphi) = \text{Mod}(\varphi) \cup \{(K,U) \in \mathcal{F}(St^\tau)/$ existe $A \in U$ con $A \subseteq \text{Mod}(\varphi)\}$. Desde este punto de vista, la compactificación $C'St^\tau$ puede ser entendida como una compactificación tipo Wallman para St^τ, que además la generaliza, pues una compactificación de Wallman exigiría fijar $K = St^\tau$ en todos los pares (K,U), lo cual no es permitido en la teoría de conjuntos que fundamenta nuestras construcciones. Por otro lado, dejar libre K, siendo aún un conjunto, es más conveniente para las necesidades de la teoría de modelos, como ya vimos.

Volviendo a los cocientes módulo equivalencia elemental, y para finalizar, debemos observar que si L no es una lógica compacta, entonces, en general, $CSt^{\tau}/\equiv$ no es homeomorfo a $St^{\tau}/\equiv$. Sin embargo, si L es compacta, entonces, probaremos en la proposición (10) siguiente que, para todo tipo $\tau, CSt^{\tau}/\equiv$ es homeomorfo a $St^{\tau}/\equiv$. Es más, el homeomorfismo construido será un s-homeomorfismo.

LEMA. Si St^{τ} es compacto, entonces, con respecto al retracto $F : CSt^{\tau} \to St^{\tau}$ tenemos lo siguiente: $(K,U) \equiv (M,W) \Leftrightarrow F(K,U) \equiv F(M,W)$.

DEMOSTRACIÓN.

Supongamos $(K,U) \equiv (M,W)$, entonces, para todo $\varphi \in L^{\tau}, (K,U) \Vdash \varphi \Leftrightarrow (M,W) \Vdash \varphi$, ie. $K \cap \text{Mod}(\varphi) \in U \Leftrightarrow M \cap \text{Mod}(\varphi) \in W$.

Probaremos que $F(K,U) \equiv F(M,W)$. Sea $\varphi \in L^{\tau}$ tal que $F(K,U) \models \varphi$, ie. $F(K,U) \in \text{Mod}(\varphi)$, entonces, $K \cap \text{Mod}(\varphi) \in U$, pues si no, tendríamos que $K \cap \text{Mod}(\neg\varphi) \in U$, luego $F(K,U) \in \text{Mod}(\neg\varphi)$ ya que, por construcción, $F(K,U) \in \bigcap\{\text{Mod}(\psi)/K \cap \text{Mod}(\psi) \in U\}$, una contradicción. Tenemos entonces que $M \cap \text{Mod}(\varphi) \in W$, por lo tanto, como $F(M,W) \in \bigcap\{\text{Mod}(\psi)/M \cap \text{Mod}(\psi) \in W\}, F(M,W) \in \text{Mod}(\varphi)$, ie. $F(M,W) \models \varphi$.

Analogamente se prueba que $F(M,W) \models \varphi$ implica $F(K,U) \models \varphi$.

Supongamos que $F(K,U) \equiv F(M,W)$, entonces, para todo $\varphi \in L^{\tau}, F(K,U) \in \text{Mod}(\varphi) \Leftrightarrow F(M,W) \in \text{Mod}(\varphi)$.

Probaremos que $(K,U) \equiv (M,W)$. Sea $\varphi \in L^{\tau}$ tal que $(K,U) \Vdash \varphi$, ie. $K \cap \text{Mod}(\varphi) \in U$, entonces, $F(K,U) \in \text{Mod}(\varphi)$ pues, por construcción, $F(K,U) \in \bigcap\{\text{Mod}(\psi)/K \cap \text{Mod}(\psi) \in U\}$, luego, $F(M,W) \in \text{Mod}(\varphi)$, de ahí $M \cap \text{Mod}(\varphi) \in W$ pues si no tendríamos que $M \cap \text{Mod}(\neg\varphi) \in W$, y como $F(M,W) \in \bigcap\{\text{Mod}(\psi)/M \cap \text{Mod}(\psi) \in W\}$, tendríamos que $F(M,W) \in \text{Mod}(\neg\varphi)$, una contradicción. Por lo tanto, $(M,W) \Vdash \varphi$.

Analogamente se prueba que $(M,W) \Vdash \varphi$ implica $(K,U) \Vdash \varphi$ ∎

PROPOSICIÓN 10. Para cualquier lógica L y cualquier tipo τ tenemos: para ser St^{τ} compacto es necesario y suficiente que $CSt^{\tau}/\equiv$ sea homeomorfo a $St^{\tau}/\equiv$.

DEMOSTRACIÓN.

SUFICIENCIA. Si $CSt^{\tau}/\equiv$ es homeomorfo a $St^{\tau}/\equiv$, entonces $St^{\tau}/\equiv$ es compacto, luego, como la proyección $\pi : St^{\tau} \to St^{\tau}/\equiv$ es un q-homeomorfismo, también es compacto St^{τ}.

NECESIDAD. Supongamos St^{τ} compacto, y consideremos el retracto $F : CSt^{\tau} \to St^{\tau}$

y la proyección $\pi : St^\tau \to St^\tau / \equiv$.

F induce la aplicación $F' : CSt^\tau / \equiv \to St^\tau / \equiv$ dada por $F'([K,U]) = \pi(F(K,U))$, donde $[K,U]$ denota la clase de equivalencia de (K,U) en CSt^τ).

Es fácil ver que F' está bien definida pues, por el lema anterior, $(K,U) \equiv (M,W) \Rightarrow F(K,U) \equiv F(M,W)$. Por otro lado, la otra implicación dada por el lema, ie. $F(K,U) \equiv F(M,W) \Rightarrow (K,U) \equiv (M,W)$, garantiza que F' es inyectiva. Por lo tanto, F' es una biyección ya que es obviamente sobreyectiva.

AFIRMACIÓN. F' es continua y abierta.

En realidad probaremos que F' es s-continua y s-abierta. Para esto tenemos que explicitar las bases de $CSt^\tau / \equiv$ y $St^\tau / \equiv$ respectivamente.

Consideremos las proyecciones canónicas $\pi_1 : St^\tau \to St^\tau / \equiv$ y $\pi_2 : CSt^\tau \to CSt^\tau / \equiv$; como ellas son q-homeomorfismos, es fácil de verificar que las colecciones $\{\pi_1[\mathrm{Mod}(\varphi)] / \varphi \in L^\tau\}$ y $\{\pi_2[\mathrm{Mod}^*(\varphi)] / \varphi \in L^\tau\}$ son bases (de clopens) de $St^\tau / \equiv$ y $CSt^\tau / \equiv$ respectivamente.

A partir de ahí es de rutina demostrar que para todo $\varphi \in L^\tau$,

$$(F')^{-1}[\pi_1[\mathrm{Mod}(\varphi)]] = \pi_2[\mathrm{Mod}^*(\varphi)]$$

y

$$(F')[\pi_2[\mathrm{Mod}^*(\varphi)]] = \pi_1[\mathrm{Mod}(\varphi)]$$

(siendo ambas equivalentes por ser F' una biyección). Ésto termina la demostración ■

AGRADECIMIENTO. Agradezco a A. M. Sette, mi orientador de doctorado, y a X. Caicedo, por el entusiasmo e invalorable estímulo ofrecidos durante el desarrollo de estos tópicos. La idea del estudio del completamiento, y subsecuente compactificación, de una lógica abstracta surgió en el seminario dado por Caicedo en UNICAMP durante su estadía como Profesor Visitante en 1989.

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A Note on Polarized Partitions

Jimena Llopis

Abstract

It is not known wether the polarized partition property

$$\begin{bmatrix} \omega \\ \omega \\ \cdot \\ \cdot \end{bmatrix} \to \begin{bmatrix} 2 \\ 2 \\ \cdot \\ \cdot \end{bmatrix}$$

is equivalent to the partition property $\omega \to (\omega)^\omega$.

In this paper we prove several facts that suggest that this equivalence is not true. We prove that the existence of a rapid filter implies that there is a function $f : [\omega]^\omega \to 2$ which does not have an homogeneous set, but whose associated function $f' : \omega^\omega \to 2$ has an homogeneous set. We also define two notions of "strong" homogeneous set for functions from ω^ω to 2 and prove the equivalence between the corresponding polarized partition property

$$\begin{bmatrix} \omega \\ \omega \\ \cdot \\ \cdot \end{bmatrix} \to^1 \begin{bmatrix} 2 \\ 2 \\ \cdot \\ \cdot \end{bmatrix}$$

and the partition property $\omega \to (\omega)^\omega$.

For the second notion of strong product partition relation, we prove the following:

$$\begin{bmatrix} \omega \\ \omega \\ \omega \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \to^2 \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ \cdot \\ \cdot \end{bmatrix} \text{ is equivalent to } \omega \to_{X,Y} (\omega)^\omega.$$

This second notion of strong partition relation still captures some product structure (as opposed to the first notion), and it narrows down the original question (in some sense) to the more natural one:

Is $\omega \to (\omega)^\omega$ equivalent to $\omega \to_{X,Y} (\omega)^\omega$?

It is easy to see that it is equivalent to $\omega \to_Y (\omega)^\omega$; however, we believe that if the Y varies with the X (as in $\omega \to_{X,Y} (\omega)^\omega$), the equivalence will not hold. One path to follow would be to study the consistency of rapid filters with the partition relation $\omega \to_{X,Y} (\omega)^\omega$.

1 Some Definitions

We will recall the definitions of the partition properties mentioned above.

1. $\omega \to (\omega)^\omega$ means that for every function $f : [\omega]^\omega \to 2$ there is an infinite subset X of ω such that for every Y in $[X]^\omega$ $f(Y)$ is constant.

2. $\begin{bmatrix} \omega \\ \omega \\ \cdot \\ \cdot \end{bmatrix} \to \begin{bmatrix} 2 \\ 2 \\ \cdot \\ \cdot \end{bmatrix}$ means that for every function $f : \omega^\omega \to 2$ there are sets $H_i (i < \omega)$ such that $card(H_i) \geq 2$ and the image of the product of the H_i's is constant.

3. Consider an increasing function $f : \omega \to \omega$. We will denote by h_f the following function:
   $h_f : \omega^\omega \to \omega^\omega$
   $h_f(\alpha)(i) = \alpha(f(i))$.
   And if H is a subset of ω^ω, $H_f = \{h_f(\alpha) : \alpha \in H\}$.
   Let $F : \omega^\omega \to 2$. A subset H of ω^ω is called a strong homogeneous set for F iff:
   (a) H is an homogeneous set for F (ie. H is the product of sets $H_i, i < \omega$; each H_i of cardinality at least 2 and $F"H$ = constant).
   (b) For every increasing function $f : \omega \to \omega$, $F"H_f$ is constant and equal to $F"H$.
   (c) H is unbounded (ie. $\bigcup H_i$ is an unbounded subset of ω).

   Then the polarized partition $\begin{bmatrix} \omega \\ \omega \\ \cdot \\ \cdot \end{bmatrix} \to^1 \begin{bmatrix} 2 \\ 2 \\ \cdot \\ \cdot \end{bmatrix}$ means that for every function $F : \omega^\omega \to 2$, there exists a strong-1 homogeneous set for F.

4. The partition relation $\begin{bmatrix} \omega \\ \omega \\ \omega \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \to^2 \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ \cdot \\ \cdot \end{bmatrix}$ means that for every function $F : (\omega)^\omega \to 2$ there is a set H (product of the H_i's) such that:
   (a) H is homogeneous for F.
   (b) There are infinitely many i's such that $card(H_i) = 2$, and infinitely many i's with $card(H_i) = 1$.
   (c) If α is an increasing sequence (element of $(\omega)^\omega$) and it is such that for every $\alpha(i) \in H_j$ for some j, and α intersects each H_j; then $F(\alpha)$ is constant (and has the same value as the elements of H).

5. The partition relation $\omega \rightarrow_{X,Y} (\omega)^\omega$ defines the following property: For every function $F : [\omega]^\omega \rightarrow 2$ there exists sets X and Y such that:

   (a) X, Y and $X - Y$ are elements of $[\omega]^\omega$.

   (b) For every set Z containing Y and contained in X, $F(Z) = F(X)$.

   We call X a Y-homogeneous set for F.

6. The partition relation $\omega \rightarrow_Y (\omega)^\omega$ means that for every function F (as above) there is an X that is Y-homogeneous for F.

7. Let F be a filter on ω, we say that F is a rapid filter if it contains the Frechet filter and:

   for every increasing function $f : \omega \rightarrow \omega$ there is an element S of the filter such that for all n in ω $card([0, f(n)] \bigcap S) < n$.

2 Some Theorems

Theorem 1
$$\begin{bmatrix} \omega \\ \omega \\ \cdot \\ \cdot \end{bmatrix} \rightarrow^1 \begin{bmatrix} 2 \\ 2 \\ \cdot \\ \cdot \end{bmatrix} \iff \omega \rightarrow (\omega)^\omega.$$

PROOF:
($\Leftarrow$) The same argument used in [1] (for the corresponding implication in the "weak" polarized partition case) works in this case.
($\Rightarrow$) Given $G : [\omega]^\omega \rightarrow 2$ define $G' : \omega^\omega \rightarrow 2$ as follows:

$$G'(\alpha) = \begin{cases} G(A_\alpha) & \text{if } \alpha \text{ is increasing} \\ 0 & \text{otherwise} \end{cases}$$

where we denote by A_α the image of α.

Take $H' = \prod H_i$ a homogeneous for G'.

Let ${h_i}^0 < {h_i}^1$ be elements of H_i. Since H' is unbounded, there exists a function $f : \omega \rightarrow \omega$ such that H'_f contains an increasing β.

We claim that the set $H = A_\beta = \{{h_{f(i)}}^1 : i < \omega\}$ is homogeneous for G.

Proposition 1 *Let F be a rapid filter on ω. Consider the following function $G : (\omega)^\omega \rightarrow 2$*

$$G(\alpha) = \begin{cases} 1 & \textit{if } \alpha \textit{ is increasing and } \bigcup(\alpha(2n), \alpha(2n+1)] \textit{ is in } F \\ 0 & \textit{otherwise} \end{cases}$$

Then G does not have a strong homogeneous set.

PROOF:
Suppose that H' is a strong homogeneous set for G. Consider G' the restriction of G to the increasing sequences of $\omega^\omega(incr(\omega^\omega))$. Identify $[\omega]^\omega$ with $incr(\omega^\omega)$ and consider G' as

a function from $[\omega]^\omega$ to 2. Take β, A_β and H exactly as in the proof of theorem 1; then H is homogeneous for $G' : [\omega]^\omega \to 2$. But we know that G' does not have an homogeneous set (see [4] or [5]).

Theorem 2 *Let F be a rapid filter. The function G defined in proposition 1 has an homogeneous set. Moreover it has an homogeneous set contained in $incr(\omega^\omega)$.*

Lemma 1 *Let F be a rapid filter on ω. There is a subset of ω with increasing enumeration $\{a_i\}_{i<\omega}$ such that:*

1. $a_0 > 0$
2. $|(a_{2i-1}, a_{2i})| \geq 2$ *for all* $i < \omega$
3. $\bigcup(a_{2i}, a_{2i+1}]$ *is in* F

PROOF: Start with $b_0 > 0$ and $b_{i+1} = b_i + 3$ and consider the set $X' = \{b_i\}_{i<\omega}$. If X' does not have property 3, repeat the argument of [4] and find a subset $X = \{a_i\}_{i<\omega}$ of X' that satisfies property 3. Obviously X satisfies properties 1 and 2.
PROOF OF THEOREM 2: Consider the set X given in Lemma 1 ($X = \{a_i\}_{i<\omega}$). Construct the set $H' = \{h_i\}_{i<\omega}$ as follows:

1. $h_0 < a_0$ (such h_0 exists because of property 1 of the lemma)

   and for $i \geq 0$ put:

2. $h_{2(2i)+1} = a_{2i}$ and

   $h_{2(2i+1)} = a_{2i+1}$

3. Finally take $h_{2(2i+1)+1}$ and $h_{2(2i+2)}$ such that:

   $a_{2i+1} < h_{2(2i+1)+1} < h_{2(2i+2)} < a_{2(i+1)}$

   For this use property 2 of the lemma.

   Take $H = \prod\{h_i, h_{i+1}\}$

For any sequence $< c_0, c_1, \ldots >$ in H we have that:
$[a_{2i}, a_{2i+1}] \subseteq [c_{2i}, c_{2i+1}]$
And therefore H is a homogeneous set for G and $G"H = 1$.

3 Remarks

1. All the arguments of section 2 can be carried with unbounded filters instead of rapid filters.

2. It is known that a rapid filter is unbounded but it is consistent to have unbounded filters and no rapid filter. (see [3] for 1) and 2))

3. The original question would then be solved if we solve the problem the consistency of the "weak" polarized partition property and the existence of unbounded filters on ω.

Theorem 3 *The partition relation* $\begin{bmatrix} \omega \\ \omega \\ \omega \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \rightarrow^2 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ \cdot \end{bmatrix}$ *is equivalent to the partition* $\omega \rightarrow_{X,Y} (\omega)^\omega$.

PROOF:
($\rightarrow$): Let $f : [\omega]^\omega \rightarrow 2$, consider f as a function from the increasing part of $(\omega)^\omega$. In this sense let H be strong-2 homogeneous for f. If H is the product of H_i, then let Y be the union of the H_i's such that $card(H_i) = 1$ and X be the union of all H_i's.
($\leftarrow$): Using similar arguments it is easy to see.

Proposition 2 (Di Prisco)*:* $\omega \rightarrow (\omega)^\omega$ *is equivalent to* $\omega \rightarrow_Y (\omega)^\omega$.

PROOF:
($\rightarrow$): This direction is clear.
($\leftarrow$): Let $F : [\omega]^\omega \rightarrow 2$. Want to find Z homogeneous for F.
Define $G : [\omega]^\omega \rightarrow 2$ as follows:
Let $g : \omega \rightarrow \omega - Y$ be a bijection. Then for each $S \in [\omega]^\omega$ define:
$G(S) = F(g^{-1}{}''(S - Y))$ if $S - Y$ is infinite; and $G(S) = 0$ otherwise.
Let X be Y-homogeneous for G and define $Z = g^{-1}{}''(X - Y)$; then it is easy to cheak that H is homogeneous for F.
Using the same type of argument, the following proposition is not hard to prove:

Proposition 3 $\omega \rightarrow (\omega)^\omega$ *is equivalent to For all* Y*,* $(\omega \rightarrow_Y (\omega)^\omega)$.

REMARKS:

1. We obtain as a curious corollary the following: if there is some Y such that $\omega \rightarrow_Y (\omega)^\omega$, then the same is true for all Y.

2. Another approach to solving the original question would be to adress the following one:

   Is $\omega \rightarrow (\omega)^\omega$ equivalent to $\omega \rightarrow_{X,Y} (\omega)^\omega$?

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La Théorie de la Valuation en Question

N. C. A. da Costa and J. Y. Béziau[1]

> "Nous sommes donc conduits à identifier la *valeur de vérité* d'une proposition avec sa dénotation. Par valeur de vérité d'une proposition, j'entends le fait qu'elle est vraie ou fausse. Il n'y a pas d'autre valeur de vérité."
> G. Frege

Dans un article récent [9] nous avons donné un exposé de la théorie de la valuation présentant les principaux résultats qui s'y rattachent. Nous voudrions ici parler du développement de cette théorie, de son lien avec d'autres théories, de l'orogine des concepts fondamentaux qu'elle emploie, afin d'éclaircir sa signification, de la motiver et de la situer au sein de la logique actuelle.

1 De la Logique Paraconsistante à la Théorie de la Valuation

Le premier d'entre nous développa à la fin des années cinquante une hiérarchie de systèmes de logiques paraconsistantes (connue sous l'appellation C_n, $1 \leq n \leq \omega$), à l'époque le terme "paraconsistant" n'existait pas, l'on parlait de systèmes formels inconsistants [5]. L'idée était de distinguer deux concepts habituellement confondus car équivalent au niveau classique, celui de consistance et celui de non-trivialité. On appelait "système formel inconsistant" un système pouvant servir de base aux théories inconsistantes et non-triviales.

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[1]Titulaire d'une bourse Lavoisier du ministère des affaires étrangères du gouvernement français durant ce travail.

Il s'agissait de systèmes formels au sens où ces logiques étaient présentées sous forme de systèmes axiomatiques de style hilbertien. La question de trouver une sémantique pour ces systèmes resta longtemps ouverte et l'on peut dire que c'est lors de l'établissement d'une sémantique pour C1, qui fut faite en 1976 par le premier d'entre nous en collaboration avec E.H. Alves [7][8], que prit vraiment naissance la théorie de la valuation; ce qui explique le fait que dans la monographie de N. Grana [13] on trouve en annexe l'exposition de cette sémantique.

On ferait cependant erreur si l'on pensait que la théorie de la valuation est intrinsèquement liée à la logique paraconsistante. Ainsi les méthodes utilisées pour créer une sémantique pour ces systèmes peuvent être utilisées avec de légères modifications pour élaborer des sémantiques pour des systèmes paracomplets, non-alèthiques, modales, intuitionnistes, etc ... [2][4][14][15][16][19]. *En fait un nouveau regard était porté sur la sémantique.*

2 Bivalence avant tout

Quelle est l'originalité de la sémantique de C1?

La sémantique de C1 est constituée par un ensemble d'applications de l'ensemble des formules de cette logique dans un ensemble à deux valeurs, le vrai et le faux, symbolisées par 1 et 0. On apelle bivaluations de telles applications. La notion de conséquence sémantique est définie sans surprise: une formule F est conséquence d'une théorie T (i.e. un ensemble de formules) si et seulement si toute bivaluation qui donne la valeur 1 à toutes les formules de T donne également la valeur 1 à F. Cette sémantique semble *a priori* orthodoxe. Pour préciser sa spécificité considérons les données suivantes.

Thèse de Frege: la dénotation d'une proposition est soit le vraie, soit le faux.

Principe de bivalence: une formule est vraie si et seulement si elle n'est pas fausse.

Principe d'analycité: la valeur d'une formule complexe depend uniquement de la valeur de ses composantes.

Principe de verifonctionnalité: la valeur d'une formule complexe est fonction (au sens mathématique du terme) de la valeur de ses composantes.

Principe de substitutionnalité: la valeur d'une formule n'est pas altérée par la substitution à une de ses composantes d'une formule qui est équivalente à cette composante.

Le principe de bivalence est la mise en application de la thèse de Frege. On appelera *sémantique bivalente* une sémantique vérifiant ce principe. Une question se pose: en quoi cette notion de sémantique bivalente est-elle restrictive? Le second d'entre nous a donné dans [3] un résultat tendant à prouver qu'elle ne l'est en rien. Notamment une logique qui a une sémantique à plus de deux valeurs a également une sémantique bivalente, comme l'avait déjà remarqué Suszko [26].

La sémantique de C1 est bivalente mais ne vérifie aucun des autres principes, cela ne l'empêche pas d'être tout à fait intuitive. Le premier d'entre nous a remarqué [9][13] que si une logique a une sémantique bivalente vérifonctionnelle c'est en quelque sorte une sous-logique de la logique classique. L'idée de vérifonctionnalité est donc trop restreinte pour servir de base à moins que l'on ne soit classiciste. Généralement l'on ne requerra que l'exigence de bivalence. Signalons au passage plusieurs faits remarquables: une logique peut très bien avoir une sémantique bivalente analytique non-vérifonctionelle et non-substitutionnelle [2][4]. Une logique qui a une sémantique non-substitutionnelle peut très bien être étudiée d'un point de vue algébrique (cf les algèbres de Curry [6][10]).

Une des idées maîtresses de la théorie de la valuation est que *le concept de bivalence suffit à lui seul à fonder la notion de sémantique.*

Une question se pose: étant donnée une logique, comment obtenir une sémantique bivalente? Y a-t-il une méthode générale? Dans le cas de la logique propositionnelle classique la sémantique est en fait constituée par les fonctions caractéristiques des ensembles maximaux non-triviaux de formules (on appellera survaluations de telles bivaluations). On constate que dans le cas de C1 il en est de même. Ce résultat est-il ou non lié à des caractéristiques propres à ces logiques? , par exemple le fait que dans l'une comme dans l'autre de ces logiques il y ait une négation classique ou bien une implication classique?, ou alors ce résultat ne dépend-il que de propriétés beaucoup plus générales, comme la monotonie? Le problème est de distinguer l'accidentel et l'universel.

Il s'agit de déterminer la classe des logiques pour lesquelles l'ensemble des survaluations constitue une sémantique adéquate. Ces logiques sont-elles les seules à posséder une sémantique bivalente, ou y en a-t-il d'autres?, mais alors quelle est leur sémantique et peut-on encore parler de "logique", n'étend-on pas trop, dans ce cas, le concept de logique?

3 Completude par Saturation

Por répondre à ces questions il faut un peu se pencher sur la preuve du théorème de complétude. On veut démontrer étant donnée une logique $\mathcal{L}$ et $\vdash_{\overline{\mathcal{M}}}$ la relation de conséquence sémantique induite par l'emsemble des survaluations que si $T \vdash_{\overline{\mathcal{M}}} F$ alors $T \vdash_{\overline{\mathcal{L}}} F$.

Considérons la contraposé: si $T \vdash\!\!/_{\overline{\mathcal{L}}} F$ alors $T \vdash\!\!/_{\overline{\mathcal{M}}} F$. Ce que nous voulons montrer c'est que si F n'est pas déductible de T alors il existe b telle que b(T) = 1 et b(F)=0, autrement dit:

(L4) Si $T \vdash\!\!/_{\overline{\mathcal{L}}} F$ alors il existe une extension maximale non-triviale T′ de T telle $F \notin T'$.

Compacité

On pourrait penser que le fait que notre logique $\mathcal{L}$ soit compacte est une condition suffisante pour démontrer L4. Mais qu'appelons-nous exactement "compacité"? En logique classique il y a plusieurs formulations équivalentes de la compacité, notamment ce que nous appellerons compacité déductive (CD) et compacité limitative (CL).

(CD) Si F est déductible de T alors elle est déductible d'une sous-théorie finie de T.

(CL) Si T est triviale alors il existe une sous-théorie finie de T qu'il l'est.

Ces deux formes de la compacité ne sont pas nécessairement équivalentes. Ni CD, ni CL ne suffisent pour démontrer L4, ni les deux à la fois. On peut même montrer que le fait que la logique soit finie (concerne un nombre finie de formules) n'est pas une condition suffisante pour démontrer L4 (tout cela est démontré dans [3]). Par contre on peut démontrer deux résultats remarquables.

Le premier est connu sous le nom de *théorème de Lindenbaum*, l'idée essentielle étant due à A.Lindenbaum, et il peut s'énoncer ainsi: CL $\Rightarrow$ L3, où L3 est le principe suivant:

(L3) Si T est non-triviale, T a une extension maximale.

Ce résultat ne suffit cependant pas, car les deux formes de la complétude:

$$T \nvdash_{\mathcal{L}} F \Rightarrow T \nvdash_{\overline{\mathcal{M}}} F$$
$$\text{T est non-triviale} \Rightarrow \text{T a un modèle}$$

ne sont pas nécessairement équivalentes. Il y a des logiques vérifiant L3 qui ne vérifient pas L4 (se reporter à [3]).

Le second résultat est une variante du théorème de Lindenbaum, due probablement à G. Asser [1], et peut s'énoncer comme suit: CD $\Rightarrow$ L2, où L2 est le principe suivant:

(L2) Si $T \nvdash_{\mathcal{L}} F$ alors il existe une extension T′ de T telle $T' \nvdash_{\mathcal{L}} F$ et telle que si G $\notin$ T′ alors $T', G \vdash_{\mathcal{L}} F$.

C'est ce résultat qui va s'avérer fondamental.

Saturation et Valuation

Disons qu'une théorie T est F-saturée lorsque F n'est pas déductible de T et lorsque F est déductible de toute extension stricte de T. Le théorème de Lindenbaum-Asser, nous dit donc que *pour que T ait une extension F-saturée il suffit que F ne soit pas déductible de T.*

On dira que T est saturée lorsqu'il existe F telle que T soit F-saturée. Cette terminologie diffère de celle employée par Asser ("vollständig in bezug auf") et ses continuateurs ("relatively maximal"). En fait ce concept de saturation et la forme du théorème de Lindembaum qui lui est propre ont été redécouvert. La nouveauté c'est qu'ils jouent un rôle fondamental auquel on ne semble pas avoir prête attention auparavant.

On peut facilement montrer qu'une théorie est maximale si et seulement si elle est F-saturée pour toute F qui n'est pas déductible d'elle. Donc toute théorie maximale est saturée. Mais toute théorie saturée est-elle maximale? En logique classique c'est effectivement le cas et l'on peut même démontrer qu'une condition suffisante pour que ces deux concepts coïncident dans une logique est qu'elle possède une négation ou une implication classique [9][21]. Mais il y a des logiques dans lesquelles il existe des théories saturées non-maximales, comme par exemple la logique positive intuitionniste.

Si on a une logique dans laquelle toute théorie saturée est maximale, alors une condition suffisante pour que l'on ait complétude avec les survaluations est la compacité déductive. Cependant dans le cas contraire, le théorème de Lindenbaum-Asser ne garantit pas la complétude sous cette condition. On peut même démontrer qu'*il ne peut le faire*. Le second d'entre nous a en effet démontré que la sémantique de la valuation (i.e. constituée par les fonctions caractéristiques des théories saturées) est minimale [3]. *Dans le cas où il y a des théories saturées non-maximales, la sémantique de la survaluation est donc incomplète.*

Un des problèmes qui restait posé concernant la sémantique de la hiérarchie Cn était de trouver une sémantique por $C\omega$. En fait $C\omega$ est une extension de la logique positive intuitionniste qui n'a ni implication classique, ni négation classique et dans laquelle il y a des théories saturées non-maximales. Cherchant une sémantique adéquate pour $C\omega$, ce concept de saturation est donc apparu comme décisif (la sémantique de $C\omega$ a été étudiée par A. Loparic [14][16]); on l'a utilisé également pour donner une sémantique bivalente aux logiques modales [15][16].

Qu'une logique déductivement compacte ait ou non des théories saturées non-maximales, elle possède une sémantique bivalente. Les valuations de par leur minimalité s'imposent comme un concept central.

Si la logique n'est pas déductivement compacte peut-on prouver le théorème de Lindenbaum-Asser? Ce n'est pas une condition nécessaire [3] mais sans la compacité déductive comment s'y prendre? En fait on voit facilement qu'au lieu de considérer la sémantique de la valuation on peut tout simplement considérer la sémantique dont les bivaluations sont les fonctions caractéristiques des théories closes (une théorie est dite close lorsque toute formule qui en est déductible lui appartient, la clôture d'une théorie clT est l'ensemble des formules qui en sont déductibles). C'est ce que l'on appelera la *sémantique de l'évaluation.* Mais quelle est la classe des logiques ainsi déterminées, n'est-elle pas trop vaste?

4 Santé par Calcul Abstrait

Calcul de Hilbert Abstrait

Nous sommes restés vague sur la notion de logique. En fait la théorie de la valuation a pris comme point de départ la notion de "calcul". C'est une généralisation des calculs hilbertiens. On donne une définition ne faisant pas intervenir la spécificité du langage. Un calcul de Hilbert abstrait $\mathcal{C}$ est une paire $\langle \mathcal{F}; \mathcal{R} \rangle$ où $\mathcal{F}$ est un ensemble de formules et où $\mathcal{R}$ est un ensemble de règles, une règle R étant une paire $\langle$T;F$\rangle$ dont le premier membre T est un ensemble de formules dites prémisses de la règle et dont le second membre F est une formule dite conclusion de la règle. A partir de ces notions on définit une relation sur $P(\mathcal{F})X\mathcal{F}$ par l'entremise du concept de démonstration. F est démontrable dans T si et seulement si il existe une suite *fini ou infini* de formules telle que F en fasse partie et que chacun des termes de la suite appartienne à T ou soit la conclusion d'une règle dont les prémisses la précèdent.

Santé

Appelons révaluation, une bivaluation respectant les règles d'un calcul de Hilbert. La sémantique de la révaluation est saine ("sound") , c'est même la sémantique la plus large que l'on puisse considérer, elle contient toute sémantique saine. En fait on montre qu'elle coïncide avec la sémantique de l'évaluation [3][9].

La Théorie de la Valuation comme Théorie des Calculs de Hilbert Abstraits

Appelons logique $\mathcal{L}$ une paire $\langle \mathcal{F}; \vdash \rangle$ où $\mathcal{F}$ est un ensemble d'objets et $\vdash$ une relation sur $P(\mathcal{F})X\mathcal{F}$.

On peut voir qu'une logique a une sémantique bivalente si et seulement si c'est un calcul de Hilbert abstrait [9]. De ce point de vue *la théorie de la valuation est la théorie des calculs de Hilbert abstraits.*

Cependant la classe des logiques correspondant aux calculs de Hilbert abstraits peut également être définie sans faire intervenir la notion de calcul, par exemple comme étant la classe des logiques vérifiant les lois de monotonie, de coupure et d'identité ou encore par un opérateur de conséquence [3]. Toutefois le concept de calcul reste fondamental pour la théorie de la valuation.

L'idée de calcul repose avant tout sur la notion de démonstration et l'on peut prétendre qu'elle garde sa valeur même dans le cas où l'on passe à l'infini.

On peut alors facilement étudier la logique du premier ordre et d'ordre supérieure de même que la logique propositionnelle quantifiée du point de vue de la théorie de la valuation. On remarquera notamment qu'un calcul admettant des démonstrations infinies peut très bien être déductivement compacte [9].

Calcul de Gentzen Abstrait

On peut également vouloir prendre comme notion de base la notion de calcul de Gentzen abstrait. Dans ces calculs, les règles sont des relations entre bithéories (i.e. paires de théories): les prémisses sont un ensemble de bithéories et la conclusion une bithéorie; quant aux démonstrations ce sont des suites de bithéories: une démonstration de F dans T est une suite de bithéories dont fait partie ⟨T;F⟩ et dont chacune est la conclusion d'une règle dont les prémisses la précèdent.

Les calculs de Hilbert sont des cas particuliers de calcul de Gentzen [9]. La théorie de la valuation semble de ce point de vue immergée dans une théorie métalogique plus générale.

5 La Théorie de la Valuation comme Théorie des Espaces Logiques

La théorie de la valuation apparaît dans un certain sens fortement liée à celle de calcul mais de par son caractère polymorphique elle permet également d'autres approches métalogiques, dont l'une a un caractère topologique.

Il est clair qu'à chaque bivaluation b on peut faire correspondre une théorie Tb dont les éléments sont exactement les formules qui prennent la valeur 1 avec b. On peut donc se passer de considérer une famille d'applications de l'ensemble des formules dans $\{0,1\}$ et considérer plutôt une famille de théories, comme la famille des théories closes, la famille des théories saturées, etc... Ainsi au lieu d'introduire des notions sémantiques on aura des notions qui ont une allure plutôt topologique. On considérera par exemple l'*Espace Logique* $< \mathcal{F}; \mathcal{T} >$ où $\mathcal{T}$ est l'ensemble des théories vérifiant: T=CIT, T⊆T′ ⇒ CIT⊆CIT′. Considérer cet espace logique ou la sémantique bivalente constituée par les fonctions caractéristiques des théories closes revient au même. Pour reprendre une distinction philosophique classique, nous dirons *en droit*, cela revient au même, *en fait* ces approches nous orientent dans des directions de recherche différentes, nous ne *penserons* pas de la même façon si l'on opte pour l'une ou pour l'autre de ces notions.

En tous les cas la théorie de la valuation, au sens où elle privilégie la notion de bivalence, ouvre la voie vers cette possible analyse.

6 Décidabilité par Table de Vérité

La théorie de la valuation permet de présenter le problème de la décidabilité de façon assez simple. *Grosso modo* une logique est décidable si et seulement si elle l'est par la méthode des tables de vérité [9][13]. La notion de table de vérité en question ici est une généralisation de celle habituelle. La première ligne de la table est constituée d'un ensemble de formule, la *sphère* de la formule F que l'on veut décider. En ce qui concerne les autres lignes de la table chacune d'entre elles est la restriction d'une bivaluation à la sphère de F et pour toute bivaluation, il existe une ligne de la table qui coïncide avec la restriction de cette bivaluation à la sphère de F (pour plus de détails voir [2][9][13][17][19]).

Dans bien des esprits la notion de table de vérité est inexorablement associée à celle de vérifonctionnalité. La méthode utilisée ici dissocie cependant ces deux notions. D'autre part cette notion de sphère d'une formule illustre un autre divorce, celui de l'analycité et de la vérifonctionnalité. La sphère d'une formule F apparaît comme l'ensemble des formules dont dépend F. Si la sphère de F est l'ensemble des sous-formules de F, la valeur de vérité de F ne dépend donc que de la valeur de ses composantes même dans le cas où la logique n'est pas vérifonctionnelle (des exemples se trouvent dans [2] et [4]).

7 La Théorie de la Valuation au sein de la Métalogique

La théorie de la valuation fait partie de ce que l'on peut appeler la métalogique au sens d'une étude générale des systémes logiques. Elle apporte un nouveau jour sur deux problèmes centraux de la métalogique: la complétude et la décidabilité.

Il y a plusieurs façons et plusieurs méthodes pour traiter de la métalogique [24][25]. L'une des plus célèbres est celle, initiée par Tarski [27], utilisant la notion d'opérateur de conséquence comme notion de base. Une autre méthode [11][22] consiste à mettre en avant plutôt la notion de démonstration. On peut partir encore de concepts plutôt algébriques [10], topologico-algébriques [23], algébrico-sémantiques comme dans le cas de l'approche matricielle [20][28][29], ou topologico-sémantiques [12]. La théorie de la valuation se trouve à la croisée de plusieurs chemins.

Ces méthodes sont différentes et elles conviennent plus ou moins bien suivant le but recherché, que l'on veuille par exemple appliquer des résultats et des concepts d'autres parties des mathématiques (l'approche en terme d'opérateur de conséquence permet d'employer des concepts topologiques), que l'on veuille étudier une classe de logiques particulières (les calculs de Gentzen sont peu propices à étudier des logiques non-associatives et la théorie de la valuation ne semblent guère pouvoir s'appliquer aux logiques linéaires) ou que l'on veuille étudier certains problèmes métalogiques spécifiques (étudier la question de la décidabilité ne requiert par nécessairement les mêmes outils que ceux employés pour explorer la question de la complétude).

Cependant ce qui fait l'unité de la métalogique c'est que toutes les méthodes employées sont plus ou moins orientées vers l'abstrait, généralisation oblige. Au sein de la métalogique l'on peut distinguer un domaine appelé *Logique Abstraite.* Là la nature des objets de la logique n'est plus prise en compte, seule importe la structure du système.

Certaines méthodes de la métalogique sont entiérement abstraites, d'autres consistent a établir des liens entre la logique abstraite et la logique "concrête". Par exemple dans la théorie de la valuation les notions de calcul et de table de vérité servent à appliquer les méthodes purement abstraites de cette théorie à la logique concrête.

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COMMENTARII DE BELLO INFINITORUM Adversus unam non sanctam Cantorianam consuetudinem

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Abstract

One of the main characteristics of any Cantorian-fashioned set theory - the proliferation of infinites, specially the cardinal ones - is in this paper revisited. It begins with a short history of the question, the main tenet of Cantor's position and some of the set-theoretical difficulties arising from this kind of theories. Critical approaches and solutions are discussed, specially the constructive solutions of Paul Lorenzen. The gist of the difficulties in a Cantorian-fashioned set theory inheres in the opposition between enumerable and non-enumerable, which is conceived there as an absolute opposition, namely **an opposition which is independent of the language in which such predicates are asserted.** From another point of view a constructivist like Lorenzen builds up a fragment of set theory with neat predicative means as a toolkit for his critical reconstruction of mathematics. When he meets with a non-enumerable set in a specified symbolic system, then he asserts the mentioned set is non-enumerable only **with regard to** (i.e. **in relation to**) the utilized symbolic system. Of course this is not a quibble. In point of fact Lorenzen gives a "layerwise" construction of symbolic systems and proves a number of theorems which assure the existence of a "linguistic layer" in which the former non-enumerable set turns into an enumerable one. Than means that there is always a one-to-one mapping between any non-enumerable set (in a specified symbolic language) and the set of natural numbers, certainly in another specified symbolic language of a higher ordinal lever. Finally the author proposes a recursive function whose limit for $n \to \omega$ can be considered as an example of a one-to-one mapping of the intended kind. The ontological trend of the paper in constructivistic. "Cantor's paradise" becomes sometimes hellish. A counsel of ontological (and epistemological) prudence, important not only for philosophers, but also for logicians and mathematicians, states: **entia non sunt multiplicanda praeter necessitatem.**

COMMENTARII DE BELLO INFINITORUM

(Adversus unam non sanctam Cantorianam consuetudinem)

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> "Il y a deux Labyrinthes fameux, où nostre raison s'égare bien souvent: l'un regarde la grande Question du Libre et du Necessaire, surtout dans la production et dans l'origine du Mal; l'autre consiste dans la discussion de la continuité, et des indivisibles, qui en paroissent les Ele'ments, et où doit entrer la consideration de l'infini. Le premier embarasse presque tout le genre humain, l'autre n'exerce que les Philosophes." LEIBNIZ[1].

El laberinto del infinito surge ya entre los griegos. Zenón ataca a los adversarios de Parménides con los recursos de la adición y la división "infinita" de segmentos y de tiempos. Pero usa un método inadecuado. Su laberinto es otro, que no menciona Leibniz en el epígrafe citado y que también aprisiona a la mente humana: el de la oposición ser-cambio. Las paradojas de Zenón no demuestran la imposibilidad del movimiento, pero las críticas, desde Aristóteles hasta Hobbes y que culminan con Stuart Mill[2] , también yerran su objetivo (como también la de Russell, muy elogiada por Borges pero igualmente inútil[3]).

Las "soluciones" matemáticas al estilo de Stuart Mill se limitan a decir que la flecha alcanza su objetivo y Aquiles a la tortuga porque las series correspondientes son convergentes e iguales en tiempo finito, pero no refutan a Zenón porque no advierten que éste usó un mal método para un buen problema. Refutan el mal método: Zenón yerra porque, o bien no sabía que series infinitas podían tener límite finito o bien consideraba convergente la serie espacial y divergente la serie temporal. En ambos casos se equivocaba. La cuestión realmente discutida era que cambio y tiempo implicaban contradicción y que ésta (conforme a los Eléatas y muchos otros) es incompatible con el ser, tesis sumamente controvertida.

Más allá de los hallazgos de Eudoxo, Euclides y Arquímedes y sus usos del método de exhaución (aniquilar la diferencia[4]), la historia moderna del infinito (todavía potencial) comienza con Galileo, quien por primera vez muestra que las "colecciones" infinitas tienen al menos una parte propia que se puede poner en correspondencia biunívoca con el todo; con ello deja de ser verdad inconcusa la octava noción común del libro primero de los Elementos de Euclides: "El todo es mayor que la parte" (incluso para infinitos potenciales)[5]. Leibniz cambia la perspectiva en una carta a Bernoulli en la que escribe: "el ... conjunto de todos los números trae consigo una contradicción si se lo concibe como un todo completo"[6]. La "contradicción" se refiere a la incompatibilidad entre la mencionada noción común y una colección infinita actual. Sin embargo, como sabemos, esa incompatibilidad ya se da entre la citada noción y el infinito potencial. Bolzano verá en ello una "paradoja del infinito"[7]. Finalmente será

Dedekind quien utilice esta propiedad para definir afirmativamente a los conjuntos infinitos de cualquier índole[8], lo que usualmente se expresa así:

$$M \text{ es infinito } \forall N.\forall f.(N \subset M \wedge N \neq M \wedge f : N \longrightarrow M \wedge \text{f es biyectiva}).$$

La principal dificultad que fatigaba al análisis matemático no era empero la de lo infinitamente grande, sino la de lo infinitamente pequeño *in actu*, desde los indivisibles de Cavalieri, hasta los infinitésimos del análisis prearitmetizado. En una célebre carta a Schumacher de 1831 dice Gauss: "Pero en lo que concierne a su demostración yo protesto ante todo contra el uso de una magnitud infinita (unendliche Grösse) como una magnitud completa, lo que en matemática jamás está permitido. El infinito es sólo una façon de parler con la cual en realidad se habla de límites." [9]

Pero la "gloria" en la tarea de eliminar los infinitésimos more arithmetico corresponderá a Cauchy y especialmente a Weierstrass, preludiados por D'Alembert en 1784[10]. Hilbert señala que la aritmetización del análisis de Weierstrass da por vez primera "un fundamento firme" a éste.[11] (El regreso de los infinitésimos con el "non-standard analysis" de Abraham Robinson se encontraba aún lejano.) Sin embargo, aunque Hilbert coincide con Gauss en que "el infinito, en el sentido de lo infinitamente pequeño y de lo infinitamente grande puede, en el caso de los procesos de límite del cálculo infinitesimal, mostrarse como una mera *façn de parler*[12], pues el análisis sólo trata con el infinito potencial[13], cuando se presenta en las formas de inferencia que tratan de "todos los números que tienen una cierta propiedad" o afirman que existen números reales que tienen una cierta propiedad" [14], es decir, "como totalidad realmente dada, completa y cerrada" [15] (procedimiento vastamente utilizado por Weierstrass), constituye un problema "que no ha sido aún completamente clarificado." [16] La afirmación de Hilbert es muy posterior a la aparición de la teoría cantoriana (es de 1925). Cantor afirmaba el infinito actual, con un énfasis antes jamás escuchado, y producía además un universo infinito de diversos infinitos cardinales y ordinales. Hilbert bautizó a ese universo como el "paraíso de Cantor". La caracterítica más importante de esta proliferación de infinitos actuales, en lo que concierne a los cardinales, es que dos cardinales transfinitos diferentes, como $\aleph_n$ y $\aleph_{n+m}$ ($n \geq 0$ y $m > 0$) no son meramente desiguales respecto del lenguaje en que se expresan esas entidades, sino que se consideran absolutamente desiguales, con independencia del lenguaje (en menor medida esto se repite con los ordinales: los transfinitos menores que ω^{ω} (o 2^{ω}) son absolutamente enumerables (abzählbar)).

Las soluciones a las antinomias que asolaron a la primitiva teoría cantoriana fueron varias. Diversas axiomatizaciones (desde Zermelo 1908) cumplieron exitosamente esa misión. Pero una condición esencial de la fundamentación ,"el requisito esencial de nuestro procedimiento" [17] - una prueba de consistencia absoluta de la teoría de conjuntos transfinitos - (por la complejidad de la teoría) no se satisfizo plenamente (y esta condición puede considerarse esencial para poder predicar "existencia matemática" a una entidad simbólica, según el mismo programa hilbertiano).

El programa de Hilbert fue llamado formalismo. Hilbert aseguraba que "nadie será capaz de expulsarnos del paraíso que Cantor ha creado para nosotros" [18] y para ello se proponía probar la consistencia de los procedimientos demostrativos que incluyen totalidades infinitas actuales mediante (1) la construcción de un lenguaje en suppositio materialis, privado de significados (inhaltlose Sprache), que sea intuitivamente manejable como sistema de figuras materiales y (2) en el cual toda prueba y toda construcción de figuras se realice mediante procedimientos finitos, aunque (3) alguna interpretación de dichas figuras admita entidades infinitas actuales. Ese finitismo metamatemático es el que le permitiría permanecer en el paraíso de Cantor. Por ello no debe sorprender el aspecto aparentemente paradojal del siguiente pasaje de Hilbert: "El infinito no se manifiesta en ninguna parte: ni está presente en la naturaleza, ni es admisible como un fundamento de nuestro pensar racional - una notable armonía entre ser y pensar. Nosotros logramos una convicción que se opone a los esfuerzos originarios de Frege y Dedekind, la convicción de que, si ha de ser posible el conocimiento científico, ciertas representaciones intuitivas e intuiciones son indispensables; la mera lógica no basta. El derecho a operar con el infinito sólo se puede garantizar por medio de lo finito." [19]

Hilbert rechazaba el infinito en la naturaleza de modo algo ingenuo[20], pero aportaba una aguda crítica a la primera antinomia kantiana.[21] Luego afirmaba un finitismo estricto en el orden del pensar que acompañaba con un fragmento de sabor parmenídeo-hegeliano y que prima facie parece incompatible con su anterior afirmación de que jamás seremos expulsados del paraíso de Cantor. La solución es simple: en su metamatemática el finitismo estricto se ejerce sobre figuras no interpretadas, aunque alguna interpretación pueda validar enunciados sobre entidades infinitas de ese paraíso. El fragmento de sabor kantiano también encuentra explicación: las representaciones de que habla ya no son las de la experiencia en las "formas puras del tiempo y el espacio", sino las de las figuras del lenguaje no interpretado y sus reglas de manipulación, aspecto intuitivo que excede a la mera lógica y que es necesario para la construcción de la ciencia.

El finitismo hilbertiano no logra plenamente su propósito. Por ejemplo, cree demostrar la hipótesis del continuo de Cantor, pero la demostración es defectuosa: habrá que esperar hasta Cohen (1963) para conocer de su independencia. Además la demostración de la consistencia absoluta de una teoría tan compleja está más allá del alcance de su finitismo estricto. Por otra parte el paraíso transfinito es más bien un infierno para el intuicionista de estricta observancia. Para éste sólo existe un concepto admisible de infinito: el de "infinito potencial", e.d. el que se construye paso a paso en la "forma pura del tiempo", ya por adición, ya por división - nunca se da el infinito completo o "actual" (e.d. real). Así sólo queda lugar para una única cardinalidad infinita potencial: la de la sucesión fundamental (Grundzahlen) 1, 2,..., n,... Como en Hilbert, los ejemplos básicos de infinitud son siempre dos: el de la sucesión "natural" y el de los puntos de un segmento continuo de recta o, por isomorfismo, el conjunto de los números reales. Pero tanto la teoría cantoriana ingenua cuanto la mayoría de sus reconstrucciones críticas pretendían demostrar que no existía biyección entre la sucesión "natural" enumerable y el conjunto de los reales - o el segmento continuo [0,1]. Si 'k' indica la "cardinalidad", "potencia" o "número de elementos" de un conjunto, 'ω' el conjunto de los naturales y 'c' el de los reales o del continuo, la tesis cantoriana, presuntamente demostrada por reducción al absurdo mediante el segundo método diagonal, afirma que $k(\omega) < k(c)$. Así, para una teoría de conjuntos de cuño cantoriano, ya inicialmente existen dos infinitos cardinales inconmensurables. Puesto que la cardinalidad del continuo es $2^k(\omega)$, de la desigualdad anterior resulta $2^k(\omega) > k(\omega)$, resultado que se generaliza inmediatamente, generando un conjunto infinito de cardinales transfinitos desiguales crecientes.[22] Tales desigualdades serán además absolutas en las teorías de conjuntos de cuño cantoriano, no relativas al lenguaje en que se construyen esos teoremas. Los intuicionistas, con Brouwer[23], rechazan esos resultados mostrando defectos en las demostraciones de los teoremas críticos: en particular el que niega la existencia de cualquier biyección entre un conjunto infinito y su conjunto de partes. Kronecker la rechazaba porque rehusaba existencia matemática a los conjuntos de partes - o conjuntos potencia - de los conjuntos infinitos. Contrario sensu Ernesto Zermelo (1908), en la primera reconstrucción exitosa de la desvencijada teoría cantoriana ingenua, admite axiomáticamente la existencia de dicho conjunto en su axioma IV (Axiom der Potenzmenge): "Para todo conjunto M existe otro conjunto $U(M)$, el conjunto potencia de M, que contiene como elementos precisamente a todos los subconjuntos de M."[24] En la axiomática conjuntista de von Neumann-Bernays-Gödel se soslaya inicialmente la dificultad admitiendo que para cada conjunto M existe una clase de sus subconjuntos $\{N : N \subset M\}$, pero no se asegura que dicha clase sea representable por un conjunto (puede ser una clase "propia" o "última"). Sin embargo luego se introduce un "Potenzmengenaxiom" semejante al de Zermelo.[25]

El axioma que asegura la existencia del conjunto potencia es "naif", pues equivale a suponer acríticamente que, si se nos da un conjunto de objetos, también se nos da el conjunto de todas las formas de enunciado sobre esos objetos. Tal conjunto habitualmente no es construíble. Lo construíble es a lo sumo un conjunto infinito potencial (recursivo o no) de enunciados y formas de enunciado sobre los objetos de un conjunto. Por ello dicho axioma amplía acríticamente el universo de los conjuntos.[26]

Bernays está plenamente conciente de ello cuando afirma que: "La representación utilizada al operar con el conjunto de partes, de que los subconjuntos de un conjunto constituyen en su totalidad un objeto matemático, no está dada por cierto sin más con el concepto de subconjunto, y en primer lugar ello no es algo evidente, cuando se considera el paso de un conjunto a su conjunto de partes como una simple operación iterable a voluntad, por ejemplo análoga con aquella del paso de un número al siguiente."[27]

La crítica de Lorenzen va más allá: "Por ello ya el "conjunto potencia" del conjunto de todos los números naturales, que aparece de modo decisivo en el análisis moderno (allí tiene incluso la potencia del continuo), carece de todo sentido definido en la matemática operativa. Cuando se habla de todos los conjuntos de números naturales se debe dar la capa (lingüística) en que estos conjuntos deben ser representados mediante formas de enunciados. Cuanto más alta la capa, tanto más conjuntos hay." [28]

Se debe recordar también que la demostración del así llamado "teorema de Cantor" $k(M) < k(U(M))$ recurre a un uso ingenuo del axioma de separación al afirmar la existencia del conjunto $Z = \{x : x \notin F(x)\}$, donde F es una hipotética función biyectiva entre M y $U(M)$. El procedimiento es notoriamente impredicativo y equivalente al segundo método diagonal [29], es decir, un procedimiento que constructivamente sólo se justifica en casos especiales, como cuando se lo torna predicativo por construcción sucesiva de capas lingüísticas.

Es leal reconocer que los estupendos edificios teóricos que - comenzando con el de Zermelo - axiomatizaron la teoría cantoriana evitando las antinomias descubiertas a partir de la de Burali-Forti, dieron carta de ciudadanía matemática al "arenarius" de los infinitos actuales. Pero también hay que advertir que dichas teorías axiomáticas difieren en muchos de sus resultados: sus clases de consecuencias no coinciden. Así en ZF es teorema $\{x : x = x\} = \emptyset$ (por inexistencia del conjunto universal), en tanto que en el sistema de von Neumann es teorema $\{x : x = x\} = V$, es decir la clase universal. Igualmente en ZF es teorema $\emptyset = \emptyset$, en tanto que en von Neumannn $\emptyset = V$. Al respecto comenta Suppes: "La diferencia radical entre $\emptyset = \emptyset$ y $\emptyset = V$ enfatiza el carácter ligeramente artificial de toda forma de teoría axiomática de conjuntos." [30] Otros autores son más severos: por ejemplo Lorenzen [31] y, recientemente y desde una perspectiva epistemológica diferente (mereología con lógica clásica subyacente), Richard Martin nos dice que "hemos sido capaces de mantenernos completamente alejados de la teoría de conjuntos con sus evidentemente incurables dolores de cabeza ontológicos y axiomáticos. Tampoco se debería olvidar que estos dolores de cabeza son congénitos y son heredados por todos los miembros contemporáneos del clan conjuntista - teoría modal, teoría de modelos, teoría de las categorías y semejantes." [32] A la luz de las dificultades consideradas, especialmente las relativas a la existencia de cardinales transfinitos absolutamente irreducibles a la enumerabilidad, desde sus comienzos ha parecido deseable desarrollar una versión de teoría de conjuntos que de algún modo no admitiera sino una sola cardinalidad transfinita (enumerable). La proliferación lingüística de cardinales transfinitos sobreviviría como un procedimiento notacional inocuo. Varios intentos se han realizado. Uno es el llamado "camino heroico" mereológico de Lesniewski y Martin [33], en el cual se prescinde de los conjuntos e incluso se pueden eliminar las clases virtuales en favor de los objetos mereológicos fundamentales. Otro de los caminos, que comienza con Kronecker y los pre- y semi-intuicionistas y llega al constructivismo contemporáneo, como el de Lorenzen, apela a la crítica de los axiomas y las reglas de deducción de las diversas teorías de conjuntos de cuño cantoriano. En su faz más técnica se puede hacer arrancar esta línea con los teoremas de Löwenheim de 1915. Este autor demuestra, en primer lugar, para la lógica cuantificacional monádica que, si una fórmula A tiene n predicadores distintos y es válida en un dominio D de al menos 2 n individuos, entonces es válida en todo dominio no-vacío de cualquier cardinalidad:

$$\text{si} \models_{D(2^n)} A, \text{entonces} \models_D A, \quad \text{para todo } D \neq \emptyset.$$

Generalizando este resultado para predicadores n-ádicos y dominios transfinitos se obtiene: "Si una fórmula es válida en un dominio infinito enumerable, entonces es válida en todo dominio no vacío." [34] Skolem contrapone este resultado y lo generaliza para un conjunto infinito enumerable de fórmulas usando su método de las formas normales prenexas (en las que todo generalizador precede a todo particularizador) y el axioma de elección. Sea 'F_ω' un conjunto enumerable de fórmulas, 'D' un dominio de objetos 'I' una función interpretación y '(D, I)' un modelo de F_ω. El teorema de Skolem se expresa de la siguiente manera: "Si el conjunto de fórmulas F_ω tiene un modelo (D, I), entonces F_ω tiene un modelo enumerable (D', I') tal que (D', I') es un submodelo de (D, I)." [35] Desde entonces se comenzó a hablar de la "paradoja de Loewenheim- Skolem". Dicho teorema es paradojal solo bajo el supuesto de que la distinción entre conjuntos infinitos enumerables y no-enumerables sea una distinción absoluta. Como además la teoría de conjuntos cantoriana se puede axiomatizar en el cálculo de predicados de primer orden, el teorema de Loewenheim-Skolem viene a decir que, si la teoría de conjuntos cantoriana tiene un modelo, entonces también tiene un modelo enumerable. Esto muestra otra vez el carácter relativo de la distinción enumerable - no-enumerable, como bien lo señalara Oskar Becker. [36] Este resultado parecía vedado por la "prueba" de la "no-enumerabilidad" del conjunto c de los reales, realizada por Cantor mediante el segundo procedimiento diagonal. Pero, además de no ser predicativo ni constructivo (no permite construir un conjunto de decimales con la potencia del continuo) [37], lo que inicialmente parece probar ese método diagonal es que la tarea de "acomodar" a todos los números reales en la "primera clase numérica ordinal" es irrealizable mediante ese método; pero de ninguna manera prueba la no-enumerabilidad del conjunto de los reales. La conclusión que de ello saca Becker es que "el soberbio edificio de la teoría de conjuntos de Cantor... parece manifestarse como una fantasmagoría, como un cierto espejismo que se desvanece al acercarse a él." [38] Lorenzen da un nuevo paso en esta crítica de la distinción enumerable - no-enumerable absoluta, mediante una construcción "en capas" de entidades, con métodos estrictamente predicativos. [39] Una vez elaborada la lógica constructiva se edifica un "lenguaje elemental" para la matemática operativa entendida como "un operar esquemático con figuras". Los átomos de ese lenguaje son (1) átomos de objeto $a_1, \ldots, a_n$, (2) variables de objeto $x_1, \ldots, x_m$, (3) símbolos de relación, (4) partículas lógicas y (5) operadores de abstracción para conjuntos, relaciones, descripciones y funciones respectivamente. Se define paso a paso qué son términos, enunciados primos, enunciados, formas de enunciado, etc. Para alcanzar "conjuntos de conjuntos", "funciones sobre conjuntos", etc., se reitera el proceso de construcción del lenguaje elemental. La "capa lingüística 0" consta de átomos y variables de objeto. La "capa 1" consta de figuras compuestas por átomos de las especies (1) a (5). En la segunda capa se introducen nuevos átomos y variables, cuyos dominios de variabilidad (finitos) se encuentran en la primera capa. Luego se da un procedimiento efectivo para traducir cada expresión o conjunto de la capa n al lenguaje de la capa $n + 1$. Esto permite prescindir de subíndices y considerar a las capas inferiores como subconjuntos de las superiores. '$X \in S_0$' significa que X es una figura de la capa 0, '$X \in S_n$' que X es una figura de la enésima capa. Se define

$$X \in S_\omega \Leftrightarrow \forall n(X \in S_n)(\text{con } n \in \omega),$$

es decir, X es una figura de la capa w si y sólo si existe un n natural tal que X es una figura de la enésima capa. $S_{\omega+1}$ será el lenguaje elemental sobre figuras de todas la capas finitas. El proceso se continua *ad libitum* hasta algún ordinal determinado de la segunda clase numérica. Allí se recuerda que, para el análisis matemático sólo se requiere superar la capa ω.[40] Cada conjunto finito de objetos básicos es representable en la primera capa. En la segunda es representable el conjunto de todos los conjuntos finitos de objetos básicos. Para alcanzar el conjunto infinito más simple, como $\{a, \{a\}, \{\{a\}\}, \ldots\}$, se requiere llegar a la capa $S_{\omega+1}$. En las capas superiores no sólo son representables nuevos conjuntos de conjuntos, sino nuevos objetos básicos de capas superiores, como muestra el ejemplo de conjunto infinito citado. Uno de los resultados fundamentales de estas *Sprachkonstruktionen* es que todas las capas son enumerables en el sentido de que existe una biyección sobre el conjunto de los enteros positivos. La biyección entre S_t y una capa enumerable es representable ya en S_{t+1}. Ese teorema se demuestra por un procedimiento de goedelización.[41] Un corolario es que en matemática constructiva no existen conjuntos absolutamente "transenumerables": los conjuntos no-enumerables hasta la capa S t ya son enumerables en S_{t+1}. Luego, todo conjunto es enumerable en una capa lingüística apropiada.[42] Así muchos resultados pierden su carácter paradojal: el teorema de Löwenheim-Skolem se torna natural. El teorema cantoriano de que el conjunto c de los reales tiene cardinalidad 2^ω no enumerable es correcto en la capa lingüística $S_{\omega+1}$ (en la que se desarrolla el procedimiento diagonal), pero 2^ω se torna enumerable en una capa superior, etc., etc. La demostración de la relatividad de los conceptos de enumerable y no-enumerable es constructiva, como la de la "enumerabilidad", en una capa superior, de todo conjunto no-enumerable en una capa lingüística determinada. Lo que Lorenzen no da es la construcción de una biyección concreta entre ω y c. Estrictamente hablando sus pruebas son de "existencia débil", por lo tanto precedidas de una doble negación. Lo que queda firme es la no-conclusividad de los teoremas cantorianos críticos y sus derivados, la relatividad de la distinción enumerable - no-enumerable y la existencia débil de la "enumerabilidad" de todo conjunto no-enumerable en una capa lingüística adecuada. Bajo circunstancias que se especificarán se podría considerar que el límite para $n \to \omega$ de la siguiente función recursiva ilustraría el aspecto de un ejemplo (entre otros) de biyección entre $\aleph_0$ y c con existencia fuerte:

$2^0 = 1$

$2^n + k = 1 + \sum_{m<n} 2^m + k(1 \leq n; 0 \leq k < 2^n; k, m, n \in \omega)$.

Es obvio que en un sistema cantoriano, como ZF, el límite para $n \to \omega$ del miembro izquierdo es $2^{\aleph_0}$, pues k es menor que 2^ω y es finito. Para el miembro derecho, por definición de exponenciación para ordinales- límite y de número- γ, es posible deducir:

$$2^{\aleph_0} = \bigcup_{n\in\omega} 2^n = 1 + \bigcup_{n\in\omega} 2^n \leq 1 + \sum_{n\in\omega} 2^n \leq 2^{\aleph_0}.$$ [43]

Recordando otro famoso teorema de ZF para ordinales límites tene- mos: $U(A) = A$ (donde A es un ordinal límite).[44] La cuestión decisiva es aquí cuál es el ordinal límite. Con $1 + \sum_{m<n} 2^m$ podemos proceder de dos maneras diferentes, de las cuales resultarán para el límite dos expresiones cardinales lingüísticamente diferentes:

(1) O bien procedemos de la manera usual en los sistemas cantorianos, conforme a la cual dicha sucesión converge hacia 2^ω, con lo cual, considerando $U(A) = A$, tenemos el resultado esperado:

$$\lim_{n\to\omega} 1 + \sum_{m<n} 2^m = 2^\omega \ (e.d.\ 2^{\aleph_0}),$$

con total beneplácito del clan cantoriano.

(2) O bien, aferrándonos al principio de permanencia de las leyes formales de Hankel, apelamos a un teorema elemental e intuitivamente razonable de las sucesiones, que dice que "si una sucesión infinita s_n (convergente o divergente) tiene un límite s, toda subsucesión infinita t_n tiene el mismo límite"[45], con lo cual podemos recordar que

$$t_n = 1 + \sum_{m<n} 2^m$$

es una sucesión cuyo rango toma los valores 1, 2, 4, 8, ..., es decir, es una subsucesión de aquella cuyo rango es el conjunto w de los naturales. Aplicando $U(A) = A$ a ω y el citado teorema para subsucesiones tenemos:

$$\lim_{n\to\omega} 1 + \sum_{m<n} 2^m = \aleph_0.$$ [46]

De todo lo anterior surge la paradoja $aleph_0 = 2^{aleph_0}$, que puede implicar modificaciones en la base de la teoría de conjuntos, a fin de conservar la vigencia transfinita del principio de permanencia de las leyes formales, pero que no comete los vicios de las pseudodemostraciones de los teoremas cantorianos críticos, relativos a las cardinalidades transfinitas absolutas. Se trataría de una antinomia si nos encontráramos en una teoría cantoriana en la cual los conceptos de enumerable y no-enumerable son absolutos. En cambio en una teoría en la que las cardinalidades transfinitas son relativas a la capa lingüística en que se expresan, sólo se conserva su carácter paradojal.[47] Adoptar el primero o el segundo de los procedimientos arriba mencionados tiene mucho de decisión teórica. La segunda decisión, que recurriendo al principio de conservación de Hankel conserva la validez de los teoremas de sucesiones para la segunda clase ordinal, puede ser objetable bajo el supuesto cantoriano del carácter absoluto de las cardinalidades transfinitas, pero no lo es una vez que se ha demostrado que las cardinalidades transfinitas son relativas a la capa lingüística en que discurra la argumentación, es decir, la construcción de la entidad. Alguien podría objetar que el término límite de la sumatoria anterior, es decir $\lim_{n \to \omega} 2^n$, es $2^{\aleph_0}$ y que por lo tanto la sumatoria que lo contiene no puede tener un límite menor. Pero aquí se debe recordar uno de los resultados más curiosos en esta materia es, como recuerda Lorenzen, que "... esta enumerabilitad y transenumerabilidad relativas se distinguen de los conceptos absolutos de la teoría de conjuntos cantoriana ante todo en que un subconjunto de un conjunto enumerable S_t no siempre es enumerable."[48] Esto es algo de lo que se oculta en el recurso al principio de permanencia de Hankel que aplicáramos más arriba respecto de la extensión de la validez de teoremas de la teoría de sucesiones a la segunda clase ordinal. Sin embargo las demostraciones de Lorenzen del carácter relativo a la capa lingüística de los predicados 'enumerablé y 'no-enumerablé y de que toda cardinalidad no enumerable es reducible a una enumerable en una capa lingüística adecuada, permiten justificar plenamente la validez de la extensión del principio de conservación de Hankel para teoremas de la teoría de sucesiones a fragmentos de la segunda clase ordinal. La reconstrucción lorenziana de (un fragmento necesario para la elaboración de la matemática constructiva de) la teoría de conjuntos resulta entonces una modificación adecuada de las teorías cantorianas que no requiere de la proliferación de cardinales transfinitos absolutamente "inconmensurables" y que permite la conservación de principios y teoremas "intuitivamente razonables", como los mencionados más arriba, para fragmentos de la segunda clase ordinal. Los anteriores resultados de los trabajos de Lorenzen y otros han tenido importantes consecuencias. Una de ellas tiene que ver con el problema tarskiano de los cardinales inaccesibles. El único cardinal absolutamente inaccesible que merece conservarse es $\aleph_0$ u ω. Los restantes se podrían conservar como una proliferación notacional inocua, no absoluta, sino relativa a la capa lingüística, de entidades de una teoría de conjuntos demasiado vasta y no constructiva que es, según expresión de Lorenzen, un "fragmento de literatura fantástica". El propósito de estas líneas ha sido fundamentalmente comentar un prejuicio cantoriano: el del carácter absoluto de la distinción enumerable-no enumerable, insistiendo en que se trata de una distinción relativa al lenguaje en que se expresa tal distinción y enfatizando la posibilidad de permanecer siempre en el ámbito de lo enumerable, recordando también el carácter esencialmente "barroco" y platónico de las teorías absolutas de cardinales transfinitos. Frente a ello ¿no sería más fácil no modificar nada de lo acostumbrado?[49] (Y conservar las teorías de cuño cantoriano.) Muchos filósofos, a los que me uno, y muchos matemáticos (constructivistas) contestarían a esta pregunta postulando un principio de tolerancia como el siguiente: Concédase, a quienes ello les plazca, que sostengan y desarrollen sus teorías de cardinales transfinitos no enumerables crecientes y de cardinales inaccesibles, en tanto eviten la trivialización de sus teorías. Pero, luego de haberse demostrado la relatividad de la distinción enumerable-no enumerable y la reducibilidad de lo último a lo primero, permítasenos a los filósofos (y a los matemáticos constructivos) permanecer en el reino de lo enumerable, suficiente en lo teórico, predicativamente construído (como se demuestra) y acorde con una antigua regla de economía (y prudencia) ontológica y epistemológica del "venerabilis inceptor (invictissimae scholae nominalium)", Guilllermo de Ockham: Entia non sunt multiplicanda praeter necessitatem.

Bahía Blanca, 14 de julio de 1992.

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Addendum Lukasiewicz dice: "Hay en matemáticas modos de inferencia - entre otros el llamado "método de la diagonal" en teoría de conjuntos - que se basan en esas tesis no aceptadas en los sistemas trivalentes e infinito-valentes. Sería interesante indagar si los teoremas matemáticos basados en el método diagonal se pueden demostrar sin tesis proposicionales como éstas." ("Observaciones filosóficas sobre los sistemas polivalentes", en Estudios de lógica y filosofía, Madrid, 1970, p. 82. Las "leyes" a que se refiere Lukasiewicz son:

$(\neg p \to p) \to p$,

$(p \to \neg p) \to \neg p$,

$(p \to q) \to ((p \to \neg q) \to \neg p)$,

$(p \to q \wedge \neg q) \to \neg p$,

$(p \to (q \leftrightarrow \neg q)) \to \neg p$, etc.

Notas

1. Leibniz, Teodicea, GP, vi, 29.

2. El propio Borges se hace eco de esta historia. Cf. "La perpetua carrera de Aquiles y la tortuga", Borges [4], 244-245.

3. Cf. Borges [4], 246-247, Russell [20], caps. 10-11 y [21], 109,113, 115, 136-149.

4. Recuérdese por ejemplo [19], vol. i, p. 670.

5. Galilei [8], viii: Discorsi e dimostrazioni matematiche intorno a due nuove scienze, p. 79.

6. Leibniz, Mathematische Schriften iii, 536. Cf. Weyl [26], 52.

7. Bolzano, Paradoxien des Unendlichen, parr. 20. Cf. Weyl [26], 52.

8. Dedekind, Was sind und was sollen die Zahlen? (1887). Cf. Weyl [26], 52.

9. Cf. Dauben [6], 120.

10. D'Alembert señala en la Encyclopédie que "la théorie de la limite est la base de la vraie métaphysique du calcul differentiel. Il ne s'agit point, comme ont le dit ordinairement, des quantités infiniment petites; il s'agit uniquement de limites des quantités finies." A su vez Cauchy usa el concepto de límite de D'Alembert para definir la derivada de una función como hoy se usa en el análisis "standard", como límite de incrementos finitos.

11. Hilbert [10], 369.

12. Hilbert [10], 370.

13. Hilbert [10], 373.

14. Hilbert [10], 369-370.

15. Hilbert [10], 369.

16. Hilbert [10], 369.

17. Hilbert [10], 383.

18. Hilbert [10], 376.

19. Hilbert [10], 392.

20. Hilbert [10], 371-372.

21. Hilbert [10], 372.

22. Es decir: $2^{2^{\cdot^{\cdot^{\cdot (\text{m veces})^{\cdot^{2^{k(\omega)}}}}}}} < 2^{2^{\cdot^{\cdot^{\cdot (\text{n veces})^{\cdot^{2^{k(\omega)}}}}}}}$ (con $m, n \in \omega$ y $m < n$).

23. Importantes precursores son Kronecker, Poincaré, Borel, etc.

24. Zermelo [27], 203.

25. Cf. von Neumann [25], 401 y Bernays [2], 10, 11, 117 y 130. Al introducir este axioma se puede evitar casi totalmente el tratar con clases. Cf. Bernays [2], p. 132.

26. Lorenzen [13], 165: "Der Vorwurf der "Naivität", der der modernen Mathematik häufig gemacht wird, gründet sich wesentlich gerade darauf, daB für die moderne Mathematik mit einer Klasse von Objekten allemal auch unproblematisch und eindeutig die Klasse aller Aussagen über diese Objekte gegeben ist. Diese unbewiesene Voraussetzung wirkt sich in der CANTORschen Mengenlehre so aus, daß für jede Menge M die Potenzmenge von M "existiert"."

27. Cf. Lorenzen [14], 227: "Die im Operieren mit der Potenzmenge verwendete Vorstellung, daß die Teilmengen einer Menge in ihrer Gesamtheit ein mathematisches Objekt bilden, ist freilich nicht ohne weiteres mit dem Begriff der Teilmenge, der seinerseits unproblematisch ist, gegeben, und erst recht ist es nicht etwas Selbstverständliches, wenn der Übergang von einer Menge zu ihrer Potenzmenge wie eine simple beliebig iterierbare Operation, analog etwa zu derjenigen des Übergangs von einer Zahl zur nächsten, betrachtet wird."

28. Lorenzen [13], 193.

29. Cf. Beth [3], 369 y Kuratowski [12], 73.

30. Suppes [24], 41, da Costa [5], 91-92, Solovay [23].

31. Lorenzen [15], 159, 160, 161, 165 y 167-168.

32. Martin [17], 211; el subrayado es mío.

33. Martin [17], cap. xii: "On Mereology and the Heroic Course".

34. Löwenheim [16], 228-251.

35. Skolem [22], 252-263.

36. Becker [1], cap. 4, parr. 7.

37. Ipse, ibidem.

38. Ipse, ibidem y da Costa [5], 89 y 91.

39. Lorenzen [13], 165-194 y [15], 160 y 168.

40. Pues la capa $\omega + 1$ es la primera que permite construir conjuntos infinitos a partir de sistemas finitos de átomos, lo que no acontece en ninguna capa S_n con $n < \omega$. Cf. Lorenzen [13], 189.

41. Lorenzen [13], 191-192.

42. Ipse [13], 193.

43. Cf. p.ej. Monk [18], 107ss.

44. Cf. p.ej. Suppes [24], 196, Th. 3. (Obviamente para ordinales finitos vale $U(A') = A$. V. idem, 134, Th. 14.)

45. Cf. p.ej. Dieudonné [7], 55-56, prop. (3.13.5) y (3.13.10) y Rey Pastor [19], tomo I, 270-271.

46. Los pasos son en detalle:

(1) t_n es una subsucesión de ω; (2) $U(\omega) = \omega$; (3) $\lim_{n\to\omega} t_n = \bigcup_{n\in\omega} 1 + \sum_{n<\omega} 2_n = \omega$

47. Cf. Suppes [24], 51 y 222. Entendemos aquí por antinomia una oposición incompatible de enunciados (o expresiones) y por paradoja un enunciado inesperado y sorprendente, aunque verdadero, pero opuesto a la "comprensión ingenua" y que, por lo tanto parece ser falso. Cf. Kondakow [11], 33 y 369, y da Costa [5], 194.

48. Cf. Lorenzen [13], 193: "Diese relative Abzählbarkeit und Überabzählbarkeit unterscheidet sich von den absoluten Begriffen der CANTORschen Mengenlehre vor allem dadurch, daß eine Untermenge einer in S_t abzählbaren Menge nicht allemal auch in S_t abzählbar ist. Während die Menge aller Grundobjekte z .B. schon in S 1 abzählbar ist, gibt es ja - wie wir gesehen haben - Mengen von Grundobjekten, die in S_1 nicht darstellbar sind, also erst recht nicht in S_1 abzählbar sind."

49. Lorenzen [14], 226.

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Quasivarieties of De Morgan algebras: RCEP

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July 21, 1992

Abstract

In this note we prove that there is a least strict quasivariety (i.e., a quasivariety which is not a variety) of De Morgan algebras and that such a quasivariety is perhaps the only strict quasivariety enjoying the relative congruence extension property.

1 Introduction

For a quasivariety $\mathcal{Q}$ and an algebra $A \in \mathcal{Q}$, let $Con_{\mathcal{Q}}(A) = \{\Theta \in Con(A) : A/\Theta \in \mathcal{Q}\}$, where $Con(A)$ denotes the set of congruence relations on A. The elements of $Con_{\mathcal{Q}}(A)$ are called $\mathcal{Q}$-congruences on A. A is said to have relative (to $\mathcal{Q}$) congruence extension property (further on RCEP) if for every subalgebra B of A, any $\mathcal{Q}$-congruence on B is the restriction of a $\mathcal{Q}$-congruence on A. $\mathcal{Q}$ has RCEP if all of its elements have this property. The purpose of this note is to prove that there is a

*Research supported by the CDCHT (project C-507-91) of the University of the Andes at Merida Venezuela

least strict quasivariety of De Morgan algebras and such a quasivariety is perhaps the only strict quasivariety enjoying RCEP. For more results in this direction we refer the reader to [3] and [6]. Recall that a De Morgan algebra is an algebra $\langle A; \wedge, \vee, ', 0, 1\rangle$ of type $(2,2,1,0,0)$ such that the reduct $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and the following identities are satisfied:

$$x'' = x \quad ; \quad (x \vee y)' = x' \wedge y'.$$

The lattice of subvarieties of De Morgan algebras is a four-element chain $\mathcal{T} \subset \mathcal{B} \subset \mathcal{K} \subset \mathcal{M}$ where $\mathcal{T}$, $\mathcal{B}$, $\mathcal{K}$, and $\mathcal{M}$ denote respectively the varieties of trivial, Boolean, Kleene and De Morgan algebras. There are three non-trivial subdirectly irreducible De morgan algebras each of which generates one of the non-trivial varieties above: $\mathcal{B}$ is generated by the two- element chain $\mathbf{2} = \{0,1\}$, $\mathcal{K}$ is generated by the three element chain $\mathbf{3} = \{0,a,1\}$ in which $a' = a$ and $\mathcal{M}$ by the four-element complemented lattice $\mathbf{4} = \{0,a,b,1\}$ in which $a' = a$ and $b' = b$. $\mathcal{B}$ satisfies the identity $x \wedge x' = 0$ and $\mathcal{K}$ sastisfies $x \wedge x' \leq y \vee y'$. For a systematic study of $\mathcal{M}$ see [1]

The strategy of the paper is based on the analysis of the two possible cases concerning the variety generated by the strict quasivariety. Section 2 is devoted to the case in which such a variety is $\mathcal{M}$. It is proved that under this assumption no strict quasivariety has RCEP. In Section 3 it is proved that, except for the quasivariety generated by the four element chain $C_4 = \{0 < a < a' < 1\}$ (which is the least strict quasivariety of De Morgan algebras), no strict quasivariety such that the least variety containing it is $\mathcal{K}$ has RCEP. We still do not know wheter or not $Q(C_4)$ has RCEP. Observe that there are not strict quasivarieties contained in $\mathcal{B}$.

The basic concepts of universal algebra can be found in [2]. We follow the notation of this book, particularly for the operators on classes of algebras. In addition, by

$A \leq_{SD} \prod_{i \in I} A_i$ we mean that A is a subdirect product of the family $\{A_i : i \in I\}$. $A \leq B$ means that A is a subalgebra of B. Any class $\mathcal{A}$ of algebras such that $\mathcal{Q}$ is the least quasivariety containing $\mathcal{A}$ is said to generate $\mathcal{Q}$ and in this case we write $\mathcal{Q} = Q(\mathcal{A})$. An algebra $A \in \mathcal{Q}$ is said to be *relatively subdirectly irreducible* or, *$\mathcal{Q}$-subdirectly irreducible*, if it can not be subdirectly embedded in a direct product of algebras of $\mathcal{Q}$ unless the composite of the embedding with one of the projections is an isomorphism. It can be shown that $A \in \mathcal{Q}$ is relatively subdirectly irreducible iff there is a least non-zero $\mathcal{Q}$- congruence on A. Such a congruence is called the *$\mathcal{Q}$-monolith* of A. We denote the class of $\mathcal{Q}$-subdirectly irreducible members of $\mathcal{Q}$ by $\mathcal{Q}_{RSI}$.

2 Quasivarieties that generate $\mathcal{M}$

In this section we prove the following proposition:

Proposition 2.1 *Let $\mathcal{Q}$ be a strict quasivariety such that $H(\mathcal{Q}) = \mathcal{M}$. Then $\mathcal{Q}$ does not have RCEP.*

We pave the way for the proof of this proposition with two lemmas.

Lemma 2.2 *Let A be a homomorphic image of* **4** *such that for all $x \in A$, $x' \neq x$. Then, either the algebra depicted in Figure 1 (i) or the one depicted in 1 (ii) is in $IS(A)$*

PROOF. Let $f : A \longrightarrow \mathbf{4}$ be a surjective homomorphism. Fix an element $u \in A$ such that $f(u) \notin \{0,1\}$. Pick $v \in A$ such that $f(u)' = f(v)$. Let $a = u \wedge u'$ and $b = v \wedge v'$. Clearly, a and b' are not comparable. Let $c = (a \vee b) \wedge a'$ and

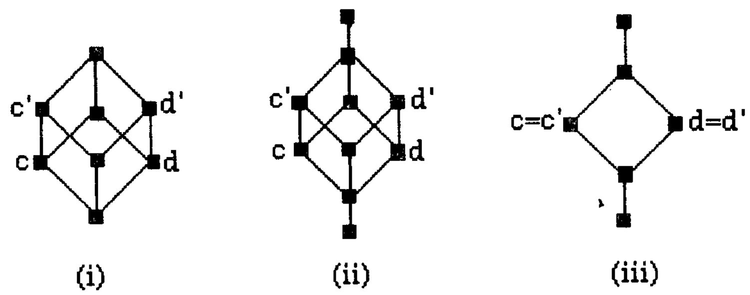

Figure 1: $C \leq A$

$d = (a \vee b) \wedge b'$. It is routine to verify the following: c and d are not comparable; $c < c'$; $d < d'$; $c' \vee d = c' \vee d' = c \vee d'$; $c \wedge d = c \wedge d' = c' \wedge d$; $c \vee d < c' \vee d'$; $c \wedge d < c' \wedge d'$. Thus, the subalgebra of A generated by c and d meets the requirements of the lemma.

Lemma 2.3 *let A be a homorphic image of* **4** *such that there exists $c \in A$ with $c = c'$. Then either* **4** *or the algebra depicted in Figure 1 (iii) is isomorphic to a subalgebra of A.*

PROOF. Let $f : A \longrightarrow \mathbf{4}$ be a surjective homomorphism. Clearly, $u = f(c) \notin \{0, 1\}$. Let v be the Boolean complement of u. Notice that $u' = u$ and $v' = v$. Fix $b \in A$ such that $f(b) = v$. Put $a = b \wedge b'$ and $d = (c \wedge a') \vee a$. One checks now that $f(d) = v$. Since u and v are not comparable, so are c and d. Using now the hypothesis about c and the fact that $a \leq a'$ one gets $d = d'$. Thus, the subalgebra of A generated by c and d meets the requirements of the lemma.

PROOF OF PROPOSITION 2.1. By Birkhoff's subdirect representation theorem, there exists $A \in \mathcal{Q} - \mathcal{K}$ such that one of its homomorphic images is isomorphic to a subalgebra of **4**. In view of the two previous lemmas, one of the algebras depicted in Figure 1 is in $\mathcal{Q}$. Let us denote such an algebra by C. Let D be the subalgebra of C generated by $t = c \vee d$. The proposition now follows from observing that the $\mathcal{Q}$-congruence on D generated by $(0, t')$ coincides with the congruence on D generated by the same element and this $\mathcal{Q}$-congruence can not be extended to a $\mathcal{Q}$-congruence on C.

3 Quasivarieties that generate $\mathcal{K}$

We start this section proving that there is a least strict quasivariety of De Morgan algebras. Notice that any strict quasivariety contains the variety $\mathcal{B}$ of Boolean algebras.

Proposition 3.1 *C_4 generates the least strict quasivariety of De Morgan algebras.*

PROOF. Let $\mathcal{Q}$ be a strict quasivariety of De Morgan algebras. Clearly $\mathcal{B} \subset \mathcal{Q}$, so, there exists $A \in \mathcal{Q}$ and $a \in A$ such that $0 < d = a \wedge a'$. Clearly $d \leq d' = a \vee a' < 1$. If $d < d'$, the subalgebra of A generated by d is isomorphic to C_4, so $C_4 \in \mathcal{Q}$. If $d = d'$, denote by D the subalgebra of A generated by d. As $\mathbf{2} \times D$ has a subalgebra isomorphic to C_4, $C_4 \in \mathcal{Q}$. Hence the proposition is established

We now recall some definitions and results from [4]. Let L be a De Morgan algebra. For a non-emty subset X of L, let $X' = \{x' : x \in X\}$. $T(L) \stackrel{\text{def}}{=} \{t \in L : t \leq t'\} = \{x \wedge x' : x \in L\}$. Denote by $n(L)$ the ideal of the underlying lattice generated by $T(L)$. L is a Kleene algebra iff $n(L) = T(L)$ iff $T(L)' = \{x \vee x' : x \in L\}$ is a filter ([4], Proposition 1.2). $\Theta(n(L))$ (respectively $\Theta(n(L)'))$ denote the least $\mathcal{D}_{0,1}$-congruence of the underlying lattice which has $n(L)$ (respectively $n(L)'$) as a congruence class (here $\mathcal{D}_{0,1}$ denotes the variety of bounded distributive lattices). More precisely, $x \equiv y\Theta(n(L))$ (respectively $\Theta(n(L)'))$ iff there exists $j \in n(L)$ (respectively

$k \in n(L)')$ such that $x \vee j = y \vee j$ (respectively $x \wedge k = y \wedge k$). Let $\beta(L)$ be the least congruence on the De Morgan algebra L such that the quotient algebra $L/\beta(L)$ is a Boolean algebra. Then $\beta(L) = \Theta(n(L)) \vee \Theta(n(L)')$ ([4],Theorem 1.3).

Lemma 3.2 *Let $\mathcal{Q}$ be a strict quasivariety cointained in $\mathcal{K}$. Let $L \in \mathcal{Q}_{RSI}$. Let α be the $\mathcal{Q}$-monolith of L. Then there exist $t, u \in n(L)$ with $t < u$ such that (t, u) generates α.*

PROOF. We first claim that one may choose a, b such that $a \equiv b\Theta(n(L))$ or $a \equiv b\Theta(n(L)')$ with the pair (a, b) generating α. To prove the claim, notice that $\beta(L) = \Theta(n(L)) \vee \Theta(n(L)')$ is a $\mathcal{Q}$-congruence, so $\alpha \subseteq \Theta(n(L)) \vee \Theta(n(L)')$. Pick $c, d \in L$ with $c < d$ such that (c, d) generates α. Thus $c \equiv d(\Theta(n(L)) \vee \Theta(n(L)'))$. It follows from this that for some $j \in n(L)$ and $k \in n(L)'$, $(c \vee j) \wedge k = (d \vee j) \wedge k$. If $c \vee j = d \vee j$ then $c \equiv d\Theta(n(L))$. In this case, take $a = c$ and $b = d$. Otherwise, take $a = c \vee j$ and $b = d \vee j$. In this case $a \equiv b\Theta(n(L)')$. This ends the proof of the claim. Assume now that $a \equiv b\Theta(n(L))$. Observe that since L as a lattice is distributive, either $a \wedge a' \neq b \wedge a'$ or $a \wedge b' \neq b \wedge b'$ or $a \wedge a' \neq b \wedge b'$. If $a \wedge b' \neq b \wedge b'$, take $t = a \wedge b' \wedge b$ and $u = (a \wedge b') \vee (b \wedge b')$. From $a \wedge b' \equiv b \wedge b'\Theta(n(L))$ and $b \wedge b' \in n(L)$ it follows that $a \wedge b' \in n(L)$. Then, because $n(L)$ is an ideal, $u, t \in n(L)$. The case $a \wedge a' \neq b \wedge a'$ is taken care of similarly. If $a \wedge a' \neq b \wedge b'$, take $u = (a \wedge a') \wedge (b \wedge b')$ and $t = (a \wedge a') \vee (b \wedge b')$. Whatever the case is, $u, t \in n(L)$ and (u, t) generates α. Finally, if $a \equiv b\Theta(n(L)')$ then $b' \equiv a'\Theta(n(L))$ and α is also generated by (b', a') and in this situation one can argue as above. Now the proof is complete.

If $A \in \mathcal{Q}$ and $a, b \in A$ then $\Theta^A_{\mathcal{Q}}(a, b)$ denotes the least $\mathcal{Q}$-congruence on A which contains the pair (a, b); i.e., the $\mathcal{Q}$-congruence generated by (a, b).

Corollary 3.3 *In the previous lemma, if $\mathcal{Q}$ has RCEP then $t = 0$.*

PROOF. Assume $t > 0$. As α is the least non-zero $\mathcal{Q}$-congruence, $t \equiv u\Theta^L_{\mathcal{Q}}(0,t)$. Now, since $\mathcal{Q}$ has RCEP, by Proposition 2.4 of [3], $t \equiv u\Theta^S_{\mathcal{Q}}(0,t)$ where $S = \{0 < t < u < u' < t' < 1\}$ is the subalgebra of L generated by $\{t,u\}$. Observe next that $\Theta^S_{\mathcal{Q}}(0,t) = \Theta^S(0,t)$ because $S/\Theta^S(0,t) \cong \{0,u,u',1\} \in \mathcal{Q}$ (Proposition 3.1). But, as it is easily checked, $u \not\equiv t\Theta^S(0,t)$, a contradiction. Then t must be 0.

Proposition 3.4 *Let $\mathcal{Q} \neq Q(C_4)$ be a strict quasivariety of Kleene algebras. Then $\mathcal{Q}$ does not have RCEP.*

PROOF. Assume on the contrary that $\mathcal{Q}$ has RCEP. Let $L \in \mathcal{Q}_{RSI}$ such that L is not a chain (such an L must exist; otherwise $\mathcal{Q}$ would be the quasivariety generated by C_4) and let α be the $\mathcal{Q}$-monolith of L. By Lemma 3.2 and Corollary 3.3 we may pick $b \in L$ with $b < b'$ such that the pair $(0,b)$ generates α. Pick $a \in L$ non-comparable to b. Now look at the two possibilities: $\underline{a \wedge b = c > 0}$; $\underline{a \wedge b = 0}$. In the first one, α is also generated by $(0,c)$ and therefore $b \equiv 0\Theta^L_{\mathcal{Q}}(0,c)$. Since by assumption $\mathcal{Q}$ has RCEP, $b \equiv 0\Theta^S_{\mathcal{Q}}(0,c)$ where $S = \{0 < c < b < b' < c' < 1\}$ is the subalgebra of L generated by $\{c,b\}$. It is evident that $\Theta^S_{\mathcal{Q}}(0,c) = \Theta^S(0,c)$, so $b \not\equiv 0\Theta^S(0,c)$ and this is a contradiction. Let us consider now the possibility $\underline{a \wedge b = 0}$. Denote by A the subalgebra of L generated by a and b. Without lost of generality we may assume that either $\underline{a < a'}$ or $\underline{a \wedge a' = 0}$. All other possibilities about the comparability of a and a' can be reduced to one of these two. Assume first that $\underline{a < a'}$. Observe that $a \vee b < (a \vee b)'$ because $n(L)$ is an ideal ($a \vee b = (a \vee b)'$ is not possible because $\mathcal{Q} \subset \mathcal{K}$). Now, since α is the least non-zero $\mathcal{Q}$-congruence and $\mathcal{Q}$ has RCEP, $b \equiv 0\Theta^A_{\mathcal{Q}}(0,a)$ (see Figure 2 (i)). Next, notice that $A/\Theta^A(0,a) \cong C_4 \in \mathcal{Q}$; so, $\Theta^A_{\mathcal{Q}}(0,a) = \Theta^A(0,a)$ and

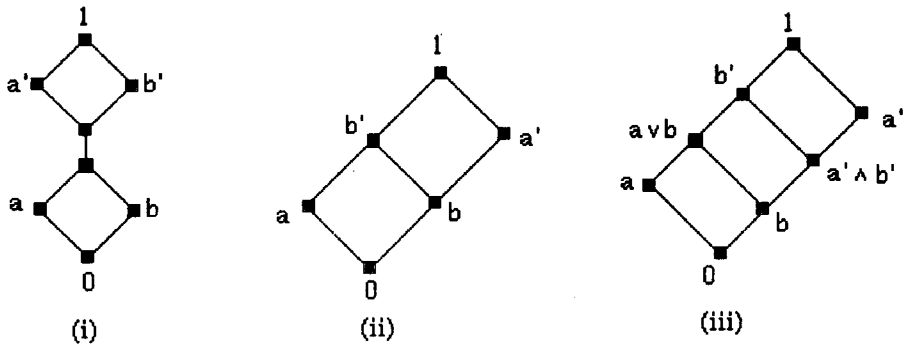

Figure 2: $A \leq L$

consequently $b \not\equiv 0\Theta^A(0,a)$ which is a contradiction. Now we consider the possibility $\underline{a \wedge a' = 0}$. By Corollary 2.5 of [7], $\Theta(x,1) = \Theta_{lat}(x,1)$, $x \in \{a, a'\}$, and these two congruences are complement of each other (see [7], Lemma 3.10). Thus $a' \wedge b > 0$ ($a' \wedge b = 0$ implies $b \equiv 0\Theta(a',1)$). If a' and b are not comparable, we proceed as in the earliest case. If $a' > b$ ($a' \leq b$ is not possible) then A looks like either the algebra depicted in Figure 2 (ii) or the one in 2 (iii). If $a \vee b = b'$ the $\mathcal{Q}$-congruence generated by $(0,a)$ on the subalgebra of A generated by a can not be extended to a $\mathcal{Q}$-congruence on A, which is a contradiction. If $a \vee b < b'$, then $b \equiv 0(\Theta^A_{\mathcal{Q}}(0,a) = \Theta^A(0,a))$, obviously a contradiction. Now the proof of the proposition is complete.

QUESTION. Does $Q(C_4)$ enjoy RCEP?

Proposition 2.9 of [3] can not be used to answer this question in the affirmative because according to Proposition 2.4 of [5], no strict quasivariety of De Morgan algebras is relatively congruence distributive. On the other hand, the method of proof of Fact 2.5 of [3] can not be used to answer it in the negative because $\mathcal{M}$ is not congruence permutable.

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Spectra and Embedding Theorems for Ordered Groups

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1 Introduction

Although this paper is enterely self contained, it can be seen as a sequel of [4]. There we developed a topological duality for a wide class of lattice ordered algebraic structures, including l-groups. Since the underlying order of those structures were distributive lattices we could base on Priestley's and Stone's dualities for distributive lattices. We begin here the extension of the theory to ordered groups. The first difficulty we found is the lacking of an appropriated notion of spectrum. In dealing with lattice ordered structures the natural choice was to take the set of prime lattice filters. The main property of this set -which do things work- is the Prime Lattice Filter Theorem, the fact that the set of prime lattice filters separates points. We introduce in Definition 2.1 the right notion of spectrum to deal with ordered groups. Again, which does things work, is the analogous of the Prime Lattice Filter Theorem, which we have called Separation Hypothesis. The main result of the paper is an ambedding theorem (Theorem 3.1) which states that an ordered group is embeddable as an ordered subgroup of an l-group of sets if and only if it satisfies the Separation Hypothesis. When we restrict ourselves to the abelian case it happens that the ordered groups with the Separation Hypothesis are the well known unperforated groups. In particular, we can reobtain in a nice way the embedding theorem for ordered abelian unperforated groups [2] (see also [1], Chapter 4).

The author wishes to thank some stimulant talks with Dr. D. Mundici about the subject, in particular with respect to the equivalence between primal subsets and subsemigroups (Theorem 3.2).

2 Preliminaries

Recall that $\mathcal{G} = \langle G, \leq, ^{-1}, \cdot, e\rangle$ is an *ordered group* if $\langle G, \leq\rangle$ is an ordered set, $\langle G, ^{-1}, \cdot, e\rangle$ is a group and $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$ for all $z \in G$. $\mathcal{G} = \langle G, \vee, \wedge, ^{-1}, \cdot, e\rangle$ is a *lattice ordered* group (l-group for short) iff $\langle G, \vee, \wedge\rangle$ is a lattice, $\langle G, ^{-1}, \cdot, e\rangle$ is a group and the following equalities hold: $a(b \vee c) = ab \vee ac$; $(b \vee c)a = ba \vee ca$; $a(b \wedge c) = ab \wedge ac$; $(b \wedge c)a = ba \wedge ca$.

Also recall that a subset P of an ordered set $\langle G, \leq\rangle$ is *increasing* if $x \in P$ and $x \leq y$ imply $y \in P$ for all $x, y \in G$. A susbset F of a lattice $\mathcal{A} = \langle A, \vee, \wedge\rangle$ is a *filter* iff F is increasing and $x, y \in F$ implies $x \wedge y \in F$; F is a *prime* lattice filter if also satisfies $F \neq \emptyset$, A and $x \vee y \in F$ implies $x \in F$ or $y \in F$.

Let's introduce the appropriated notion of spectrum for ordered groups.

Definition 2.1 *We'll say that a subset P of a group G (not neccesarily ordered) is primal if it satisfies: $ab \in P$ and $cd \in P$ imply $ad \in P$ or $cb \in P$.*

If $\mathcal{G}$ is an ordered group the *primal spectrum* of $\mathcal{G}$ is the family of all increasing and primal subsets of G. We denote it by $S(G)$. Note that $\emptyset$ and G belong to $S(G)$.

Proposition 2.1 *Let $\mathcal{G}$ be an ordered group. Then the usual product of sets $P \cdot Q = \{xy : x \in P \text{ and } y \in Q\}$ defines a binary function on $S(G)$ which is order-preserving in each variable with respect to set theoretical inclusion.*

Proof. Let $P, Q \in S(G)$; to prove that $P \cdot Q$ is increasing let $a \in P \cdot Q$ and take $b \geq a$; since $a = xy$, with $x \in P$ and $y \in Q$, we have $x^{-1}b \geq y$; then, as Q is increasing, $x^{-1}b \in Q$ and from this follows at once that $b \in P \cdot Q$.

Now, suppose that $ab \in P \cdot Q$ and $cd \in P \cdot Q$ and let x, y, w, z be such that $x, w \in P$, $x, z \in Q$, $ab = xy$ and $cd = wz$. Then, $(x^{-1}a)b \in Q$ and $(w^{-1}c)d \in Q$. Since Q is primal, $(x^{-1}a)d \in Q$ or $(w^{-1}c)b \in Q$, and from this we obtain that $ad \in P \cdot Q$ or $cd \in P \cdot Q$. We have proved that $P \cdot Q$ is also primal; then $P \cdot Q \in S(G)$. The fact that $\cdot$ is order preserving in each variable follows at once. □

Lemma 2.1 *Let $\mathcal{G}$ be an ordered group. Let $P \in S(G)$ and for each $a \in G$ let's define $P_a = \{x : x^{-1}a \notin P\}$. Then:*

i) $P_a \in S(G)$

ii) P_a is the greatest Q in $S(G)$ (with respect to $\subseteq$) such that $a \notin Q \cdot P$.

iii) If $a, b \in G$, then either $P_a \subseteq P_b$ or $P_b \subseteq P_a$.

iv)

$$P \cdot Q = \bigcup_{x \in P} (Q_e)_x$$

Proof. Let $x \in P_a$ and take $y \geq x$; then, $x^{-1} \geq y^{-1}$, and $x^{-1}a \geq y^{-1}a$. Since $x^{-1}a \notin P$ and P is increasing, $y^{-1}a \notin P$; then $y \in P_a$. This proves that P_a is an increasing subset. To prove that P_a is also primal, let x, y, w, z be such that $xy \in P_a$ and $wz \in P_a$ and suppose that $xz \notin P_a$ and $wy \notin P_a$. Then $(xz)^{-1}a \in P$, and $(wy)^{-1}a \in P$, which can be rewritten as $z^{-1}(x^{-1}a) \in P$ and $y^{-1}(w^{-1}a) \in P$. Since P is primal, $z^{-1}w^{-1}a \in P$ or $y^{-1}x^{-1}a \in P$. The first case contradicts $xy \in P_a$; the second one contradicts $wz \in P_a$.

To prove ii) note first that $a \in P_a \cdot P$ implies that there exist $x \in P_a$ and $b \in P$ such that $a = xb$; then $x^{-1}a \in P$, which contradicts $x \in P_a$. Then, $a \notin P_a \cdot P$. Now, let $Q \in S(G)$ such that $a \notin Q \cdot P$ and take $x \in Q$; if $x^{-1}a \in P$, we would have $a \in Q \cdot P$; then, $x^{-1}a \notin P$; we have obtained $Q \subseteq P_a$.

To prove iii), suppose that there exist $P \in S(G)$ and $a, b \in G$ such that $P_a \not\subseteq P_b$ and $P_b \not\subseteq P_a$. Take $x \in P_a \setminus P_b$ and $y \in P_b \setminus P_a$; then $x^{-1}a \notin P$, $x^{-1}b \in P$, $y^{-1}b \notin P$ and $y^{-1}a \in P$, which contradicts the fact that P is primal.

Finally, note that $y \in P \cdot Q$ iff for some $x \in P$, $x^{-1}y \in Q$ iff for some $x \in P$, $(y^{-1}x)^{-1} \in Q$ iff for some $x \in P$, $y^{-1}x \notin Q_e$ iff for some $x \in P$, $y \in (Q_e)_x$ iff

$$y \in \bigcup_{x \in P} (Q_e)_x$$

This proves iv). □

Note that in particular, if we take $a = e$, the assignment $P \to P_e$ defines a unary function g on $S(G)$ which satisfies:

1) $g(g(P)) = P$

2) If $P \subseteq Q$, then $g(Q) \subseteq g(P)$.

Also, from Lemma 2.1 iv) above, we have that

$$P \cdot Q = \bigcup_{x \in P} (g(Q))_x$$

Theorem 2.1 *Let $\mathcal{G}$ be an ordered group and for each $a \in G$ let's denote by $\sigma(a)$ the family of primal increasing subsets of G containing a. Then:*

a) *i)* $\sigma(a^{-1}) = (g^{-1}[\sigma(a)])^c$ *(where ^{c} stands for the set theoretical complement respect to $S(G)$).*

ii)

$$\sigma(ab) = \bigcap_{P \in \sigma(a^{-1})} \{Q : P \cdot Q \in \sigma(b)\}$$

b) Denoting $\Phi(P, Q) = P \cdot Q$, $\sigma(ab)$ is the greatest subset $W \subseteq S(G)$ such that $\sigma(a^{-1}) \times W \subseteq \Phi^{-1}[\sigma(b)]$.

Proof. To prove i) note that $P \in \sigma(a^{-1})$ iff $a^{-1} \in P$ iff $a^{-1}e \in P$ iff $a \notin P_e = g(P)$ iff $g(P) \notin \sigma(a)$ iff $P \notin g^{-1}[\sigma(a)]$.

To prove the left inclusion of ii), let $Q \in \sigma(ab)$ and take $P \in \sigma(a^{-1})$; since $ab \in Q$, $b = a^{-1}c$ for some $c \in Q$. Since $a^{-1} \in P$, $b \in P \cdot Q$ and this equals to say $P \cdot Q \in \sigma(b)$. For the opposite inclusion, let

$$Q \in \bigcap_{P \in \sigma(a^{-1})} \{Q : P \cdot Q \in \sigma(b)\}$$

and assume for the moment that $ab \notin Q$. Then $a^{-1} \in Q_b = \{x : x^{-1}b \notin Q\}$, which equals to say that $Q_b \in \sigma(a^{-1})$. Now, by the choice of Q, we have that $Q_b \cdot Q \in \sigma(b)$. From this we obtain $b \in Q_b \cdot Q$, which contradicts Lemma 2.1 ii).

To prove b) note that from the above discussion $\sigma(a^{-1}) \times \sigma(ab) \subseteq \Phi^{-1}[\sigma(b)]$; suppose now that $\sigma(a^{-1}) \times W \subseteq \Phi^{-1}[\sigma(b)]$ and $W \not\subseteq \sigma(ab)$. Then, there is $Q \in W$ such that $ab \notin Q$. Since $(Q_b, Q) \in \sigma(a^{-1}) \times W$, $b \in \Phi(Q_b, Q) = Q_b \cdot Q$, which is a contradiction. □

Theorem 2.2 (Representation Theorem) *Let $\mathcal{G} = \langle G, \leq, ., ^{-1}, e\rangle$ be an ordered group and suppose that the spectrum $S(G)$ satisfies the following Separation Hypothesis: (SH) if $a \not\leq b$, there is $P \in S(G)$ such that $a \in P$ and $b \notin P$ Then the family of sets $\sum(S(G)) =$*

$\{\sigma(a) : a \in G\}$, equipped with the set theoretical inclusion, the inverse operation $\sigma(a)^{-1} = (g^{-1}[\sigma(a)])^c$, the product

$$\sigma(a)\sigma(b) = \bigcap_{P\in\sigma(a)^{-1}} \{Q : P \cdot Q \in \sigma(b)\}$$

and the unit $\sigma(e)$ is an ordered group isomorphic to G.

Proof. Note first that under the Separation Hypothesis (SH), σ establishes an order-isomorphism between the ordered sets $\langle G, \leq\rangle$ and $\langle \Sigma(S(G)), \subseteq\rangle$: In fact, if $a \neq b$, $a \not\leq b$, or $b \not\leq a$; let's suppose, for example, that a $\not\leq b$; then, there is $P \in S(G)$ such that $a \in P$ and $b \notin P$, which means that $\sigma(a) \neq \sigma(b)$. Since $\Sigma(S(G))$ is the image of σ, σ is a bijection between G and $\Sigma(S(G))$. Now, if $a \leq b$, $\sigma(a) \subseteq \sigma(b)$, because the members of $S(G)$ are increasing sets; conversely, if $\sigma(a) \subseteq \sigma(b)$, using again the Separation Hypothesis, we obtain $a \leq b$.

Note now that from Theorem 2.1 i) it holds that $\sigma(a^{-1}) = \sigma(a)^{-1}$; also, we have from ii) of the same Theorem that

$$\sigma(ab) = \bigcap_{P\in\sigma(a^{-1})} \{Q : P \cdot Q \in \sigma(b)\};$$

using $\sigma(a^{-1}) = \sigma(a)^{-1}$, we obtain $\sigma(ab) = \sigma(a)\sigma(b)$. Then $\sum = \langle \Sigma(S(G)), \subseteq, ^{-1}, \cdot, \sigma(e)\rangle$ is an ordered group isomorphic to $\mathcal{G}$. □

3 Extending Σ to an l-group.

Let's consider the group $\sum$ defined above, whose elements are the sets $\sigma(a)$ of $\Sigma(S(G))$. Note that $\Sigma(S(G))$ with the order given by inclusion can be extended in a natural way to a distributive lattice by taking all the finite unions of finite intersections of these sets. Let's denote by $\mathcal{L}$ this distributive lattice. We shall prove that also the inverse operation $^{-1}$ and the product $\cdot$ of $\sum$ can be extended to $\mathcal{L}$ in order to obtain an l-group. Let's begin with the following:

Lemma 3.1 *Let $A, B, C \in \mathcal{L}$; then:*

a)

$$\bigcap_{P\in A\cup B} \{Q : P \cdot Q \in C\} = \bigcap_{P\in A} \{Q : P \cdot Q \in C\} \cap \bigcap_{P\in B} \{Q : P \cdot Q \in C\}$$

b)

$$\bigcap_{P\in A} \{Q : P \cdot Q \in B \cap C\} = \bigcap_{P\in A} \{Q : P \cdot Q \in B\} \cap \bigcap_{P\in A} \{Q : P \cdot Q \in C\}$$

c)

$$\bigcap_{P\in A\cap B} \{Q : P \cdot Q \in C\} = \bigcap_{P\in A} \{Q : P \cdot Q \in C\} \cup \bigcap_{P\in B} \{Q : P \cdot Q \in C\}$$

d)

$$\bigcap_{P\in A}\{Q : P\cdot Q\in B\cup C\} = \bigcap_{P\in A}\{Q : P\cdot Q\in B\}\cup\bigcap_{P\in A}\{Q : P\cdot Q\in C\}$$

Proof. Note that conditions a) and b) hold simply by set theoretical properties of union and intersection. To prove c) and d) recall that any element of $\mathcal{L}$ can be written as a finite union of finite intersections of elements from $\Sigma(S(G))$. Let's suppose then that

$$A=\bigcup_{i=1}^{n}\bigcap_{j=1}^{m(i)}\sigma(a_{ij})$$

$$B=\bigcup_{k=1}^{r}\bigcap_{l=1}^{g(k)}\sigma(a_{kl})$$

and

$$C=\bigcup_{u=1}^{t}\bigcap_{v=1}^{w(u)}\sigma(a_{uv})$$

To prove condition c) suppose first that there is

$$Q\in\bigcap_{P\in A\cap B}\{Q : P\cdot Q\in C\}$$

such that

$$Q\notin\bigcap_{P\in A}\{Q : P\cdot Q\in C\}\cup\bigcap_{P\in B}\{Q : P\cdot Q\in C\}$$

Then, there exist $P_1\in A$ and $P_2\in B$ such that $P_1\cdot Q\notin C$ and $P_2\cdot Q\notin C$. Since

$$C=\bigcup_{u=1}^{t}\bigcap_{v=1}^{w(t)}\sigma(a_{uv})$$

for all u such that $1\le u\le t$ there is $v(u)\le w(u)$ such that $P_1\cdot Q\notin\sigma(a_{uv(u)})$ and there is $v'(u)\le w(u)$ such that $P_2\cdot Q\notin\sigma(a_{uv(u)})$. From Lemma 2.1 ii), we have that $P_1\subseteq Q_{uv(u)}$ and $Q_{uv(v)}\cdot Q\notin\sigma(a_{uv(u)})$ for each u such that $1\le u\le t$. Note now that from Lemma 2.1iii), the set $\{Q_{uv(u)} : 1\le u\le t\}$ is totally ordered by inclusion. Let's call Q_o the smallest element of this set. Note that the elements of $\mathcal{L}$ are increasing with respect to inclusion; since we have that $P_1\in A$ and $P_1\subseteq Q_o$, we have that $Q_o\in A$. Since . is order preserving in each variable and $Q_{uv(u)}\cdot Q\notin\sigma(a_{uv(u)})$ for each u such that $1\le u\le t$, it follows that $Q_o\cdot Q\notin\sigma(a_{uv(u)})$ for all u such that $1\le u\le t$. Then $Q_o\cdot Q\notin C$. In a similar way, if we call Q'_o the smallest element of the chain $\{Q_{uv'(u)} : 1\le u\le t\}$, we have that $Q'_o\in B$ and $Q'_o\cdot Q\notin C$. Using again Lemma 2.1 iii) we have that $Q_o\subseteq Q'_o$ or $Q'_o\subseteq Q_o$. Let's choose the biggest one. (Suppose, for example, that it is Q'_o). Since A is an increasing set with respect to inclusion, and $Q_o\subseteq Q'_o$, $Q'_o\in A$. Then $Q'_o\in A\cap B$ and $Q'_o\cdot Q\notin C$, which is a contradiction with our assumption that

$$Q\in\bigcap_{P\in A\cap B}\{Q : P\cdot Q\in C\}$$

The opposite inclusion follows directly.

To prove condition d) suppose that there is

$$Q \in \bigcap_{P \in A} \{Q : P \cdot Q \in B \cup C\}$$

such that

$$Q \notin \bigcap_{P \in A} \{Q : P \cdot Q \in B\} \cup \bigcap_{P \in A} \{Q : P \cdot Q \in C\};$$

then there exist $P_1 \in A$ and $P_2 \in A$ such that $P_1 \cdot Q \notin B$ and $P_2 \cdot Q \notin C$. Since

$$B = \bigcup_{k=1}^{r} \bigcap_{l=1}^{g(k)} \sigma(a_{kl})$$

we have that for all k such that $1 \leq k \leq r$, there is $l(k) \leq s(k)$ such that $P_1 \cdot Q \notin \sigma(a_{kl(k)})$. Since

$$C = \bigcup_{u=1}^{t} \bigcap_{v=1}^{w(t)} \sigma(a_{uv})$$

we have that for all u such that $1 \leq u \leq t$ there is $v(u) \leq w(u)$ such that $P_2 \cdot Q \notin \sigma(a_{uv(u)})$. Let's consider, as above, the sets $\{Q_{kl(k)} : 1 \leq k \leq l\}$ and $\{Q_{uv(u)} : 1 \leq u \leq t\}$. We already know that they are totally ordered with respect to inclusion. Let Q_o be the smallest element of the first chain and Q'_o the smallest element of the second one. Then, using as above Lemma 2.1 ii) and the fact that . is order-preserving in each variable, we have that $Q_o \in A$ and $Q_o \cdot Q \notin B$; also, $Q'_o \in A$ and $Q'_o \cdot Q \notin C$. Again from Lemma 2.1 iii) we have that $Q_o \subseteq Q'_o$ or $Q'_o \subseteq Q_o$. Let's choose now the smallest one. Suppose, for example, that it is Q_o Since C is order-preserving with respect to inclusion, $Q'_o \cdot Q \notin C$ and $Q_o \subseteq Q'_o$, we have that $Q_o \cdot Q \notin C$. Then $Q_o \cdot Q \notin B \cup C$. Since $Q_o \in A$ we have derived a contradiction with the assumption

$$Q \in \bigcap_{P \in A} \{Q : P \cdot Q \in B \cup C\}.$$

The opposite inclusion follows directly. □

Now, let's extend the operations of $\sum$ to $\mathcal{L}$ in the natural way: If $A \in \mathcal{L}$, define $A^{-1} = (g^{-1}[A])^c$ and for $A, B \in \mathcal{L}$, define

$$AB = \bigcap_{P \in A^{-1}} \{Q : P \cdot Q \in B\}$$

Note that $(A \cap B)^{-1} = (g^{-1}[A \cap B])^c = (g^{-1}[A] \cap g^{-1}[B])^c = (g^{-1}[A])^c \cup (g^{-1}[B])^c = A^{-1} \cup B^{-1}$. In a similar way, $(A \cup B)^{-1} = A^{-1} \cup B^{-1}$. From this it is easy to prove that $A^{-1} \in \mathcal{L}$; using these equalities and Lemma 3.1 above, it can be readily seen that also $AB \in \mathcal{L}$.

Moreover, from $(A \cap B)^{-1} = A^{-1} \cup B^{-1}$ and Lemma 3.1 a) we can write the following equalities for A, B, $C \in \mathcal{L}$:

1)

$$(A\cap B)C=\bigcap_{P\in(A\cap B)^{-1}}\{Q:P\cdot Q\in C\}=\bigcap_{P\in A^{-1}\cup B_{-1}}\{Q:P\cdot Q\in C\}=$$

$$\bigcap_{P\in A^{-1}}\{Q:P\cdot Q\in C\}\cap\bigcap_{P\in B^{-1}}\{Q:P\cdot Q\in C\}=AC\cap BC$$

In a similar way, using $(A\cup B)^{-1}=A^{-1}\cup B^{-1}$ and b), c) and d) of Lemma 3.1, we can also derive:

2) $A(B\cap C)=(AB)\cap(AC)$;

3) $(A\cup B)C=AC\cup BC$ and

4) $A(B\cup C)=AB\cup AC$.

Let's prove now that our product is associative. Recall first that the product in $S(G)$ is associative, i.e., $P\cdot(Q\cdot R)=(P\cdot Q)\cdot R$ for all P, Q,$R\in S(G)$. Note also that the following hold:

i) $g(P\cdot Q)\cdot P\subseteq g(Q)$;

ii)$P\cdot g(Q\cdot P)\subseteq g(Q)$

To prove i) let $x\in g(P\cdot Q)\cdot P$ and suppose that $x\notin g(Q)$; choose $y\in g(P\cdot Q)$ and $z\in P$ such that $x=yz$. Since $y\in g(P\cdot Q)=(P\cdot Q)_e$, $y^{-1}\notin P\cdot Q$. Since $x\notin g(Q)=Q_e$, $x^{-1}\in Q$. From $z\in P$, $x^{-1}\in Q$ and $y^{-1}\notin P\cdot Q$, we obtain $y^{-1}=zx^{-1}$; then $x=yz$, a contradiction. Condition ii) follows similarly.

We are ready now to prove $A(BC)=(AB)C$ for all A, B, $C\in\mathcal{L}$

For the left inclusion let

$$Q\in A(BC)=\bigcap_{P\in A^{-1}}\{R:P\cdot R\in BC\}$$

and suppose that

$$Q\notin(AB)C=\bigcap_{P\in(AB)^{-1}}\{R:P\cdot R\in C\}$$

Choose $P_1\in(AB)^{-1}$ such that $P_1\cdot Q\notin C$. Since $P_1\in(AB)^{-1}=(g^{-1}[AB])^c$,

$$g(P_1)\notin AB=\bigcap_{P\in A^{-1}}\{R:P\cdot R\in B\}$$

and we can choose $P_2\in A^{-1}$ such that $P_2\cdot g(P_1)\notin B$. Then, $g(P_2\cdot g(P_1))\in B^{-1}$. Since $P_2\in A^{-1}$ and

$$Q\in\bigcap_{P\in A^{-1}}\{R:P\cdot R\in BC\}$$

we can derive

$$P_2\cdot Q\in BC=\bigcap_{P\in B^{-1}}\{R:P\cdot R\in C\}$$

Since $g(P_2\cdot g(P_1))\in B^{-1}$, it follows that $g(P_2\cdot g(P_1))\cdot(P_2\cdot Q)\in C$. As $\cdot$ is associative, $(g(P_2\cdot g(P_1))\cdot P_2)\cdot Q\in C$. Note now that, from condition i) above, $g(P_2\cdot g(P_1))\cdot P_2\subseteq g(g(P_1))=P_1$. Since $\cdot$ is order-preserving in each variable and C is increasing with respect to inclusion, from $(g(P_2\cdot g(P_1))\cdot P_2)\cdot Q\subseteq P_1\cdot Q$. We derive $P_1\cdot Q\in C$, a contradiction.

To prove the opposite inclusion, let

$$Q \in (AB)C = \bigcap_{P \in (AB)^{-1}} \{R : P \cdot R \in C\}$$

and suppose that

$$Q \notin A(BC) = \bigcap_{P \in A^{-1}} \{R : P \cdot R \in BC\}$$

Choose $P_1 \in A^{-1}$ such that

$$P_1 \cdot Q \notin BC = \bigcap_{P \in B^{-1}} \{R : P \cdot R \in C\}$$

Then, there is $P_2 \in B^{-1}$ such that $P_2 \cdot (P_1 \cdot Q) \notin C$. Since $P_2 \cdot (P_1 \cdot Q) = (P_2 \cdot P_1) \cdot Q$, $(P_2 \cdot P_1) \cdot Q \notin C$. Let's prove that $P_2 \cdot P_1 \in (AB)^{-1}$: note that this equals to prove that

$$g(P_2 \cdot P_1) \notin AB = \bigcap_{P \in A^{-1}} \{R : P \cdot R \in B\}.$$

Since $P_1 \in A^{-1}$, it will enough to show that $P_1 \cdot g(P_2 \cdot P_1) \notin B$. From the statement ii) above, $P_1 \cdot g(P_2 \cdot P_1) \subseteq g(P_2)$. Since $P_2 \in B^{-1}$, $g(P_2) \notin B$. As B is an increasing set with respect to inclusion, it follows that $P_1 \cdot g(P_2 \cdot P_1) \notin B$. Now, since we have proved that $P_2 \cdot P_1 \in (AB)^{-1}$ and, from our hypothesis

$$Q \in \bigcap_{P \in (AB)^{-1}} \{R : P \cdot R \in C\},$$

we derive $(P_2 \cdot P_1) \cdot Q \in C$, a contradiction.

Let's prove finally that $A^{-1}A = \sigma(e)$ for all $A \in \mathcal{L}$. Since $g^2 = Id$, note that $(A^{-1})^{-1} = A$. Then

$$A^{-1}A = \bigcap_{P \in A} \{Q : P \cdot Q \in A\}$$

and we have to show that

$$\bigcap_{P \in A} \{Q : P \cdot Q \in A\} = \sigma(e).$$

Let's write, as in Lemma 3.1.

$$A = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m(i)} \sigma(a_{ij}).$$

For the left inclusion take

$$Q \in \bigcap_{P \in A} \{Q : P \cdot Q \in A\}$$

and suppose that $Q \notin \sigma(e)$. This equals to say that $e \notin Q$; since $a^{-1} \cdot a = e$ for all $a \in G$, note that $a \in Q_a = \{x : x^{-1}a \notin Q\}$ for all $a \in G$. We already know, from Lemma 2.1 iii) that for each i, $1 \leq i \leq n$, the set $S_i = \{Q_{a_{ij}} : 1 \leq j \leq m(i)\}$ is totally ordered by inclusion. Also, as we have observed, $a_{ij} \in Q_{a_{ij}}$. Let's denote Q_i the maximum of each set S_i. Then $a_{ij} \in Q_i$ for all j such that $1 \leq j \leq m(i)$ and, from this, $Q_i \in A$. Since $Q_i = Q_{a_{ij_0}}$ for some j_0, $Q_i \cdot Q \notin \sigma(a_{ij_0})$; then, for each i, $1 \leq i \leq n$,

$$Q_i \cdot Q \notin \bigcap_{j=1}^{m(i)} \sigma(a_{ij}).$$

Again from Lemma 2.1 iii), the set $\{Q_i : 1 \leq i \leq n\}$ is a chain. Let's denote Q_o the smallest element of this set. Since $Q_o = Q_i$ for some i, $Q_o \in A$. Since $\cdot$ is order-preserving with respect to inclusion $Q_o \cdot Q \subseteq Q_i \cdot Q$ for all i, $1 \leq i \leq n$; since

$$\bigcap_{j=1}^{m(i)} \sigma(a_{ij})$$

is an increasing set with respect to inclusion and

$$Q_i \cdot Q \notin \bigcap_{j=1}^{m(i)} \sigma(a_{ij}),$$

$$Q_o \cdot Q \notin \bigcap_{j=1}^{m(i)} \sigma(a_{ij})$$

for all i, $1 \leq i \leq n$; then, $Q_o \cdot Q \notin A$. Since $Q_o \in A$, we have derived a contradiction with the assumption that

$$Q \in \bigcap_{P \in A} \{Q : P \cdot Q \in A\}$$

To prove the opposite inclusion, let $Q \in \sigma(e)$. This means that $e \in Q$; from this it follows that $P \subseteq P \cdot Q$ for all $P \in S(G)$. Now take $P \in A$. Since A is an increasing set with respect to inclusion and $P \subseteq P \cdot Q$, we have that $P \cdot Q \in A$. We have proved that

$$Q \in \bigcap_{P \in A} \{Q : P \cdot Q \in A\}$$

Thus, we have proved that $\mathcal{L}$ with the set theoretical union and intersection, the inverse $A^{-1} = (g^{-1}[A])^c$, the product

$$AB = \bigcap_{P \in A^{-1}} \{Q : P \cdot Q \in B\}$$

and the unit $\sigma(e)$ is a lattice ordered group. Moreover, since the lattice order extend the inclusion and the group operations extend those of $\sum$, $\sum$ is an ordered subgroup of $\mathcal{L}$. We are ready now to prove the following:

Theorem 3.1 (Embedding theorem for ordered groups). *Let $\mathcal{G} = \langle G, \leq, ^{-1}, \cdot, e \rangle$ be an ordered group and let $S(G)$ be the primal spectrum of $\mathcal{G}$. Then the following are equivalent:*

i) For each a, $b \in G$, if $a \not\leq b$, there is $P \in S(G)$ such that $a \in P$ and $b \notin P$. (Separation Hypothesis).

ii) $\mathcal{G}$ is embeddable as an ordered subgroup of an l-group of sets, whose underlying lattice operations are the usual union and intersection of sets.

iii) $\mathcal{G}$ is embeddable as an ordered subgroup of an l-group.

Proof. If the Separation Hypothesis holds, we have from Theorem 5. that $\mathcal{G}$ is isomorphic to $\sum$. From the construction above, $\sum$ can be seen as an ordered subgroup of the l-group $\mathcal{L}$, whose underlying lattice operations are union and intersection. Then, ii) holds.

Of course, ii) implies iii). To prove that iii) implies i) let's suppose, stripping away inessentials, that $\mathcal{G}$ is an ordered subgroup of an l-group A. Let's prove that for each prime lattice filer $P \subseteq \mathcal{A}$, $P \cap G$ is a primal increasing subset of G: since the filters of $\mathcal{A}$ are increasing subsets of $\mathcal{A}$, $P \cap G$ is an increasing subset of G. Now, take a, b, c, $d \in G$ and suppose that ab, $cd \in P \cap G$. If we denote respectively by $\vee$ and $\wedge$ the join and the meet lattice operations of $\mathcal{A}$, we have that $ab \leq a(b \vee d)$ and that $cd \leq c(b \vee d)$. Since P is a filter of $\mathcal{A}$, $a(b \vee d) \in P$, $c(b \vee d) \in P$, and also $a(b \vee d) \wedge c(b \vee d) \in P$. Since $\mathcal{G}$ is an l-group $a(b \vee d) \wedge c(b \vee d) = (a \wedge c) \cdot (b \vee d)$; then $(a \wedge c) \cdot (b \vee d) \in P$. As in an l-group it also holds that $(a \wedge c)b \vee (a \wedge c)d = (a \wedge c)b \vee (a \wedge c)d$, we obtain that $(a \wedge c)b \vee (a \wedge c)d \in P$. Since P is prime, $(a \wedge c)b \in P$ or $(a \wedge c)d \in P$. If the first case holds, as $a \wedge c \leq c$, we obtain $cb \in P$. Since $\mathcal{G}$ is a subgroup of $\mathcal{A}$ and c, $b \in G$, $cb \in P \cap G$. If the second case holds, as $a \wedge c \leq a$, we obtain in a similar way that $ad \in P \cap G$. Then $P \cap G$ is primal.

Now let a, $b \in G$ such that $a \not\leq b$. Since G is an ordered subgroup of the l-group $\mathcal{A}$, we know, by the Prime Lattice Filter Theorem, that there is a prime lattice filter P of $\mathcal{A}$ such that $a \in P$ and $b \notin P$. Then, $P \cap G$ is a primal increasing subset of G such that $a \in P \cap G$ and $b \notin P \cap G$. Thus, i) is satisfied. □

Let's show now that Separation Hypothesis can be rephrased in terms of subsemigroups of G

Definition 3.1 *We shall say that a subsemigroup S of G is saturated iff for all $x \in G$, $x \in S$ or $x^{-1} \in S$. Note that saturated semigroups contains the unit e.*

Theorem 3.2 *Let G be an ordered group. Then the following are equivalent:*

i) $S(G)$ satisfies the Separation Hypothesis.

ii) If $a \not\leq e$, there is a decreasing and saturated subsemigroup S such that $a \notin S$.

Proof. Suppose first that $S(G)$ satisfies the Separation Hypothesis and let $a \in G$ be such that $a \not\leq e$. Then, there is $P \in S(G)$ such that $a \in P$ and $e \notin P$. Thus, the family $\mathcal{P}$=$\{P \in S(G) : a \in P \text{ and } e \notin P\}$ is non void. It is easy to verify that the union of an ascendent chain of primal increasing subsets of G is again a primal increasing subset of G, and from this, it's inmediate that also $\mathcal{P}$ satisfies the ascendent chain condition. Then, there exists, by Zorn's Lemma, a maximal element $Q \in \mathcal{P}$. Let's prove that Q^c is a decreasing and saturated semigroup of G. Since Q is an increasing subset, it's inmediate that Q^c is decreasing. To prove that Q^c is multiplicative, let's show first that Q satisfies: $(*)$ $x \in Q$ and $y \notin Q$ imply $y^{-1}x \in Q$.

Assume for a moment that there exist $x \in Q$ and $y \notin Q$ such that $y^{-1}x \notin Q$. Then, $y \in Q_x$. From Lemma 2.1 iii), $Q_x \subseteq Q$ or $Q \subseteq Q_x$. Since $y \in Q_x$ and $y \notin Q$, $Q \subsetneq Q_x$.

Since $x \in Q$, $e \notin Q_x$. Then, $Q_x \in \mathcal{P}$ and $Q \subset Q_x$, which contradicts the maximality of Q. Then, $(*)$ is proved.

Now, let x, $y \in Q^c$ and suppose that $xy \in Q$. From $(*)$ above, we would have $x^{-1}(xy) = y \in P$, which is a contradiction. Thus, if x, $y \in Q^c$, $xy \in Q^c$. We have proved that Q^c is a subsemigroup of G.

Let's prove finally that Q^c is saturated. Let $x \in G$ and suppose that $x \notin Q^c$ and $x^{-1} \notin Q^c$. Then, $x \in Q$ and $x^{-1} \in Q$. Since Q is primal, from $xe \in Q$ and $ex^{-1} \in Q$, we derive $e \in Q$, which is a contradiction. Then, for all $x \in G$, $x \in Q^c$ or $x^{-1} \in Q^c$.

To prove the converse let's show first that if S is a decreasing and saturated subsemigroup of G, S^c is a primal and increasing subset of G: in fact, suppose that there exist a, b, c, $d \in G$ such that $ab \in S^c$, $cd \in S^c$ but $ad \notin S^c$ and $cb \notin S^c$. Then, $ad \in S$ and $cb \in S$. Note now that $d^{-1}b \notin S$ (If $d^{-1}b \in S$, $(ad)(d^{-1}b) = ab \in S$). Also, $b^{-1}d \notin S$ (If $b^{-1}d \in S$, $(cb)(b^{-1}d = cd \in S)$. Then $(d^{-1}b) \notin S$ and $(d^{-1}b)^{-1} \notin S$, which is a contradiction with the assumption that S is saturated. Then, S^c is a primal subset of G. Since S is decreasing, $S^c \in S(G)$. Then, S^c is a primal subset of G. Since S is decreasing, $S^c \in S(G)$. Then, from ii), we have for $a \not\leq e$ that there is $P \in S(G)$ such that $a \in P$ and $e \notin P$. To prove the separation hypothesis let now a, $b \in G$ be such that $a \not\leq b$. Then, $b^{-1}a \not\leq e$ and we can take $P \in S(G)$ such that $b^{-1}a \in P$ and $e \notin P$. Recall, from Lemma 3.ii) that $P_a \in S(G)$. Since $e \notin P$, $a \in P_a$; since $b^{-1}a \in P$, $b \notin P_a$; thus, the separation hypothesis is satisfied. □

From this, Theorem 3.1 can be reformulated as follows:

Theorem 3.3 *Let $\mathcal{G} = \langle G, \leq, ^{-1}, \cdot, e\rangle$ be an ordered group and let $S(G)$ be the primal spectrum of $\mathcal{G}$. Then the following are equivalent:*

i) If $a \not\leq e$, there is a decreasing and saturated subsemigroup S such that $a \notin S$.

ii) $\mathcal{G}$ is embeddable as an ordered subgroup of l-group of sets, whose underlying lattice operations are the usual union and intersection of sets.

iii) $\mathcal{G}$ is embeddable as an ordered subgroup of an l-group.

From this we can reobtain a well known embedding theorem for abelian ordered groups ([1], Corollaire 4.5.6) in the following version:

Theorem 3.4 (Embedding theorem for abelian ordered groups). *Let $\mathcal{G} = \langle G, \leq, ^{-1}, \cdot, e\rangle$ be an abelian ordered group. Then the following are equivalent:*

i) $\mathcal{G}$ is unperforated (i, e., if $a^n \leq e$ for some $n \in N$, $a \leq e$)

ii) $\mathcal{G}$ is embeddable as an ordered subgroup of an l-group of sets, whose underlying lattice operations are the usual union and intersection of sets.

iii) $\mathcal{G}$ is embeddable as an ordered subgroup of an l-group.

Proof. Let $\mathcal{G}$ be an ordered abelian unperforated group and let $a \in G$ such that $a \not\leq e$. From Theorem 3.3, to prove ii) it enoughs to show that there exists a decreasing and saturated subsemigroup $\mathcal{S}$ such that $a \notin \mathcal{S}$. Let's consider the family $\mathcal{S} = \{\mathcal{S} \subseteq \mathcal{G} : \mathcal{S}$

is a decreasing subsemigroup and for all $n \geq 1$, $a^n \notin S\}$. Since G is unperforated and $a \not\leq e$, the cone $(e] = \{x \in G : x \leq e\}$ is a member of $\mathcal{S}$; then $\mathcal{S}$ is non void. It is easy to verify that $\mathcal{S}$ satisfies the ascendent chain condition. Then, by Zorn's Lemma, there is a maximal decreasing subsemigroup S_M such that for all $n \in N$, $a^n \notin S_M$. Let's prove that S_M is saturated: suppose, on the contrary, that there is some $x \in G$ such that $x \notin S_M$ and $x^{-1} \notin S_M$. Since G is abelian, note that $(x, S_M] = (\{x^n y : n \in N$ and $y \in S_M\}] = \{z \in G$: for some $y \in S_M$, and for some $n \in N$, $z \leq x^n y\}$ is a decreasing subsemigroup of G: it's inmediate, from the definition, that $(x, S_M]$ is a decreasing subset of G; also, if z_1, $z_2 \in (x, S_M] = (\{x^n y : n \in N$ and $y \in S_M\}]$, let y_1, $y_2 \in S_M$, and n, $m \in N$ such that $z_1 \leq x^n y_1$ and $z_2 \leq x^m y_2$. Then $z_1 z_2 \leq (x^n y_1)(x^m y_2)$ and, from conmutativity, $z_1 z_2 \leq x^{n+m}(y_1 y_2)$. Since S_M is a subsemigroup, $y_1 y_2 \in S_M$. Thus, $z_1 z_2 \in (\{x^n \cdot y : n \in N$ and $y \in S_M\}]$. We have proved that $(x, S_M]$ is a decreasing subsemigroup of G. Since $e \in S_M$, $x \in (x, S_M]$; since $y = x^0 y$, $S_M \subseteq (x, S_M]$. Then, $S_M \subset (x, S_M]$. In a similar way, $S_M \subset (x^{-1}, S_M]$. Since S_M is a maximal element of $\mathcal{S}$, for some n, $m \geq 1$, $a^n \in (x, S_M]$ and $a^m \in (x^{-1}, S_M]$. Now, let r, $t \in N$ and y_1, $y_2 \in S_M$ be such that $a^n \leq x^r y_1$ and $a^m \leq (x^{-1})^t y_2$. Let $R = \{p \in N$: for some $n \geq 1$ and for some $y \in S_M$, $a^n \leq x^P y\}$. Since $r \in R$, $R \neq \emptyset$. Let's denote by u the least element of R. In a similar way, let $T = \{q \in N$: for some $n \geq 1$ and for some $z \in S_M$, $a^m \leq (x^{-1})^t z\}$. Since $t \in T$, $T \neq \emptyset$. Let's denote by v the least element of T. Now, choose n, $m \geq 1$, and y, $z \in S_M$ such that $a^n \leq x^u y$ and $a^m \leq (x^{-1})^v z$. Then, $a^{n+m} \leq x^{u-v} yz$. Now, if $u > v$, since $u - v < u$ and $yz \in S_M$, we would have a contradiction with the choice of u. Similarly, if $u < v$, since $v - u < v$, we have a contradiction with the choice of v because $a^{n+m} \leq (x^{-1})^{v-u} yz$. Finally, if $u = v$, we obtain $a^{n+m} \leq yz$. Since $yz \in S_M$, which is a decreasing subset, we derive $a^{n+m} \in S_M$, again a contradiction. Then S_M is saturated. Since $a \notin S_M$, our proof is ended. □

REMARK: Notice that the results we have collected in this paper paves us the way toward topological dualities for ordered groups with the separation hypothesis in the line of the Stone duality for Boolean algebras, of the Priestley for distributive lattices, and of our recently developed duality for l-groups. As we have pointed, the main difference with those cases is the lacking of an underlying distributive lattice for the algebraic structure. A second difficulty that arises (the last obstacle to overcome in order to obtain such duality) is to find an appropriated topological characterization of the sets $\sigma(a)$. That will be the excuse for our next research.

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TOPICOS SOBRE ALGEBRAS MODALES 4-VALUADAS

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Resumen. En este artículo recopilamos algunos resultados sobre las álgebras modales 4-valuadas y de otras clases de álgebras relacionadas con ellas.

En 1979 A. Monteiro definió las álgebras modales 4-valuadas como una generalización de las álgebras de Lukasiewicz 3-valuadas.

Este autor conjeturó que estas álgebras serían contrapartidas algebraicas de cálculos proposicionales modales definidos sintacticamente. La existencia de tales cálculos fue probada por J. M. Font y por A. V. Figallo conjuntamente con A. Ziliani.

El estudio de la variedad de las álgebras modales 4-valuadas fue comenzado por I. Loureiro, durante la última estadía en Portugal del Profesor A. Monteiro.

1. *La Variedad M_4 de la álgebras modales 4 - valuadas*

Las álgebras de Lukasiewicz trivalentes fueron introducidas por G.C. Moisil en 1941 (ver[21]) y A. Monteiro en 1964 mostró que pueden ser definidas como álgebras $(A, \wedge, \vee, \sim, \nabla, 1)$ de tipo (2,2,1,1,0) que satisfacen las identidades:

M0) $x \vee 1 = 1$,

M1) $x \wedge (x \vee y) = x$,

M2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,

M3) $\sim \sim x = x$,

M4) $\sim (x \vee y) = \sim x \wedge \sim y$,

M5) $\sim x \vee \nabla x = 1$,

M6) $\nabla x \wedge \sim x = \sim x \wedge x$,

M7) $\nabla (x \wedge y) = \nabla x \wedge \nabla y$, [23].

En 1966 L.Monteiro [26] demostró que M0 es consecuencia de M1,...,M7 y que estos son independientes. Para exhibir la independecia de M7 consideró el álgebra $\mathcal{M}_4 = (\mathbb{M}_4, \wedge, \vee, \sim, \nabla, 1)$, donde $\mathbb{M}_4 = \{0, a, b, 1\}$, y las operaciones están dadas por las tablas

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

x	$\sim x$	∇x
0	1	0
a	a	1
b	b	1
1	0	1

A. Monteiro, según me lo manifestó verbalmente, motivado en el ejemplo anterior consideró la variedad $\boldsymbol{M_4}$ de álgebras generada por las identidades M1,...,M6, que llamó álgebras modales tetravalentes (o 4-valuadas), y en 1978 durante su última estadía en Portugal le sugirió a I. Loureiro estudiar la variedad $\boldsymbol{M_4}$.

Los resultados básicos sobre $\boldsymbol{M_4}$ pueden ser consultados en [4,5,6,7,8,14,15,16,16,17, 18,19].

1.1. $\boldsymbol{M_4}$ - *Congruencias*

Con $Con(A)$ y $\mathfrak{P}(A)$ designaremos los conjuntos de las congruencias y las partes de A respectivamente.

Para cada $A \in \boldsymbol{M_4}$ la involución de Birula-Rasiowa $\phi : \mathfrak{P}(A) \to \mathfrak{P}(A)$ está definida por $\phi(X) = A - n(X)$, donde $-$ es la diferencia de conjuntos y $n(X) = \{\sim x : x \in P\}$.

Sea π una familia de filtros primos tal que si $P \in \pi$ entonces $\phi(P) \in \pi$ y sea $R_\pi \subseteq A^2$ la relación binaria definida sobre A tal que $(a,b) \in R_\pi$ si y solo si se verifican las dos condiciones siguientes:

R1) $P \in \pi$ y $a \in P$ implícan $b \in P$,

R2) $P \in \pi$ y $b \in P$ implícan $a \in P$.

Entonces $R_\pi \in Con(A)$ y de esta manera se pueden obtener todas las congruencias de A [19]. Nosotros, en [4] hemos descripto a las congruencias de A de manera más explícita.

En efecto, si para toda $A \in \boldsymbol{M_4}$ consideramos la operación $\to$ por medio de la fómula

$$x \to y = \nabla \sim x \vee y \text{ [14] (implicación débil)},$$

y llamamos sistemas deductivos (s.d.) a los subconjuntos de A que cumplen

(1) $1 \in D$,

(2) si $a, a \to b \in D$ entonces $b \in D$,

tenemos que

$$Con(A) = \{R(D) : D \in \mathbb{D}(A)\},$$

donde $\mathbb{D}(A)$ es el conjunto de todos los s.d. de A, y

$$R(D) = \{(a,b) : a \to b \ , \ b \to a \ , \sim (a \wedge b) \to \sim b \ , \sim (a \wedge b) \to \sim a \in D\}.$$

1.2. M_4 - *Algebras simples*

Sea $A \in \boldsymbol{M_4}$ con más de un elemento y $\mathbb{P}(A)$ la familia de todos los filtros primos de A. Si $P \in \mathbb{P}(A)$ tenemos que $\pi = \{P, \phi(P)\}$ es cerrada por ϕ y la M_4 - álgebra cociente A/R_π es simple . Más aún A/R_π es isomorfa a una subálgebra no degenerada de $\mathcal{M}_4$.

En [19] se probó que existe un conjunto no vacío X y un M_4 - homomorfismo inyectivo $\alpha : A \rightarrow \mathbb{M}_4^X$.

Por otra parte el operador $\rightarrow$ verifica las siguientes propiedades:

C1) $x \rightarrow (y \rightarrow x) = 1,$

C2) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1,$

C3) $1 \rightarrow x = x,$

C4) $((x \rightarrow y) \rightarrow x) \rightarrow x = 1.$

De C1,...,C4 y un resultado de A. Monteiro [25] tenemos que $\boldsymbol{M_4}$ es deductivamente semisimple, esto es, $\{1\} = \bigcap \{M : M \in E(A)\}$, donde $E(A)$ es el conjunto de todos los s.d. maximales de A.

De este último hecho y teniendo en cuenta nuestra descripcción de $Con(A)$ se puede obtener una demostración diferente a la indicada por I. Loureiro de que $\boldsymbol{M_4}$ es una variedad algebraicamente semisimple.

1.3. *Un cálculo proposicional modal 4 - valuado*

J. M. Font en [10] ha desarrollado un cálculo proposicional en términos de los conectivos $\wedge, \vee, \sim, \Box$ y la constante, utilizando los siguientes esquemas y reglas

$\alpha \vdash \alpha,$

$\alpha \vdash \Box\alpha,$

$\vdash \alpha \vee \sim\Box\alpha,$

$$\frac{\Gamma \vdash \alpha}{\Gamma, \beta \vdash \alpha},$$

$$\frac{\Gamma \vdash \alpha \quad \Gamma, \alpha \vdash \beta}{\Gamma \vdash \beta},$$

$$\frac{\Gamma, \alpha, \beta \vdash \gamma}{\Gamma, \alpha \wedge \beta \vdash \gamma},$$

$$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta},$$

$$\frac{\Gamma, \alpha \vdash \gamma \quad \Gamma, \beta \vdash \gamma}{\Gamma, \alpha \vee \beta \vdash \gamma},$$

$$\frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta},$$

$$\frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta},$$

$$\frac{\alpha \vdash \beta}{\sim \beta \vdash \sim \alpha},$$

$$\frac{\Gamma, \alpha \vdash \beta}{\Gamma, \sim \sim \alpha \vdash \beta},$$

$$\frac{\Gamma \vdash \alpha}{\Gamma \vdash \sim \sim \alpha},$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \alpha},$$

$$\frac{\Gamma, \alpha, \sim \alpha \vdash \beta}{\Gamma, \alpha, \sim \Box\alpha \vdash \beta},$$

$$\frac{\Gamma \vdash \alpha \wedge \sim \alpha}{\Gamma \vdash \alpha \wedge \sim \Box\alpha},$$

donde Γ es un conjunto cualquiera de fómulas y $\Gamma \vdash \alpha$ si y solo si existe un conjunto finito $\Gamma_o \subseteq \Gamma$ tal que $\Gamma_o \vdash \alpha$, siendo $\vdash$ la operación usual de derivabilidad. Este autor ha considerado la matriz $S(\mathcal{M}_4^*) = \langle \mathcal{M}_4^*, \mathfrak{D} \rangle$ tal que:

i) $\mathcal{M}_4^* = \langle \mathbb{M}_4 , \wedge, \vee, \sim, \triangle . 1 \rangle$,

ii) $\mathfrak{D} = \{1\}$ es el conjunto de los valores designados,

iii) $\triangle$ está definido por la fómula $\triangle x = \sim \nabla \sim x$,

y ha demostrado que las siguientes condiciones son equivalentes:

$\Gamma \vdash \alpha$,

$\Gamma \models \alpha$,

siendo $\models$ la noción de consecuencia semántica sobre $S(\mathcal{M}_4^*)$.

2. *Otras clases de álgebras relacionadas con las álgebras modales 4 – valuadas*

2.1. *Los n – Reticulados Generalizados*

La noción de n – Reticulado introducida por H. Rasiowa en [28] (ver también [1,2,3,22, 30]) corresponde a una contrapartida algebraica de la lógica constructiva con negación fuerte considerada por D. Nelson en [27] y A. Markov en [20] .

A. Monteiro y D. Brignole [1,2] han caracterizado a los n – Reticulados como álgebras $(A, \wedge, \vee, \rightarrow, \sim, 1)$ de tipo (2,2,2,1,0) tales que el reducto $(A, \wedge, \vee, 1)$ es un reticulado distributivo con último elemento 1 que satisfacen los axiomas adicionales

N1) $\sim\sim x = x,$

N2) $\sim(x \wedge y) = \sim x \vee \sim y,$

N3) $(\sim x \wedge x) \wedge (\sim y \vee y) = \sim x \wedge x,$

N4) $x \rightarrow x = 1,$

N5) $(x \rightarrow y) \wedge (\sim x \vee y) = \sim x \vee y,$

N6) $x \wedge (x \rightarrow y) = x \wedge (\sim x \vee y),$

N7) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z),$

N8) $x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z.$

Teniendo en cuenta N1, N2 y N3 podemos decir que el reducto $(A, \wedge, \vee, \sim, 1)$ es un álgebra de de Morgan que satisface la condición de linealidad N3, es decir son álgebras de Kleene.

Por otra parte A. Monteiro determinó la subvariedad de los n – reticulados semisimples y en [24] probó que estas álgebras *coinciden* con las álgebras de Lukasiewicz trivalentes.

Nosotros hemos generalizado este resultado, para lo cual hemos definido la variedad de los n – Reticulados generalizados (o NRG – álgebras) como álgebras $(A, \wedge, \vee, \rightarrow, \sim, 1)$ de tipo (2,2,2,1,0) tales que el reducto $(A, \wedge, \vee, 1)$ es un reticulado distributivo con último elemento 1 que satisfacen los axiomas adicionales N1, N2, N4,...,N8 (ver [5]), y hemos probado que la variedad de las NRG – álgebras deductivamente semisimples coincide con las variedad de las álgebras modales 4-valuadas.

Más precisamente.

Diremos que $(A, \wedge, \vee, \rightarrow, \sim, 1)$ es una NRG – álgebra deductivamente semisimple si satisface la propiedad:

S) $(x \rightarrow y) \rightarrow x = x.$

Entonces, si A es una NRG – álgebra deductivamente semisimple y ponemos

$$\nabla x = \sim x \rightarrow \sim 1,$$

tenemos que $(A, \wedge, \vee, \sim, \nabla, 1)$ es un álgebra modal 4 – valuada y se verifica

$x \to y = \nabla \sim x \vee y.$

Reciprocamente, si $(A, \wedge, \vee, \sim, \nabla, 1)$ es un álgebra modal 4 – valuada y ponemos

$x \to y = \nabla \sim x \vee y,$

tenemos que $(A, \wedge, \vee, \to, \sim \vee, 1)$ es una NRG-álgebra deductivamente semisimple y se verifica

$\nabla x = \sim x \to \sim 1.$

En [9] hemos desarollado un cálculo proposicional $\mathcal{L}$ en términos de los conectivos $\to, \wedge, \sim$ la regla de modus ponens y ciertos esquemas.

En este cálculo se tiene que la congruencia entre fórmulas queda determinada por el conjunto T de los teoremas del cálculo del siguiente modo:

$$\alpha \approx \beta \text{ si y solo si } \alpha \to \beta,\ \beta \to \alpha,\ \sim(\alpha \wedge \beta) \to \sim \beta,\ \sim(\alpha \wedge \beta) \to \sim \alpha \in T,$$

El álgebra de Lindenbaum $\mathcal{L}/\approx$ es una NRG – álgebra deductivamente semisimple.

2.2. *Reticulados con negaciones y p – álgebras de Morgan modales*

En 1969 J. Varlet [29] caracterizó de una manera muy elegante a las álgebras de Lukasiewicz trivalentes como álgebras $(A, \wedge, \vee, *, +, 0, 1)$ de tipo $(2,2,1,1,0,0)$ donde $(A, \wedge, \vee, 0, 1)$ es un reticulado distributivo con primer elemento 0 y último elemento 1 y se satisfacen las siguientes propiedades adicionales:

V1) $x \wedge x^* = 0.$

V2) $(x \wedge y)^* = x^* \vee y^*,$

V3) $0^* = 1,$

V4) $x \vee x^+ = 1,$

V5) $(x \vee y)^+ = x^+ \wedge y^+,$

V6) $1^+ = 0,$

V7) $x^* = y^*$ y $x^+ = y^+$ implican $x = y.$

Las operaciones $\sim$ y ∇ quedan determinadas por las fórmulas

$\sim x = (x \vee x^*) \wedge x^+,$

$\nabla x = x^{**}.$

Esto es, J. Varlet caracterizó a la álgebras de Lukasiewicz 3-valuadas como álgebras de Stone dobles que satisfacen V7. Por otra parte V7 puede ser reeplazado por la identidad:

V8) $(x \wedge x^{+}) \wedge (y \vee y^{*}) = x \wedge x^{+}$.

Tratando de lograr un resultado análogo para el caso de las álgebras modales 4-valuadas en [7] hemos considerado la clase de los reticulados con negaciones como álgebras $(A, \wedge, \vee, *, +, 0, 1)$ de tipo $(2,2,1,1,0,0)$ donde $(A, \wedge, \vee, 0, 1)$ es un (0,1)-reticulado distributivo y se satisfacen las propiedades adicionales

B1) $x \wedge x^{*} = 0$,

B2) $x \vee x^{+} = 1$,

B3) $x^{*} \wedge x^{*+} = 0$,

B4) $x^{+} \vee x^{+*} = 1$,

B5) $(x \wedge y)^{+} = x^{+} \vee y^{+}$,

B6) $(x \vee y^{+})^{+} = x^{+} \wedge y^{++}$,

B7) $(x \vee y)^{*} = x^{*} \wedge y^{*}$,

B8) $(x \wedge y^{*})^{*} = x^{*} \vee y^{**}$,

y hemos probado que si en un reticulado con negaciones $(A, \wedge, \vee, *, +, 0, 1)$ ponemos

(i) $\sim x = (x \vee x^{*}) \wedge x^{+}$,

(ii) $\nabla x = x^{**}$,

tenemos que $(A, \wedge, \vee, \sim, \nabla, 1)$ es una M_4-álgebra si y solo si se verifican

B9) $(x \vee y) \wedge (x \vee y^{+}) \leq x \vee x^{*}$,

B10) $x \wedge x^{+} \wedge y \wedge y^{+} \leq (x \vee y)^{+}$.

Observemos que (i) e (ii) son las indicadas por Varlet para el caso de las álgebras de Lukasiewicz trivalentes.

Si consideramos la variedad $\mathfrak{P}$ de las álgebras $(A, \wedge, \vee, \sim, *, 1)$ de tipo $(2,2,1,1,0)$ llamadas álgebras de de Morgan seudocomplementadas (o p-álgebras), entonces ella contiene una subvariedad $\mathfrak{M}$ que hemos llamado p-álgebras modales [7] y son aquellas álgebras de de Morgan que satisfacen la propiedad

M) $x \vee \sim x \leq x \vee x^{*}$.

$\mathfrak{M}$ es una subvariedad propia de $\mathfrak{P}$ tal que, si para cada $(A, \wedge, \vee, \sim, *, 1) \in \mathfrak{M}$ ponemos

$\nabla x = \sim(\sim x \wedge x^{*})$

tenemos que $(A, \wedge, \vee, \sim, \nabla, 1) \in \boldsymbol{M_4}$.

Para demostrar la última afirmación, previamente hemos caracterizado las M_4-álgebras como álgebras $(A, \wedge, \vee, \sim, *, 1)$ de tipo $(2,2,1,1,0)$ tales que el reducto $(A, \wedge, \vee, \sim, 1)$ es un álgebra de de Morgan y se verifican las identidades

T1) $x \wedge x^* = 0$,

T2) $x \vee x^* = x \vee \sim x$.

En este caso la operación ∇ se recupera por la fórmula

$$\nabla x = \sim x^*.$$

3. *Las I - álgebras generalizadas*

Recordemos que otra contrapartida algebraica de la lógica trivalente de lukasiewicz, en términos de implicación y negación, la constituyen las álgebras de Wajsberg trivalentes. Las identidades que relacionan a las álgebras de Lukasiewicz trivalentes con esta clase de álgebras son

A) $x \vee y = (x \succ y) \succ y$,

B) $\sim x = x \succ 0$,

C) $\nabla x = \sim x \succ x$,

D) $x \wedge y = \sim(\sim x \vee \sim y)$.

En [6] hemos generalizado este resultado para el caso de las álgebras modales 4-valuadas. En efecto para toda $A \in \boldsymbol{M_4}$ además de la operación $\rightarrow$ ya considerada se pueden definir otros operadores de implicación del siguiente modo:

(1) $x \supset y = \sim x \vee y$ [11],

(2) $x \mapsto y = (x \rightarrow y) \wedge (x \supset \nabla y)$,

(3) $x \succ y = (x \mapsto y) \wedge ((x \supset y) \rightarrow (\Delta \sim x \vee y))$.

Entonces $\vee, \sim, \nabla$ y $\wedge$ verifican las identidades A), B), C) y D).

Además sobre $\mathbb{M}_4$, la operación $\succ$ tiene la tabla siguiente

$\succ$	0	a	b	1
0	1	1	1	1
a	a	1	b	1
b	b	a	1	1
1	0	a	b	1

El resultado anterior conduce al problema de caracterizar a las álgebras modales $4-$valuadas en términos de las operaciones $\succ$ y $\sim$.

Nosotros en [6] hemos resuelto este problema, lo que nos ha llevado a definir nuevas clases de álgebras.

Un álgebra $(A, \succ, 1)$ de tipo $(2,0)$ es una $I-$álgebra generalizada (o $G-$álgebra) si se satisfacen las propiedades:

G1) $1 \succ x = x$,

G2) $x \succ 1 = 1$,

G3) $(x \succ y) \succ y = (y \succ x) \succ x$,

G4) $x \succ (y \succ z) = 1$ implica $y \succ (x \succ z) = 1$.

Representaremos con $\boldsymbol{G}$ a esta clase de álgebras.

A continuación vamos a indicar dos ejemplos importantes de $G-$álgebras que justifican el nombre que hemos elegido para designarlas.

En primer lugar recordemos que una contrapartida algebraica del cálculo implicativo de Lukasiewicz infinito valuado lo constituyen las $I-$álgebras, las cuales pueden ser descriptas como álgebras $(A, \succ, 1)$ de tipo $(2,0)$ que satisfacen las identidades

$$1 \succ x = x\ ,$$
$$(x \succ y) \succ ((y \succ z) \succ (x \succ z)) = 1,$$
$$(x \succ y) \succ y = (y \succ x) \succ x,$$
$$((x \succ y) \succ (y \succ x)) \succ (y \succ x) = 1.$$ (ver[13])

Entonces tenemos que toda $I-$álgebra es una $G-$álgebra.

Más generalmente si $\boldsymbol{BCK}$ es la clase de las $BCK-$álgebras de Iseki [12] y $\boldsymbol{CBCK}$ es la subclase de las $BCK-$álgebras conmutativas de Tanaka [13] y $(A,*,0) \in \boldsymbol{CBCK}$, poniendo $x \succ y = y*x$ para cada $x, y \in A$ tenemos que $(A, \succ, 0) \in \boldsymbol{G}$.

Finalmente tenemos que $(\mathbb{M}_4, \succ, 1)$ es una $G-$álgebra que no es una $I-$álgebra.

Sea $(A, \succ, 1) \in \boldsymbol{G}$, entonces

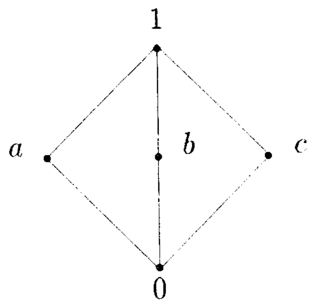
G5) la relación $\leq$ definida por $x \leq y$ si $x \succ y = 1$ es una relación de orden sobre A para la cual 1 es el último elemento. Más aún $(A, \leq)$ es un conjunto reticulado superior, donde el supremo de $x, y \in A$ es el elemento $x \vee y = (x \succ y) \succ y$.

G6) Si A tiene primer elemento 0 , entonces es un conjunto reticulado inferior, donde el ínfimo de $x, y \in A$ es el elemento $x \wedge y = \sim(\sim x \vee \sim y)$, siendo $\sim x = x \succ 0$.

Consideremos ahora la G – álgebra $(A, \succ, 1)$ con $A = \{0, a, b, c, 1\}$ y $\succ$ definida por

$\succ$	0	a	b	c	1
0	1	1	1	1	1
a	a	1	b	c	1
b	b	a	1	c	1
c	c	a	b	1	1
1	0	a	b	c	1

Entonces $(A, \leq)$ tiene el siguiente diagrama de Hasse



y es claro que no es distributivo

Con $\boldsymbol{G^o}$ designaremos la clase de álgebras $(A, \succ, 0, 1)$ de tipo (2,0,0) tales que $(A, \succ, 1) \in \boldsymbol{G}$ y se satisface la propiedad:

G7) $0 \succ x = 1$.

Para toda $A \in \boldsymbol{G^o}$ se verifican las siguientes propiedades

G8) $\sim \sim x = x$,

G9) $\sim (x \vee y) = \sim x \wedge \sim y$,

G10) $(x \succ z) \vee (y \succ z) \leq (x \wedge y) \succ z$.

La propiedad G10 nos sugiere considerar una clase particular de G^o – álgebras.

Diremos que $A \in \boldsymbol{G^o}$ es distributiva si satisface la identidad:

G11) $(x \succ z) \vee (y \succ z) = (x \wedge y) \succ z$.

Observemos que toda G^o – álgebra distributiva tiene asociada una estructura de

Observemos que toda G° – álgebra distributiva tiene asociada una estructura de álgebra de Morgan.

Indicaremos con $\boldsymbol{DG^{\circ}}$ a la clase de las G° – álgebras distributivas.

Todavía nos hacen falta dos selecciones más para llegar a la clase de álgebras que estamos buscando.

Para toda $A \in \mathbf{G}$ definimos los operadores $\Rightarrow_i$, $i = 0,1,2,\ldots$, del siguiente modo:

$$x \Rightarrow_0 y = y \ ,$$
$$x \Rightarrow_{i+1} y = x \succ (x \Rightarrow_i y).$$

Diremos que $A \in \boldsymbol{G}$ es 3 – valuada (o G_3 – álgebra) si satisface la identidad:

G12) $(x \Rightarrow_2 y) \vee x = 1.$

Indicaremos con $\boldsymbol{DG_3^{\circ}}$ a la clase de las G_3 – álgebras con primer elemento que son distributivas.

Finalmente diremos que $A \in \boldsymbol{DG_3^{\circ}}$ es modal si satisface la identidad:

G13) $(\sim x \succ x) \succ 0 = (\sim x \succ x) \Rightarrow_2 0.$

En [6] hemos probado el siguiente resultado:

Si $A \in \boldsymbol{DG_3^{\circ}}$ es modal, poniedo

$$\sim x = x \succ 0,$$
$$\nabla x = \sim x \succ x,$$
$$x \vee y = (x \succ y) \succ y,$$
$$x \wedge y = \sim(\sim x \vee \sim y)$$

tenemos que $(A, \wedge, \vee, \sim, \nabla, 1)$ es un álgebra modal 4 – valuada donde la operación $\succ$ se puede recuperar por la fórmula (3) vista anteriormente.

Utilizando el teorema de representación de Loureiro se puede demostrar el resultado recíproco.

Conclusión

Tenemos así que:

C1) $1 \succ x = x,$

C2) $x \succ 1 = 1,$

C3) $(x \succ y) \succ y = (y \succ x) \succ x$,

C4) $x \succ (y \succ z) = 1$ implica $y \succ (x \succ z) = 1$,

C5) $((x \succ (x \succ y)) \succ x) \succ x = 1$,

C6) $\sim 1 \succ x = 1$,

C7) $x \succ \sim 1 = \sim x$,

C8) $(\sim(\sim x \vee \sim y) \succ z) \succ ((x \succ z) \vee (y \succ z)) = 1$,

donde $a \vee b$ es una abreviatura de $(a \succ b) \succ b$, es una axiomática para las álgebras modales valuadas en términos de las operaciones $\{\succ, \sim, 1\}$.

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Topics in Vector Logic

Eduardo Mizraji

1 Introduction

The analysis of the cognitive functions of human nervous system uses a large variety of mathematical approaches (Scott 1976, Anderson and Rosenfeld 1989). Recently, a simple model for a context-dependent associative memory has provided us with a vectorial representation of propositional calculus (Mizraji 1989, 1992a). Within this representation the classical truth values map on two vectors, and the logical functions are executed by matrix operators.

In the present work we describe the basic operators and we analyse the extension of binary operations to non-binary input vectors. This extension generates a bidimensional many-valued vector logic with infinite truth values. We analyse the validity of the axioms of Boolean algebras in the framework of this many-valued logic. Then we study the validity of some classical implicational theorems of binary propositional calculus in the many valued vector logic. Finally, we describe two applications: the matrix Fredkin gate and the vector modal operations.

2 Basic vector logic

Suppose that the two classical truth values "true", T, and "false", F, map on two Q-dimensional vectors s and n, respectively. The vector space is defined over the field of real numbers. For simplicity, we assume that these vectors are orthonormal. Hence, $\langle s,s\rangle = \langle n,n\rangle = 1$, and $\langle s,n\rangle = \langle n,s\rangle = 0$, where $\langle u,v\rangle = u^T v$ is the scalar product between vectors u and v; w^T is the transpose of matrix w.

Starting from the set $\mathcal{C} = \{s,n\}$ we define the set $\mathcal{L} = \{\Phi s + (1-\Phi)n : \Phi \in [0,1]\}$; we name "fuzzy vectors" the vectors $u \in \mathcal{L}$.

Let us define now the following square matrices:
$I = ss^T + nn^T$, $K = ss^T + sn^T$, $M = ns^T + nn^T$, $N = ns^T + sn^T$

Notice that matrix I acts like a unitary matrix on the subspace generated by $\{s,n\}$. On the other hand, matrix N satisfies $Ns = n$ and $Nn = s$, with $NN = I$. This matrix N verifies the same formal properties of classical negation: $\neg T = F$, $\neg F = T$ and $\neg\neg T = T$. Please also note that:

$$KI = KK = KM = KN = K,$$

$$MI = MK = MM = MN = M,$$

$$NK = M, \qquad NM = K.$$

When these matrices operate on vectors s, n we have:

$Ks = Kn = s$, $Ms = Mn = n$, $s^T K = n^T m = s^T + n^T$, $n^T K = s^T m = 0$.

This set of square matrices allows a compact definition of operators that reproduce the logical properties of the classical binary functions of propositional calculus. The resulting binary matrix operators act over the Kronecker product between input vectors, and when inputs belong to the set $\{s, n\}$, these operations have formal properties identical to the classical ones.

The Kronecker product between two arbitrary matrices $A = [a_{ij}]$ of order $p \times q$ and $B = [b_{ij}]$ of order $p' \times q'$ is defined by the following expression (Bellman 1962, Graham 1981)

$$A \otimes B \equiv [a_{ij} B]$$

that is a matrix of order $(pp') \times (qq')$.

Let p and q be two arbitrary propositions. In classical logic the conjunction $p \wedge q$ satisfies the following identities between truth values: $(T \wedge T) = T$, $(T \wedge F) = (F \wedge T) = (F \wedge F) = F$. On the other hand, the disjunction $p \vee q$ is characterized by the following truth table: $(T \vee T) = (T \vee F) = (F \vee T) = T$, $(F \vee F) = F$. In the following, we describe the vectorial versions of these operations.

In the system $\{s, n\}$ the conjunction is executed by the matrix $C = I \otimes s^T + M \otimes n^T$.

Note that $C(s \otimes s) = s$ and $C(s \otimes n) = C(n \otimes s) = C(n \otimes n) = n$.

The disjunction is operated by the matrix $D = K \otimes s^T + I \otimes n^T$, with $D(s \otimes s) = D(s \otimes n) = D(n \otimes s) = s$ and $D(n \otimes n) = n$.

From matrices I, N, C and D it is possible to construct new matrix operators capable of performing other logical functions as the implication, the equivalence or the Sheffer connective (Mizraji 1992b).

3 Vector logic and Boolean algebras.

A Boolean algebra over a set B is defined by $\mathbf{B} = \langle B, +, \cdot, ' \rangle$ where the set B has at least two bounds, 0 and 1; $+$ and $\cdot$ are binary operations on B, and $'$ is a unitary operation on B. The basic properties of these algebras are as follows:

b1) Idempotence:	$a + a = a$, $a \cdot a = a$
b2) Commutativity:	$a + b = b + a$, $a \cdot b = b \cdot a$
b3) Associativity:	$(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
b4) Absortion:	$a + (a \cdot b) = a$, $a \cdot (a + b) = a$
b5) Distributivity:	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, $a + (b \cdot c) = (a + b) \cdot (a + c)$
b6) Universal bounds:	$a + 0 = a$, $a + 1 = 1$; $a \cdot 1 = a$, $a \cdot 0 = 0$
b7) Complementarity:	$a + a' = 1$, $a \cdot a' = 0$, $1' = 0$
b8) Involution:	$a'' = a$
b9) Dualization:	$(a + b)' = a' \cdot b'$, $(a \cdot b)' = a' + b'$

The propositional operations of conjunction $\wedge$, disjunction $\vee$ and negation $\neg$, constitute a Boolean algebra.

We now define the following structures:

$$\text{Bin} = \langle \mathcal{C}, (C, \otimes), (D, \otimes), N \rangle,$$

$$\text{Pol} = \langle \mathcal{L}, (C, \otimes), (D, \otimes), N \rangle.$$

It is possible to demonstrate that

a) the binary vector logic **Bin** is a Boolean algebra;

b) the many-valued vector logic **Pol** is not a Boolean algebra: only properties b2, b3, b6, b8 and b9 are true.

The commutativity is valid por any pair u, $v \in \mathcal{L}$: $C(u \otimes v) = C(v \otimes u)$, $D(u \otimes v) = D(v \otimes u)$.

The associativity of the operations performed by C and D implies $C(I \otimes C) = C(C \otimes I)$, $D(I \otimes D) = D(D \otimes I)$.

Property b6 has matrix version $D(u \otimes n) = u$, $D(u \otimes s) = s$, $C(u \otimes s) = u$, $C(u \otimes n) = n$.

Property b8 is a consequence of the fact that operator N is involutory for subspace $\mathcal{L}$, with $NNu = u$.

Property b9 generates the matrix version of De Morgan's laws: $C = ND(N \otimes N)$, $D = NC(N \otimes N)$.

4 Implicational theorems

The classical implication, $p \supset q$, is described by the following correspondences: $(T \supset T) = (F \supset T) = (F \supset F) = T$ and $(T \supset F) = F$. A vectorial version of implication is provided by the matrix $L = K \otimes s^T + N \otimes n^T$.

Note that $L(s \otimes s) = L(n \otimes s) = L(n \otimes n) = s$ and $L(s \otimes n) = n$.

We describe now some of the basic implicational theorems of binary propositional calculus (for a complete analysis see Couturat [1905] 1980, Lukasiewicz [1921] 1970). Let the letters p, q and r correspond to arbitrary propositions. They satisfy these identities,

i1) $(p \supset r) \wedge (q \supset r)$ eq. $[(p \vee q) \supset r]$

i2) $(r \supset p) \wedge (r \supset q)$ eq. $[r \supset (p \wedge q)]$

i3) $(p \supset q)$ eq. $[p \equiv (p \wedge q)]$

i4) $(p \supset q)$ eq. $(\neg p \vee q)$

i5) $(p \supset q)$ eq. $(\neg q \supset \neg p)$

i6) $[(p \wedge q) \supset r]$ eq. $[(q \wedge \neg r) \supset \neg p]$

i7) $[(p \wedge q) \supset r]$ eq. $p \supset (q \supset r)]$

i8) $(p \supset q)$ eq. $[p \supset (p \supset r)]$

We now redefine our binary many-valued vector logics as

$$\text{Bin} = \langle \mathcal{C}, (L, \otimes), N \rangle$$

$$\text{Pol} = \langle \mathcal{L}, ((L, \otimes), N \rangle.$$

In what follows we will determine which of these eight theorems remain valid in the framework of binary and many-valued vector logics **Bin** and **Pol**. We use the following representation: $h(1,k)$ and $h(2,4)$ are vertor versions that correspond to the first and second members of the equivalence ik, with k=1, ... , 8. We will verify that:
a) the binary vector login **Bin**, satisfy all the implicational equivalences i1 to i9;
b) in the many-valued vector logic **Pol**, only equivalences i4, i5, i6 and i7 remain valid.

The definition of matrix L assures the satisfaction of these theorems for the vector logic **Bin**. In the case of the vector logic **Pol**, the theorems i1, i2, i3 and 18 are not satisfied because $h(1,j)$ and $h(2,j)$ depend on different vector compositions [eg. $h(1,1)$ depends on the Kronecker product of four vectors and $h(1,2)$ depends on the Kronecker product of three vectors]. The valid equivalences are now proved:

I) $h(1,4) = L(u \otimes v)$ and $h(2,4) = D(Nu \otimes v)$. From the properties of the Kronecker product $h(2,4) = D(N \otimes I)(u \otimes v)$. But $L = D(N \otimes I)$, thus, $h(1,4) = h(2,4)$.

II) $h(1,5) = L(u \otimes v)$ and $h(2,5) = L(Nv \otimes Nu)$. From $h(2,5)$ and the commutativity of D, we have $h(1,5) = D(N \otimes I)(u \otimes v) = D(Nu \otimes v) = D(v \otimes Nv) = D(NNv \otimes Nu) = D(N \otimes I)(Nv \otimes Nu) = L(Nv \otimes Nu)$. Hence $h(1,5) = h(2,5)$.

III) $h(1,6) = L[C(u \otimes v) \otimes w]$ and $h(2,7) = L[C(v \otimes Nw) \otimes Nu]$. The properties of the Kronecker product implies $h(1,6) = L(C \otimes I)(u \otimes v \otimes w)$. The application of property i5 to $h(2,6)$ gives: $h(2,6) = L[NNu \otimes NC(v \otimes Nw)] = L[u \otimes NC(I \otimes N)(v \otimes w)$. But $NC(I \otimes N) = L$. Hence, $h(2,6) = L(I \otimes L)(u \otimes v \otimes w)$. It can be proved from the basic definitions that $L(C \otimes I) = L(I \otimes L)$. Consequently, $h(1,6) = h(2,6)$.

IV) $h(1,7) = L[C(u \otimes v) \otimes w]$ and $h(2,7) = L[u \otimes L(v \otimes w)]$. Note that this is the same case as i6. Hence, $h(1,7) = h(2,7)$.

Let us point out that a composed logical formula generates in vector logic the same sequence of operators and variables that is obtained writing this formula in Polish notation.

5 Fredkin gate in vector logic.

The Fredkin gate executes a reversible transformation of three binary inputs in three binary outputs. This gate generates, given an appropriate configuration of inputs, the logical operations of negation, conjunction, disjunction and implication (Toffoli 1977, Fredkin y Toffoli 1982). In what follows, we will show the construction of a matrix operator that extends the computational abilities of the Fredkin gate to fuzzy vectors.

We first define a commutation matrix: $R = s \otimes I \otimes s^T + n \otimes I \otimes n^T$.

Note that for any pair of vectors $u, v \in \mathcal{L}$, $R(u \otimes v) = v \otimes u$.

We construct now the following matrix: $F = ss^T \otimes I^{[2]} + nn^T \otimes R$.

The exponent [2] designates the second Kronecker power $I \otimes I$ (Bellman 1961). When the input vectors belong to the set $\{s, n\}$, the matrix F establishes the same input-output relations as the Boolean Fredkin operator. The matrix Fredkin gate is also a reversible operator for vectors belonging to the set $\mathcal{L}$, because $F[F(u \otimes v \otimes w)] = u \otimes v \otimes w$.

Let us define a filter that selects the third output position: $\Phi_3 = (s+n)^T \otimes (s+n)^T \otimes I$. The following results hold:

$$\begin{aligned} Nu &= \Phi_3 F(u \otimes s \otimes n) \\ C(u \otimes v) &= \Phi_3 F(u \otimes n \otimes v) \\ D(u \otimes v) &= \Phi_3 F(u \otimes v \otimes s) \\ L(u \otimes v) &= \Phi_3 F(u \otimes s \otimes v) \end{aligned}$$

These equalities demonstrate that the matrix Fredkin gate is able to operate on fuzzy vectors, thus generating the many-valued vector logic **Pol**.

The appropriate input to produce a particular logic function can be splitted, as the following example shows: To compute the implication L we write $u \otimes s \otimes v = (I \otimes s \otimes I)(u \otimes v)$.

Defining $\delta_{L3} = I \otimes s \otimes I$, we obtain the interesting identity, $L = \Phi_3 F \delta_{L3}$.

6 Vector modal operators.

In many-valued logics where a proposition p can take truth values α belonging to the interval $[0, 1]$, the following definitions for the operators "possibility" , $\Diamond\alpha$, and "necessity", $\Box\alpha$, have been proposed (Rescher 1969):

$$\Diamond\alpha = \begin{cases} 0 & \text{if } \alpha = 0 \\ 1 & \text{if } \alpha \neq 0 \end{cases}$$

$$\Box\alpha = \begin{cases} 0 & \text{if } \alpha \neq 1 \\ 1 & \text{if } \alpha = 1 \end{cases}$$

In what follows we provide a construction for the modal operators of a many-valued vector logic **Pol**.

Given the following sequence, $u_1 = D(u \otimes u)$, $u_r = D(u_{r-1} \otimes u_{r-1})$, we define

$$\Diamond u \equiv \lim_{r \to \infty} \{u_r\}.$$

In the same way, for the sequence $v_1 = C(v \otimes v)$, $v_r = C(v_{r-1} \otimes v_{r-1})$ we define

$$\text{v} \equiv \lim_{r \to \infty} \{v_r\}.$$

These vector modal operators perform the following functions:

$$\Diamond u = \begin{cases} n & \text{if } u = n \\ s & \text{if } u \neq n \end{cases}$$

$$\Box v = \begin{cases} n & \text{if } v \neq s \\ s & \text{if } v = s \end{cases}$$

The proof uses the scalar projection of these vectors (Mizraji 1992b), defined by

$$\Diamond\alpha = s^T \Diamond u, \qquad \Box\alpha = s^T \Box v.$$

We have $\Diamond\alpha = \lim_{t\to\infty}[1 - (1-\alpha)^t]$, $\Box\alpha = \lim_{t\to\infty} \alpha^t$, where $t = 2^r$. Consequently, these modal vector functions generate the classical many-valued modal functions.

Finally, notice how these vector modal operators are linked by the equation
$\Box u = N\Diamond(Nu)$.

The proof is immediate:
$\Box u = \lim_{r\to\infty} C(u_r \otimes u_r) = \lim_{r\to\infty} ND(N\otimes N)(u_r \otimes u_r) = N[\lim_{r\to\infty} D(Nu_r \otimes Nu_r)] = N\Diamond(Nu)$

This proof shows that, in basic vector logic, the classical Aristotelian relation between necessity and possibility is a consequence of the De Morgan relation between conjunction and disjunction.

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A Closure for Symmetric Heyting Algebras

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Abstract

A DH-algebra (symmetric Heyting algebra) is a Heyting algebra endowed with a De Morgan algebra structure. DH-algebras have two negations: the pseudocomplement $*$ and the De Morgan operator $'$. A. Monteiro calls Moisil normal algebras those DH-algebras $\mathcal{A}$ that satisfy: for each $x \in A, x^* \leq x'$. We call $\mathcal{N}$ the class of DH-algebras of normal type, which fullfil the above condition only for complemented elements of A. The aim of this paper is to study the class of DH-algebras that admit a boolean additive closure and its relation whith the class $\mathcal{N}$.

1 Introduction

An algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, ', 0, 1\rangle$ is a symmetric Heyting algebra ([9], Ch III, 2) or a De Morgan Heyting algebra, briefly DH-algebra ([10], 2), if $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a De Morgan algebra and $\langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra.

Given a DH-algebra $\mathcal{A}$ and an $x \in A$ we denote $x^* = x \rightarrow 0$ (the pseudocomplement of x). A.Monteiro ([9]) defines a *normal algebra* or "Moisil normal algebra" to be a DH-algebra $\mathcal{A}$ that satisfies:

(M) *For each* $x \in A$, $x^* \leq x'$.

Let ν be the operator: $\nu^0(x) = x, \nu^{n+1}(x) = (\nu^n(x))'^*, n \geq 0$. We have defined ([5]) *n-normal algebras* by the prescription: $\mathcal{A}$ is n-normal if for every $x \in A, \nu^{n+1}(x) \leq \nu^n(x)$. This condition is equivalent to: (M) holds only for every z of the form $z = (\nu^n(x))$, for some x of A ([6], 1.7). Therefore, normal algebras become 0-normal algebras.

$B(\mathcal{A})$ will denote the center of $\mathcal{A}$ (the subalgebra of boolean or complemented elements of $\mathcal{A}$) and $K(\mathcal{A})$ the subalgebra of the elements x of $B(\mathcal{A})$ such that $x^* = x'$ (strong boolean elements). It is easy to see that:

$$B(\mathcal{A}) = \{x \in A : \nu^2(x) = x\}$$

and

$$K(\mathcal{A}) = \{x \in A : \nu(x) = x\}.$$

In [6] we showed that if A is finite then A is n-normal for some n if and only if:

(N) $B(\mathcal{A}) = K(\mathcal{A})$.

We call $\mathcal{A}$ of *normal type* if (N) holds in $\mathcal{A}$. The "only if" part of the above theorem is true even for A infinite. In general, (N) is equivalent to assume that (M) holds only for $x \in B(\mathcal{A})$. Let $\mathcal{N}$ be the class of DH-algebras that satisfy (N).

An important result ([9], Ch II) for finite De Morgan algebras has the following extension for finite DH-algebras: every finite DH-algebra $\mathcal{A}$ is isomorphic to the algebra Π^+ of increasing subsets of a poset Π (being Π the set of join-irreducible elements of $\mathcal{A}$) endowed of an anti-isomorphism ψ, such that $\psi \circ \psi$ is the identity of Π . The De Morgan structure of $\mathcal{A}$ is determined by ψ . We have proved ([6]) that $\mathcal{A}$ is in $\mathcal{N}$ if and only if every component of the Hasse diagram of Π is invariant for ψ (see §2 of this paper). The components are exactly the atoms of $B(\Pi^+)$, so if A is finite then $\mathcal{A}$ satisfies (N) if and only if (M) holds only for the atoms of $B(\mathcal{A})$. This result is also valid for every algebra Π^+ of increasing subsets of an infinite poset Π and, in general, for every algebra $\mathcal{A}$ such that $B(\mathcal{A})$ is atomic and complete. In such algebras it is possible to "aproximate" each element $x \in A$ by the "nearest" boolean $c(x)$, that would be the infimum of the coatoms greater than x. The assignment obtained c is a closure operator (in fact, a quantifier) whose range is $B(\mathcal{A})$. Conversely, we can define an operator c satisfying these properties by means of identities or quasi-identities and consider DH-algebras with such additional unary operator c. We obtain a variety ecDH and a quasivariety cDH (see 2.1). Every DH-algebra of finite range in $\mathcal{N}$ (see 2.4, example 3) is an example of cDH-algebra. In particular, every algebra of SDH_n.

We characterize simples, subdirectly irreducibles in ecDH and the cDH-congruences by means of some kind of filters. $Con_K(\mathcal{A})$ will denote the set of all K-congruences of $\mathcal{A}$. If $\mathcal{A}$ is of finite range and belongs to $\mathcal{N}$, then $Con_{DH}(\mathcal{A}) = Con_{cDH}(\mathcal{A})$ (see 3.8). We prove that cDH has CEP, is semisimple and also has EDPC (see 3).

In section 4 we show some relations between DH-algebras and cDH-algebras. For example: every DH-subdirectly irreducible algebra belongs to cDH.

It is well known the Priestley duality between lattices and topological ordered spaces. In [4] "Priestley relations" are defined and also it is proved that the dual of a Priestley relation in a Priestley space is a 0-preserving join-homomorphism in the corresponding lattice. In particular, Priestley equivalences correspond to boolean additive closure operators. From some results of [4] we can obtain (see 4.6) a characterization of the DH-algebras that admit a boolean additive closure, i.e., DH-algebras in which there exists a structure of cDH-algebra.

2 cDH and ecDH-algebras

Let $\mathcal{A}$ be a DH-algebra, $x \in A$, n an integer. We define, as in [10], $t_n(x)$ as follows: $t_0(x) = x$, $t_{n+1}(x) = t_n(x) \wedge \nu^{n+1}(x)$. From now on we will use freely the property: for every $x \in A$, $\nu^2(x) \leq x$, which is a particular case of 1.10, Ch III of [9]. From this property we deduce that for every $x \in A$ and for every integer n: $t_{n+1}(x) = \nu^n(x) \wedge \nu^{n+1}(x)$. In particular, if x is a boolean element, then $t_n(x) = t_1(x)$, for $n \geq 1$. Also, we can observe that $t_{n+1}(x) = t_n(x)$ if and only if $t_n(x) \in K(\mathcal{A})$, since $t_{n+1}(x) = \nu(t_n(x))$.

DEFINITION 2.1 *An algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, ', c, 0, 1\rangle$ is a cDH-algebra if $\langle A, \vee, \wedge, \rightarrow, ', 0, 1\rangle$ is a DH-algebra and c is an unary operator that satisfies the following conditions:*

(C1) c is an additive closure operator, that is:

$$x \leq c(x),\ c(c(x)) = c(x),\ c(x \vee y) = c(x) \vee c(y).$$

(C2) $c(0) = 0$.

(C3) $c(x) \vee (c(x))^* = 1$

(C4) If $c(x' \wedge x^{*\prime}) = 0$ *then* $c(x) \wedge c(x^*) = 0$.

If $\mathcal{A}$ satisfies also:

(C5) $(c(x))^* = (c(x))'$, *then $\mathcal{A}$ is called it of normal type.*

REMARK 2.2 *It is easy to see that, if (C1) and (C2) hold, then condition* $c(x' \wedge x^{*\prime}) = 0$ *is equivalent to "$x \in B(\mathcal{A})$". So, the meaning of (C4) is "if $x \in B(\mathcal{A})$ then $c(x) \leq (c(x^*))^*$", or (by using (C1)) "if $x \in B(\mathcal{A})$ then $c(x) = x$".*

Also the operator c is a quantifier, since it satisfies: $c(x \wedge c(y)) = c(x) \wedge c(y)$.

In order to define s, the dual operator of c, we take: $s(x) = (c(x'))'$. It is clear that s is an interior operator in $\mathcal{A}$.

If $\mathcal{A}$ is a DH-algebra and x, y are elements of A then we denote the smallest element z of $B(\mathcal{A})$ such that $x \vee z \geq y$, in case it exists, by $x \Leftarrow y$.

PROPOSITION 2.3 *Let A be a DH-algebra. Let c be a map $c : A \to A$. Then, the set of conditions (C1), (C2), (C3) and (C4) is equivalent to*

$$(I)\ c(x) = 0 \Leftarrow x \qquad \text{and} \qquad (II)\ c(A) = B(\mathcal{A}).$$

Proof. Suppose that the map c satisfies (C1), (C2), (C3) and (C4). Then (taking into account remark 2.2) $c(A) = B(\mathcal{A})$. By [1], ChII, 4.11, $c(x) = 0 \Leftarrow x$.

Conversely, in virtue of the same theorem, it follows that (C1) hold true and it is obvious that c satisfies also (C2), (C3) and (C4). □

The conditions (C1), (C2), (C3) and (C4) are quasi-identities ([3], Ch V, 2.24 and 2.25) so the class of cDH-algebras (that we will denote also cDH) is a quasivariety. Therefore, the class is closed under the class operators I, P, S and P_U and contains a trivial algebra.

EXAMPLES 2.4

1) If $\mathcal{A}$ is a DH-algebra and $B(\mathcal{A})$ is a complete subalgebra of $\mathcal{A}$, then $\mathcal{A}$ is a cDH-algebra, with c defined by:

$$c(x) = \bigwedge\{b \in B(\mathcal{A}) : b \geq x\}, \text{ for every } x \in A.$$

As a particular case, we have:

2) Let Π be a partially ordered set and Π^+ the class of all increasing subsets of Π (subsets x such that $p \in x$, $q \geq p$ implies $q \in x$). It is easy to see that Π^+ is a complete sublattice of 2^Π. Π^+ is indeed a Heyting algebra ([7], Ch 8, 8.4) whose implication for $x, y \in \Pi^+$ can be expressed by: $x \to y = (x \cap y^c]^c$, where ^{c} means set complementation and $(u] = \{p \in \Pi : p \leq q, \text{ for some } q \in u\}$. If there exists an anti-isomorphism $\psi : \Pi \to \Pi$ such that $\psi \circ \psi = id_{\Pi^+}$, then Π^+ admits a De Morgan negation given by :

$x' = (\psi(x))^c$. In fact, $'$ is a De Morgan operator in 2^Π ([9], Ch II, §1) that is closed in Π^+. So $\langle \Pi^+, \cup, \cap, \rightarrow, ', \emptyset, \Pi \rangle$ is a DH-algebra. Moreover, the boolean elements are exactly the increasing and decreasing subsets of Π. Indeed, if $x \in \Pi^+$ and $x \vee x^* = \Pi$, then $(x]^c = x^c$. This characterization allows us to establish that $B(\Pi^+)$ is a complete subalgebra of Π^+.

3) Given a DH-algebra $\mathcal{A}$ let (S) be:

(S) For every $x \in A$ there exists an integer k such that $t_k(x) \in K(A)$.

It is easy to verify that (S) holds if and only if $\mathcal{A}$ is of *finite range* ([10], §2), i.e. for every $x \in A$ there exists an integer k such that: $\nu^k(x \wedge \nu(x)) = \nu^{k+1}(x \wedge \nu(x))$.

If $\mathcal{A}$ is a DH-algebra verifying (S) and $B(\mathcal{A}) = K(\mathcal{A})$, then we can prove that $\mathcal{A}$ is a cDH-algebra of normal type. We define $s(x) = t_k(x)$ (then $c(x) = (t_k(x'))'$) where k is the least integer i such that $t_i(x) = t_{i+1}(x)$.

Let x be an element of $K(\mathcal{A})$, so $x' \in K(\mathcal{A})$ and $t_1(x') = x'$ which implies: $x = c(x)$. This proves (II) of 2.3.

Given an element x of A, we have: $t_k(x') \leq x'$, for every k. So $c(x) \geq x$. Let $b \in K(\mathcal{A})$ and $b \geq x$. Then $b' \leq x'$, so: $t_k(b') = b' \leq t_k(x')$ which implies $b \geq c(x)$. Thus, we have proved (I) of 2.3.

In [10] Sankappanavar introduces DH_n (respectively SDH_n) the subvariety of DH satisfying the identity: $t_n(x) = t_{n+1}(x)$ (respectively $\nu^n(x) = \nu^{n+1}(x)$). Algebras of DH_n and SDH_n are particular cases of algebras of DH that satisfy (S). For every integer n, if $\mathcal{A}$ belongs to SDH_n then $\mathcal{A}$ is a cDH-algebra of normal type, furthermore: $\mathcal{A}$ is n-normal.

4) If $\mathcal{A} = \langle A, \vee, \wedge, ', D_1, \ldots, D_{n-1}, 0, 1 \rangle$ is an n-valued Luckasiewicz algebra (see [1], where the notation is $+$ instead of $\vee$, . instead of $\wedge$), then the reduct $\langle A, \vee, \wedge, ', 0, 1 \rangle$ is a De Morgan algebra. Also, Iturrioz in [8] showed that there exists an implication $\rightarrow$ such that $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra. So, $\langle A, \vee, \wedge, \rightarrow, ', 0, 1 \rangle$ is a DH-algebra. If we define, for $x \in A : c(x) = D_1(x)$ then it follows from conditions L_2 , L_4 , L_6 , L_7 , L_8 ([1], 11.5) that $\mathcal{A}$ is a cDH-algebra of normal type.

5) Let $\mathcal{A}$ be a DH-algebra that satisfies the identity: $x^* \vee x^{**} = 1$ (Stone condition). Let c be defined in $\mathcal{A}$ by the prescription: $c(x) = x^{**}$. Then it is easy to see that (I) and (II) hold.

We are going to define a subclass of cDH which is an equational class.

DEFINITION 2.5 *Let ecDH be the class of DH-algebras that satisfy (C1), (C2), (C3) and: (C4′)* $(c(x) \wedge c(x^*))^{*\prime} \leq c(x' \wedge x^{*\prime})$

We observe that the inequality (C4′) implies (C4). So, ecDH is a variety which is a subclass of cDH . We will show that some of the examples of cDH-algebras that we exhibit in 2.4 are, in fact, in ecDH.

Suppose that $\mathcal{A}$ is a DH-algebra and that $B(\mathcal{A})$ is not only complete, but atomic. Then, we can prove that the same operator c defined in 2.4 satisfies (C4′).

Let b be a boolean element of $\mathcal{A}$. Then ([2], ChIV, 5.4) b is the l.u.b. of the set of atoms of $B(\mathcal{A})$ that are less than or equal to b. Since $B(\mathcal{A})$ is complete and the generalized

De Morgan's Laws hold, b can be expressed as the g.l.b. of the set of coatoms greater than or equal to b. Thus for each $x \in A$, $c(x)$ turns out to be the infimum of the coatoms of $B(\mathcal{A})$ that are greater than or equal to x, that is: $c(x) = \bigwedge C_x$, being $C_x = \{z : z \in B(\mathcal{A})$, z coatom, $z \geq x\}$.

Therefore:

$$c(x) \wedge c(x^*) = (\bigwedge C_x) \wedge (\bigwedge C_{x^*}) = \bigwedge(C_x \cup C_{x^*})$$

It is immediate from De Morgan's Laws for $*$ and $'$ that:

$$(c(x) \wedge c(x^*))^{*\prime} = \bigwedge C$$

with $C = \{z^{*\prime} : z \in B(\mathcal{A}),\ z \text{ coatom } z \geq x \text{ or } z \geq x^*\}$.

On the other hand : $c(x' \wedge x^{*\prime}) = \bigwedge C_y$, with $y = x' \wedge x^{*\prime}$.

Now we prove that $C_y \subseteq C$.

If $z \in C_y$, then: $x \vee x^* \geq z'$, i.e. $(x \vee x^*) \wedge z' = z'$. So: $((x \vee x^*) \wedge z') \vee z'^* = 1$, that is, $(x \wedge z') \vee ((x^* \wedge z') \vee z'^*) = 1$.

Moreover: $(x \wedge z') \wedge ((x^* \wedge z') \vee z'^*) = 0$. Therefore: $x \wedge z' \in B(\mathcal{A})$ and, since z' is an atom, $x \wedge z' = 0$ or $x \wedge z' = z'$. In the first case $z' \leq x^*$, so $z'^* \geq x^{**} \geq x$ and hence $z = (z'^*)^{*\prime}$ belongs to C (z'^* is also a coatom). If $z' \leq x$, we have $z'^* \geq x^*$ and also z belongs to C. Therefore: $\bigwedge C \leq \bigwedge C_y$, that is, (C4') holds.

The second example of 2.4 is in ecDH since $B(\Pi^+)$ is atomic. The *components* τ of Π are the equivalence classes of the transitive closure of the relation "p is comparable with q". Recall that every component is an increasing and decreasing set of Π, therefore is an element of $B(\Pi^+)$. Also, it is easy to see that every boolean element is a union of components and the components are the atoms of $B(\Pi^+)$.

Clearly, (C4') also holds for c defined as in the examples 4 and 5 because $c(x) \wedge c(x^*) = 0$ in both cases.

REMARK 2.6 *We note that every subdirectly irreducible DH-algebra $\mathcal{A}$ is an ecDH-algebra. Indeed, in this case ([10], 4.7 and 6.2) $B(\mathcal{A})$ is $\{0,1\}$ or $\{0, a = a', a^*, 1\}$. So, since $B(\mathcal{A})$ is finite, is atomic and complete.*

3 Filters and congruences

In this section we show that the congruences of cDH-algebras are determined by some special kind of filters.

DEFINITION 3.1 *Let $\mathcal{A}$ be a cDH-algebra. We will call c-filter a filter F such that:*

(cF1) If $x \in F$ then $\nu(x) \in F$.

(cF2) If $x \in F$ then $s(x) \in F$.

$cF(\mathcal{A})$ will denote the lattice of c-filters of $\mathcal{A}$.

We observe that (cF1) means that F is a "kernel" ([5]) or "regular filter" ([10], 3.1). It is well known that congruences of DH-algebras are determined by kernels ([10], 3.3, [9] Ch III, 4.5 and 4.7).

LEMMA 3.2 *Let $\mathcal{A}$ be a cDH-algebra and let F be a filter of $\mathcal{A}$ satisfying (cF1). Then the following conditions are equivalent:*

(i) *(cF2)*

(ii) $F = [F \cap B(\mathcal{A}))$

(iii) $x \to y \in F$ *implies* $c(x) \to c(y) \in F$

(iv) $x^* \in F$ *implies* $(c(x))^* \in F$

Proof. Assuming (cF2) it is clear that , if $x \in F$, then $s(x) \in F \cap B(\mathcal{A})$ and $s(x) \leq x$. So, we have *(ii)*.

If we assume *(ii)*, then given $x \to y \in F$ there exists a $b \in F \cap B(\mathcal{A})$ such that $b \leq x \to y$. That is: $b \wedge x \leq y$. Since c is a quantifier and $c(b) = b$, it is clear that $b \wedge c(x) \leq c(y)$, so $b \leq c(x) \to c(y)$, from which $c(x) \to c(y) \in F$.

(iv) follows as a particular case of *(iii)*.

To show that *(iv)* implies *(i)*, consider $x \in F$. By (cF1): $\nu(x) \in F$ and by *(iv)*, $(c(x'))^* \in F$. Again by (cF1) we have that $\nu((c(x'))^*) \in F$, that is: $(c(x'))' \in F$. □

COROLLARY 3.3 *Let $\mathcal{A}$ be a cDH-algebra and let F be a filter of $\mathcal{A}$.*

(i) If $\mathcal{A}$ is of normal type and F satisfies (cF2) then F satisfies (cF1).

(ii) If (S) holds in $\mathcal{A}$ and F satisfies (cF1) then F satisfies (cF2).

Proof. Since for every $x \in A$, $s(x)$ belongs to $K(\mathcal{A})$, we have $\nu(s(x)) = s(x)$, hence $s(x) \leq \nu(x)$. So, *(i)* holds.

Let $\mathcal{A}$ satisfy (S), F satisfy (cF1). Then, for $x \in F$ and some k, $t_k(x) \in K(\mathcal{A}) \cap F$ and so $[F \cap K(\mathcal{A})) = F$, i.e., *(ii)* holds. □

LEMMA 3.4 *Let $\mathcal{A}$ be a cDH-algebra and let $\theta \in Con_{cDH}(\mathcal{A})$. Then $1/\theta$ is a c-filter of $\mathcal{A}$.*

Proof. Since θ is a DH-congruence, $1/\theta$ is a filter that satisfies (cF1). Also, if $x \in 1/\theta$ then $x' \in 0/\theta$, so $c(x') \in 0/\theta$ and therefore $s(x) \in 1/\theta$. □

LEMMA 3.5 *Let $\mathcal{A}$ be a cDH-algebra . For $F \in cF(\mathcal{A})$, let $\theta(F)$ be the relation : $(x,y) \in \Theta(F)$ if and only if $x \wedge f = y \wedge f$ for some $f \in F$. Then $\theta(F)$ is a cDH-congruence on $\mathcal{A}$ and $1/\theta(F) = F$.*

Proof. It is well known that $\theta(F)$ is a DH-congruence on $\mathcal{A}$ such that $1/\theta(F) = F$. Let $(x,y) \in \theta(F)$. So, since s is an interior operator: $s(x) \wedge s(f) = s(y) \wedge s(f)$, for some $f \in F$. Since F is a c-filter, we have that $(s(x), s(y)) \in \theta(F)$, that is, $\theta(F)$ preserves s. By duality it also preserves c. □

THEOREM 3.6 *Let $\mathcal{A}$ be a cDH-algebra . Then: $Con_{cDH}(\mathcal{A}) \simeq cF(\mathcal{A})$.*

Proof. We claim that the function $\theta : cF(\mathcal{A}) \to Con(\mathcal{A})$ defined by $F \mapsto \theta(F)$ is an isomorphism. If $\eta \in Con(\mathcal{A})$ then, in view of lemma 3.5 it suffices to prove $\eta = \theta(1/\eta)$. The proof is similar to the proof given in [1], lemma 4, IX.4, for Heyting algebras. □

The following lemma characterizes the c-filter $F_c(G)$ generated by a subset G of A.

LEMMA 3.7 *Let $\mathcal{A}$ be a cDH-algebra and let $G \neq \emptyset$, $G \subseteq A$. Then: $F_c(G) = \{y \in A : y \geq s(x_1 \wedge \ldots \wedge x_n) \wedge \nu(s(x_1 \wedge \ldots \wedge x_n))$ for $n \in \omega$, $x_1, \ldots, x_n \in G\}$.*

In particular, for $G = \{x\}$:

$$F_c(x) = [s(x) \wedge \nu(s(x)))$$

and, for a filter F of $B(\mathcal{A})$:

$$F_c(F) = \{y \in A : y \geq t_1(b) \text{ for some } b \in F\}$$

It follows immediately from 3.6 and 3.7 that cDH has CEP (congruence extension property).

COROLLARY 3.8 *Let $\mathcal{A}$ be a cDH-algebra . Then:*

(i) If $\mathcal{A}$ is of normal type, then $Con_{cDH}(\mathcal{A}) \simeq Con_B B(\mathcal{A})$.

(ii) If $\mathcal{A}$ satisfies (S) then every DH-congruence is a cDH-congruence.

Proof. (i) We observe that $cF(\mathcal{A}) \simeq F(B(\mathcal{A}))$, where: $F(B(\mathcal{A})) = \{F : F \text{ is a filter of } B(\mathcal{A})\}$ Indeed, the map $\phi : cF(\mathcal{A}) \to F(B(\mathcal{A}))$, $\phi : F \mapsto F \cap B(\mathcal{A})$ turns out to be an isomorphism whose inverse ψ is defined by: $\psi(H) = F_c(H)$.

(ii) DH-congruences on $\mathcal{A}$ correspond isomorphically to kernels of $\mathcal{A}$ that are exactly the c-filters of $\mathcal{A}$ (by 3.3 (ii)). □

Also, we can prove:

COROLLARY 3.9 *The variety ecDH has equationally definable principal congruences (EDPC).*

Proof. We claim that $\theta(x,y)$, the principal congruence generated by (x,y), is of the form $\theta(F)$, with $F = F_c((x \to y) \wedge (y \to x))$. This follows from noting that $(x,y) \in \theta(F)$ and $F \subseteq 1/\theta(x,y)$. □

4 Simple, subdirectly irreducible and indecomposable algebras

The main result of this section is the characterization of simple algebras in ecDH.

THEOREM 4.1 *Let $\mathcal{A}$ be a subdirectly irreducible ecDH-algebra. Then $\mathcal{A}$ is simple.*

Proof. It has already been proved ([10], 4.5) that if $\mathcal{A}$ is a directly indecomposable DH-algebra then $K(\mathcal{A}) = \{0,1\}$. It is plain that the result also holds for ecDH. So $K(\mathcal{A}) = \{0,1\}$ for $\mathcal{A}$ subdirectly irreducible in ecDH.

Suppose now that F is a minimum c-filter, $x \in F$. Therefore: $F = [s(x) \wedge \nu(s(x)))$ and since $s(x) \wedge \nu(s(x)) \in K(\mathcal{A})$ it follows that F is trivial. □

THEOREM 4.2 *Let $\mathcal{A}$ be a cDH-algebra. Then $\mathcal{A}$ is simple if and only if $B(\mathcal{A}) = \{0,1\}$ or $B(\mathcal{A}) = \{0, a = a', a^*, 1\}$.*

Proof. The "only if" part follows since $K(\mathcal{A}) = \{0,1\}$ and $b' \wedge b^* \in K(\mathcal{A})$ for every $b \in B(\mathcal{A})$.

On the other hand, let F be a c-filter of $\mathcal{A}$ and suppose that $B(\mathcal{A}) = \{0, a = a', a^*, 1\}$. By (cF1) $a \notin F \cap B(\mathcal{A})$ and $a^* \notin F \cap B(\mathcal{A})$. So, by 3.2 (ii) F is trivial. When $B(\mathcal{A}) = \{0,1\}$ the result follows from 3.8 (i). □

From [10] (4.5, 4.6 and 6.2) and the above result we have:

COROLLARY 4.3 *Let $\mathcal{A}$ be a cDH-algebra. Then:*

(i) $\mathcal{A}$ is directly indecomposable in DH if and only if $\mathcal{A}$ is simple in cDH.

(ii) If $\mathcal{A}$ is subdirectly irreducible in DH then $\mathcal{A}$ is simple in cDH.

We now point out some of the relationships between DH-algebras and cDH-algebras, which can be obtained taking into account 2.6.

PROPOSITION 4.4 *If $\mathcal{A}$ is a DH-algebra and $\{A_i\}_{i\in I}$ is a family of subdirectly irreducible DH-algebras such that $A = \prod_{i\in I} A_i$, then $\mathcal{A} \in ecDH$.*

COROLLARY 4.5 *Every DH-algebra is a subalgebra (in DH) of an ecDH-algebra.*

REMARK 4.6 *Let $\mathcal{A}$ be a DH-algebra, $X(\mathcal{A})$ the Priestley space of $\mathcal{A}$. From 2.6, 2.9 and 2.10 of [4] we can deduce the following condition for a DH-algebra to be a cDH-algebra:*

There exists an additive closure operator c on $\mathcal{A}$ such that $c(\mathcal{A}) = B(\mathcal{A})$ if and only if $X(\mathcal{A})$ satisfies (i) and (ii):

(i) For every $P \in X(\mathcal{A})$ the set $\{Q : Q \in X(\mathcal{A}), P \cap B(\mathcal{A}) = Q \cap B(\mathcal{A})\}$ is closed and decreasing in $X(\mathcal{A})$.

(ii) For every clopen increasing set $\mathcal{U}$ of $X(\mathcal{A})$ the set $\{Q : Q \in X(\mathcal{A}), Q \cap B(\mathcal{A}) = R \cap B(\mathcal{A})$ for some R in $\mathcal{U}$ $\}$ is also clopen and increasing.

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Are Logical Principles Context-Independent?

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In memoriam of Andres Raggio

There are two fundamental meta-logical dogmas which are widely and uncritically accepted: logical knowledge is a priori and logical truths are necessary. For many people logical principles are inscribed in our very notion of rationality and hence cannot be questioned in a rational perspective. These principles are supposedly norms to which any proper reasoning must abide, independently of the matter reasoned upon.

In this paper I want to challenge these common places. I want to argue for the view that logical principles such as the principle of the excluded-middle or even the principle of identity, harbor certain hidden presuppositions and hence cannot be valid outside their scope.

My strategy will be to adopt a phenomenological perspective and, following Husserl, to get involved in some sort of logical archeology whose aim is to uncover below the sedimentary deposits laid down by tradition the soil from which logical principles spring.

"Logical Investigations" and "Experience and Judgment" are two books where Husserl deals extensively with logic, but it is in "Formal and Transcendental Logic" (FTL) that the question of a genealogy of logical principles is more accurately dealt with. Hence I shall be exclusively concerned with this last text.

Husserl recognizes three distinct levels in logical studies, at the first is morphology, whose concern is to establish rules by which complex propositions are obtained from simpler propositions. Morphology is a logical grammar whose task is to prevent nonsense (Unsinn), that is, a chaotic assembling of significations devoid of any unitary signification.

At the second is the logic of consequence or logic of non-contradiction. One of its tasks is to prevent formal countersense (Widersinn), which is a formal incompatibility in the judgment, independently of the material of the judgment. For example, judgments like "All A's are B's, including some that are not B's" is formal countersense. At the third level is the logic of truth, whose task is to establish the formal laws of possible truth.

Alongside this threefold division of logic, Husserl also distinguishes a threefold notion of judgment. In the first place comes the confused and vague judgment, devoid of any distinct signification, in the second, the distinct judgment, possessing a fixed signification, and in the third, the clear judgment, the judgment recognized as true due the presence to consciousness of the affairs' judgment about exactly as are described in the judgment.

Whereas the logic of non-contradiction is involved with distinct judgments, for truth is not its concern, the logic of truth is occupied with clear judgments. As we will see, the principles of logic have different versions in each of these two levels of formal logic. In the first level, that of the logic of non-contradiction, logical principles are aimed at ruling over the preservation of distinctiveness, in the second level, that of truth-logic, logical principles set a priori conditions for the adequation of judgments to things.

The word "judgment" has an ambiguity that serves Husserl's intentions well. For, it denotes, on the one hand, the product of an act, and on the other, that act itself, the act of judging. As any act, judging has a typical polarity: on one side a noematic, objetive pole, on the other, a noetic, subjective pole. On the noematic side stands the judgment as a thing, as an object, on the noetic side resides the subject and his intentional experiences by virtue of which that object is constituted.

Therefore, either in the logic of consequence or in truth-logic Husserl recognizes a splitting of logical

principles into two possible formulations, an objective and a subjective expression. And it is precisely in the subjective formulation that Husserl finds the hidden presuppositions at work at the basis of logical principles, and which function as certain decrees of the subjectivity supporting the corresponding presuppositions on the objective pole.

First let's take that most basic principle of formal logic, the principle of identity. Its subjective expression, in the logic of consequence, can be stated as follows: I, or anyone judging in the mode of distinctiveness, can hold the same judgment in different circumstances. What is the idealizing presupposition at work here? Quite obviously, it is the presupposition of the ideal identity of the judgment as pure sense. As a consequence the judgment stands as an enduring possession, a true object.

The objective expression of this principle en the logic of consequence has probably its best formulation in the Fregean principle of the self-subsistence of "thoughts".

This presupposition which usually goes unnoticed has a profound objectifying function, that of disconnecting the judgment from mental occurrences.

In truth-logic the principle of identity in subjective from says that when a judgment is true, it is true once and for all. Given that I once lived through the experience of the adequation of the judgment to things, I am free to hold this judgment as true independently of renewed experiences of that adequation again. The principle has the function of freeing subjectivity from the temporal condition of its processes.

It is worth noticing that even certain constructive versions of logic, which are very critical of basic logical principles, denying their validity in all contexts, such as intuitionistic logic, accept uncritically the idealizing presuppositions at work behind the principle of identity.

We found in relation to the principle of identity the idealization of the "once and for all". An even more basic idealization is that of the "and so forth", present from the most elementary level of logic, the level of morphology, to its most advanced provinces such as formal ontology. If the "once and for all" frees subjectivity from the constraints of time, the "and so forth" frees it from the limitations of finitude.

S. Bachelard [] notes that to assume this idealization is tantamount to give full power to the virtual constructions, in logic or mathematics, and that Husserl's emphasis on the character of this principle as a norm, puts him beyond the bounds of intuitionism.

Let's now turn to another basic logical principle, the law of contradiction. Its subjective expression in the logic of consequence is this: of two judgments that (immediately or mediately) contradict one another, only one can be accepted by any judger whatever in a proper or distinct unitary judgment - where "accepting" here means "accepting as a distinct judgment but not necesarily as a clear judgment". Any judger accepting a judgment must accept its analytic consequences.

Its objective expression can have the following form: every contradictory judgment is "excluded" by the judgment it contradicts. Every judgment that is an analytic consequence of another is "included" in it.

The inclusion and exclusion mentioned in the objective expression of the law are formal delimitations in the realm of distinct judgment corresponding to the norms of acceptance expressed in the subjective formulation.

But it is when we move to the logic of truth that it is possible to uncover another idealizing presupposition. Now the law of contradiction has the following objective expression: contradictory judgments cannot both be true (or both false). Its subjective counterpart has the form: when a judgment can be animated by a positive adequation to things (i.e., when it is true), its negation is not only excluded a priori as a judgment but cannot itself also be led a priori to such an adequation.

The law of contradiction is complemented by the law of the excluded-middle which says in an objective formulation (in truth-logic) that any judgment is either true or false. In conjunction with the law of contradiction the "or" in the law of the excluded-middle is an exclusive "or". Its subjective formulation says that any judgment admits of being brought to a positive or negative adequation, that is, that any judgment can be shown to be either true or false.

Now, the idealization becomes clear. On the subjective side the presupposition is that judgments not only have a fixed truth-value but we are in principle able to decide which it is.

The modality alluded to here gains an ontological expression in the objective form of the idealizing presupposition, that judgments are either true-in-themselves or false-in-themselves.

It is interesting to notice that whereas logicians are quite willing to give up the subjective side of the presupposition, very few are eager to let their objective version go. And this is because they usually don't see both presuppositions as faces of the same coin. To show that they are in fact only different versions of the same presupposition is one of the tasks of the genetic analysis carried out by phenomenology.

Also, according to Husserl such a presupposition is an *a priori* condition for the possibility of science. Quite simply, before embarking into a search for truth, one has to be convinced that truth can be known.

Up to now we have seen that logical principles harbor certain idealizing presuppositions, and consequently that those principles wouldn't be valid any longer were these presuppositions to be called off. And this has actually been done. By intuitionistic logic for example, which has rejected the assumption of the truth-in-itself and the falsity-in-itself of judgments, together with the presupposition of the intrinsic decidability of judgments. And it has done this in a more consistent way than classical logic.

But there's a still deeper presupposition hidden behind logical principles to be uncovered and with it the extent to which these principles are related to the sense with which certain domains are given.

Logic is, for Husserl, the theory of science. Judgments which form the subject matter of logic are scientific judgments, animated by cognitive interests. Hence they must always refer ultimately to a domain of individuals. Complex as they may be these judgments can always be descomposed into elementary judgments bearing upon individual objects.

Scientific judgments are teleologically oriented by the ideal of truth. Hence the most basic scientific judgments refer to a domain of objects in the mode of certainty. Modal variations are derived forms of certainty.

These simple remarks are enough to show the extent to which the disclosure of states-of-affairs in a basic domain of objects, which is what I mean by experience, is in the horizon of every conceivable judgment.

But experience is also at the origin of judgments. In "Experience and Judgment" Husserl investigates this problem deeply.

Therefore formal logic must always be supplement by a theory of experience. The formation of sense which we call experience must be intimately correlated to the sense attached to the logical principles.

Let's consider the judgment "the color red is greater than two". What is wrong with it? It is formed according to the rules of grammar and the rules of morphology, and it is not a formal countersense, so it seems to be a legitimate judgment. It is true or false? It can be seen a priori that it cannot be either.

So, the principle of the excluded-middle doesn't apply to it. If one wants the principle to be valid one has to rule out the above sentence as an acceptable judgment. Therefore together with nonsense and formal or analytic countersense we have to accept that judgments also can be material or synthetic countersenses.

For logical principles to be valid we have to postulate that proper judgments have to have sense as to content. But for this to be a legitimate requirement of formal logic, material sense cannot be a matter for decision *a posteriori*. It has to be determined a priori whether judgments have synthetic sense or not. And this requirement leads us to questions about intentional constitution and the acts of subjectivity in its transcendental role.

Disqualifying as proper judgments that lack synthetic sense is the task of a presupposition of a fundamental nature which our analysis brought out. Namely, that the domain of individual objects to which judgments refer must be a priori ordered according to mutual compatibilities and incompatibilities. That the color red is not a magnitude comparable to the number two is not a matter of fact, but a matter of principle.

Another way of putting this is by saying that there cannot be in principle any experience, in a broad sense, presenting a state-of-affairs corresponding to the sense of the judgment "the color red is grater than two". And this is so just because of what the words "red", "greater" and "two" mean.

I then propose the following equivalence: a judgment has material sense if, and only if, its content is in principle accessible to a unitary experience of the appropriate type.

In the words of Husserl this is expressed as follows: "the ideal existence of the judgment presupposes the ideal existence of the content of the judgment... the ideal existence of the content of the judgment is tied to the conditions of the unity of possible experience". [FTL paragraph 89].

In our discussion we have been using the expression "a priori" and "in principle" in a somewhat careless way, and anyone with a certain familiarity with transcendental philosophy knows that they always lead us to the transcendental realm, to an inquiry the workings of subjectivity in its transcendental role, namely, that of bestowing meaning.

Any conceivable objective domain, be it nature or the ideal domain of mathematics, is conceived with a certain sense. Its objects have pre-determined possibilities of appearing as they are, that is, in the way they are thought to be, according to their nature.

A material object, for instance, can be the object of sense perception, whereas an ideal object cannot.

Objective domains are intentionally constituted and it is in the intentionality of the constituting subjectivity that we have to look for their sense.

The above sentence can be grossly misinterpreted. It can be read sometimes as an expression of ontological idealism, sometimes as sheer psychologism. Let's make things clear.

Intentional constitution means the constitution of objects as intentional objects, that is, the bestowing of meaning upon objects independently of their ontological constitutition. And this is the role played by transcendental subjectivity, which is just good old subjectivity playing a specific role. But this subjectivity can't always be reduced to an isolated consciousness, it is often embodied in an entire community embedded in culture and history. Denying the transcendental approach in philosophy is to deny that sense has a genesis and that the genesis of sense is historical.

It is precisely the sense with objective domains are posited that determines what does and what does not count as a possible experience in the domain.

And it is this determination which ultimately defines synthetic sense and countersense.

We now come to an end in our regress, from logic to the experience and from there to transcendental subjectivity.

Before jumping to conclusions let's look at an example.

Sets, are given as ideal objects on the basis of the pre-given objects that figure as their elements. Once this basic clarification has been accomplished an essential legality is disclosed in the domain of sets: nothing can be a member of a set which presupposes this set for its existence. In particular, a set cannot be a member of itself. This is essentially the vicious-circle principle of Poincare and Russell. This principle is not an *ad hoc* stipulation to bypass paradox, but the expression of a law of essence, that is, a consequence of the very idea of set.

It also belongs to this idea that any set can be decomposed into a multiplicity of individuals pre-given as its most fundamental components (although in mathematics one can dispense with these elements).

It is precisely this view that is at the basis of the iterative concept of set and gives support to the usual set-theoretic axioms. In particular the axiom of foundation.

Or rather, these axioms are nothing but the development of such a view. If the axioms force themselves upon us as being true, as Göedel believes, it's because they're nothing but the consequences of the sense we attribute to the notion of set.

It's now easy to see what is wrong with Russel's paradox. The proposition leading to the paradox doesn't make sense as to content. Hence the logical principle are not valid for it.

I believe that what has been said far is enough to make my point clear: logical principles cannot have unconditional validity in all domains. The principle of classical logic cannot be established independently of an analysis of meaning, which is just another way of saying that they cannot be set up independently of a phenomenological analysis of positing acts.

What consequences can one draw from this conclusion for the philosophy of logic?

Firstly, that logical principles are not necessary, in the sense that they may not be, and often are not, valid in all possible domains.

Secondly, that they are a priori, though in a weak way. For even though they are not established in the course of experience, and so are a priori, logical principles are not independent of the sense attributed to the concept of experience, they are sensitive to what may count in any given domain as a possible experience.

FREE MMI_3 – ALGEBRAS

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Abstract

A. V. Figallo [6] introduced the notion of MMI_{n+1} – algebras as a generalization of the monadic Tarski algebras considered by A. Monteiro and L. Iturrioz in [13].

The MMI_{n+1} – algebras represent an algebraic counterpart of a fragment of the monadic $(n+1)$ – valued functional calculus of Lukasiewicz.

In this paper we determine the structure of the MMI_3 – algebras with a finite set of free generators.

1. Preliminaries.

In [9] Y. Komori considered the I – algebras as an algebraic counterpart of the infinite – valued implicative calculus of Lukasiewicz (see also [20, 21, 22, 23]).

In [6] A. V. Figallo defined the $(n+1)$ – valued modal I – algebras (or MI_{n+1} – algebras) and the monadic MI_{n+1} – algebras (or MMI_{n+1} – algebras).

The MI_{n+1} – algebras are an algebraic version of the propositional calculus where the primitive conectives are the $(n+1)$ – valued implication $(\rightarrow)$ of Lukasiewicz and the modal operators $\sigma_1, \sigma_2, \ldots, \sigma_n$ considered by G. C. Moisil in order to generalize the notion of three – valued Lukasiewicz algebra as an algebraic counterpart of the $(n+1)$ – valued propositional calculus of Lukasiewicz.

The MMI_{n+1} – algebras represent an algebraic aproximation of a fragment of the monadic $(n+1)$ – valued functional calculus of Lukasiewicz. A very important class of MMI_{n+1} – algebras are those which have first element.

The structure of the free MMI_{n+1} – algebra with first element finitely generated was described in [6] and it rests to resolve the determination of the free MMI_{n+1} – algebra finitely generated.

In [5] it is solve for $n = 1$. The MMI_2 – algebras coincide with the monadic Tarski algebras considered in [13].

In this paper we determine the structure of the free MMI_3 – algebra finitely generated.

Next we give some notions that we need in what follows.

1.1. Definition. A 3-valued Lukasiewicz algebra (or L_3-algebra) is an algebra $(A, \wedge, \vee, \sim, \nabla, 1)$ of type $(2,2,1,1,0)$ where $(A, \wedge, \vee, \sim, 1)$ is a De Morgan algebra and it satisfies the following identities:

L1) $\sim x \vee \nabla x = 1$,

L2) $\sim x \wedge \nabla x = x \wedge \sim x$,

L3) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$. [10,17]

1.2. Definition. A monadic 3-valued Lukasiewicz algebra (or ML_3-algebra) is an algebra $(A, \wedge, \vee, \sim, \nabla, \exists, 1)$ of type $(2,2,1,1,1,0)$ such that $(A, \wedge, \vee, \sim, \nabla, 1)$ is an L_3-algebra and the following identities are satisfied:

ML1) $\exists 0 = 0$,

ML2) $x \wedge \exists x = x$,

ML3) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,

ML4) $\exists \nabla x = \nabla \exists x$,

ML5) $\exists \triangle x = \triangle \exists x$, where $\triangle x = \sim \nabla \sim x$. [16]

1.3. Definition. An implicative Lukasiewicz algebra (or I-algebra) is an algebra $(A, \rightarrow, 1)$ of type $(2,0)$ which satisfies the following identities:

I1) $x \rightarrow x = 1$,

I2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,

I3) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

I4) $((x \rightarrow y) \rightarrow (y \rightarrow x)) \rightarrow (y \rightarrow x) = 1$. [6,9,19]

1.4. Definition. An I-algebra is an I_3-algebra if it satisfies the identity:

I5) $(x \rightarrow (x \rightarrow y)) \rightarrow x = x$. [12]

In what follows we write $x \succ y$ instead of $x \rightarrow (x \rightarrow y)$.

The notion of MI_{n+1}-algebra introduced in [6] is equivalent, in the case $n = 2$, to the following one.

1.5. Definition. An algebra $(A, \rightarrow, \sigma_1, \sigma_2, 1)$ of type $(2,1,1,0)$ is an MI_3-algebra if $(A, \rightarrow, 1)$ is an I_3-algebra and it satisfies the identities:

M1) $\sigma_1 x \rightarrow y = x \succ y$, M2) $\sigma_j x \vee (\sigma_j x \to y) = 1,\ j = 1,2,$ where $x \vee y = (x \to y) \to y,$

M3) $\sigma_j(\sigma_k x \to \sigma_k y) = \sigma_k x \to \sigma_k y,\ 1 \leq j,k \leq 2,$

M4) $(\sigma_1 x \to \sigma_1 y) \to ((\sigma_2 x \to \sigma_2 y) \to (x \to y)) = 1,$

M5) $\sigma_2 x \to (\sigma_1 x \vee \sigma_1(x \to y)) = 1,$

M6) $\sigma_1(x \to y) \to (\sigma_j x \to \sigma_j y) = 1,\ j = 1,2.$ [6,12]

For more details of this theory we direct the reader to [3,4,6,7,8,11,12].

1.6. Definition. Let A be an I-algebra. A function $\forall : A \to A$ is an U-operator on A if it satisfies:

U1) $\forall x \leq x,$

U2) $\forall(x \vee \forall y) = \forall x \vee \forall y,$

U3) $\forall(x \to y) \leq \forall x \to \forall y,$

U4) $\forall(\forall x \to \forall y) = \forall x \to \forall y.$ [6]

1.7. Definition. An MMI_3-algebra is an algebra $(A, \to, \sigma_1, \sigma_2, \forall, 1)$ of type $(2,1,1,1,0)$ where $(A, \to, \sigma_1, \sigma_2, 1)$ is an MI_3-algebra, $\forall$ is an U-operator on A and it verifies:

UM1) $\forall \sigma_j x = \sigma_j \forall x,\ \ j = 1,2.$

1.8. The following results show the relationship between the MMI_3-algebras and ML_3-algebras:

(i) If $\mathcal{A} = (A, \to, \sigma_1, \sigma_2, \forall, 1)$ is an MMI_3-algebra and there exists $0 \in A$ such that $0 \to x = 1$ for all $x \in A$, then defining $\sim x = x \to 0$, $\nabla x = \sigma_2 x$, $\exists x = \sim \forall \sim x$, $x \vee y = (x \to y) \to y$ and $x \wedge y = \sim(\sim x \vee \sim y)$ it follows that $L(\mathcal{A}) = (A, \wedge, \vee, \sim, \nabla, \exists, 1)$ is an ML_3-algebra.

(ii) If $\mathcal{A} = (A, \wedge, \vee, \sim, \nabla, \exists, 1)$ is an ML_3-algebra then defining $\sigma_1 x = \sim \nabla \sim x$, $\sigma_2 x = \nabla x$, $\forall x = \sim \exists \sim x$ and $x \to y = (\nabla \sim x \vee y) \wedge (\nabla y \vee \sim x)$, it follows that $M(\mathcal{A}) = (A, \to, \sigma_1, \sigma_2, \forall, 1)$ is an MMI_3-algebra.

We shall denote by $\boldsymbol{L_3}$, $\boldsymbol{ML_3}$, $\boldsymbol{I}$, $\boldsymbol{I_3}$, $\boldsymbol{MI_3}$ and $\boldsymbol{MMI_3}$ the varieties of algebras defined above respectively.

1.9. Definition. Let $A \in \boldsymbol{MMI_3}$. $D \subseteq A$ is a monadic deductive system (m.d.s.) of A, if it satisfies:

D1) $1 \in D,$

D2) if $x, x \to y \in D$ then $y \in D,$

D3) if $x \in D$ then $\forall x \in D$.

In what follows we write $x \Rightarrow y$ instead of $\forall x \succ y$.

1.10. Lemma. If $A \in \boldsymbol{MMI_3}$ then it holds:

UM2) $x \Rightarrow \forall x = 1$,

UM3) $1 \Rightarrow x = x$,

UM4) $x \Rightarrow (y \Rightarrow x) = 1$,

UM5) $(x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x \Rightarrow y) \Rightarrow (x \Rightarrow z)) = 1$,

UM6) $((x \Rightarrow y) \Rightarrow x) \Rightarrow x = 1$.

We shall denote by $\mathbb{D}_M(A)$ the set of all m.d.s. of A.

1.11. Lemma. Let $A \in \boldsymbol{MMI_3}$. Then the following conditions are equivalent:

(i) $D \in \mathbb{D}_M(A)$,

(ii) $D \subseteq A$ verifies D1 and D2'') if $x, x \Rightarrow y \in D$ then $y \in D$.

We shall denote by $Con(A)$ the set of all MMI_3-congruences of A and if $R \in Con(A)$ we shall denote by x_R the equivalence class of x, $x \in A$.

1.12. Lemma. [6] Let $A \in \boldsymbol{MMI_3}$ and $D \in \mathbb{D}_M(A)$.

(i) If $R(D) = \{(x, y) \in A^2 \colon x \to y \in D,\ y \to x \in D\}$ then $R(D) \in Con(A)$ and $1_{R(D)} = D$.

(ii) If $R \in Con(A)$, there exists only one $D \in \mathbb{D}_M(A)$ such that $R = R(D)$ and $D = 1_{R(D)}$.

Let $Hom(A, B)$ be the set of all MMI_3-homomorphism from A into B. In what follows if $h \in Hom(A, B)$ we say that h is an homomorphism from A into B.

1.13. Theorem. If $h \in Hom(A, B)$ then $Ker(h) = \{x \in A \colon h(x) = 1\}$, called the kernel of h, has the following properties:

(i) $Ker(h) \in \mathbb{D}_M(A)$,

(ii) $(x, y) \in R(Ker(h))$ if and only if $h(x) = h(y)$,

(iii) $A/Ker(h)$ and $h(A)$ are isomorphic MMI_3-algebras.

1.14. Examples. Let k be a positive integer.

(i) Let $\mathbb{B}_k = (B_k, \wedge, \vee, -, 0, 1)$ be the Boolean algebra with k atoms. $\mathfrak{B}_k$ denotes the monadic Boolean algebra $(B_k, \wedge, \vee, -, \exists, 0, 1)$ where $\exists 0 = 0$ and $\exists x = 1$, for all $x \in B_k$, $x \neq 0$. If we define for $x, y \in B_k$, $x \to y = -x \vee y, \forall x = -\exists - x, \sigma_1 x = \sigma_2 x = x$ then $\mathfrak{B}_k^* =$

$(B_k, \to, \sigma_1, \sigma_2, \forall, 1) \in \boldsymbol{MMI_3}$.

(ii) Let $(T, \wedge, \vee, \sim, \nabla, 1)$ be the three-valued Lukasiewicz algebra where $T = \{0, a, 1\}$, $0 < a < 1$ and $\sim, \nabla$ are defined by means of the following tables

x	$\sim x$	∇x
0	1	0
a	a	1
1	0	1

Let $\mathbb{L}_k = (T^k, \wedge, \vee, \sim, \nabla, 1)$ be the product algebra. $\mathcal{L}_k$ denotes the monadic 3-valued
Lukasiwicz algebra $(T^k, \wedge, \vee, \sim, \nabla, \exists, 1)$, where for all $x \in T^k$

$$\exists x = \begin{cases} 0, & \text{if } x = 0, \\ c, & \text{if } 0 < x \le c, \\ 1, & \text{if } x \nleq c. \end{cases}$$

If we define $x \to y = (\nabla \sim x \vee y) \wedge (\nabla y \vee \sim x)$, $\sigma_1 x = \sim \nabla \sim x$, $\sigma_2 x = \nabla x$ then $\mathcal{L}_k^* = (T^k, \to, \sigma_1, \sigma_2, \forall, 1) \in \boldsymbol{MMI_3}$.

Let $A \in \boldsymbol{MMI_3}$. We shall denote by $\mathbb{E}_M(A)$ the set of all maximal m.d.s. of A.

1.15. Remarks. Taking into account UM3,... ,UM6, 1.12 and results of A. Monteiro [13] (see also [14]) , we can state that for all non trivial $A \in \boldsymbol{MMI_3}$ it holds:

(1) If $D \in \mathbb{D}_M(A)$ is proper, then $D = \cap \{M \in \mathbb{E}_M(A)\colon D \subseteq M\}$. In particular $\cap \{M \in \mathbb{E}_M(A)\} = \{1\}$.

(2) The following statements are equivalent:
   (i) A/M is simple,
   (ii) $M \in \mathbb{E}_M(A)$,
   (iii) If $a \in A - M$ and $b \in A$ then $a \Rightarrow b \in M$.

(3) If S is a non trivial MMI_3-subalgebra of A then
   (i) $\mathbb{D}_M(S) = S \cap \mathbb{D}_M(A)$,

(ii) $\mathbb{E}_M(S) = \{M \cap S \colon M \in \mathbb{E}_M(A), S \not\subseteq M\}$.

1.16. Theorem. If $A \in \boldsymbol{MMI_3}$ is no degenerated, then A is a subdirect product of simple MMI_3-algebras.

1.17. Theorem. Let $A \in \boldsymbol{MMI_3}$ be finitely generated and $D \in \mathbb{D}_M(A)$. The following conditions are equivalent:

(i) $D \in \mathbb{E}_M(A)$,

(ii) there exists an MMI_3-epimorphism h from A onto $\mathfrak{B}_k^*$ or $\mathfrak{L}_k^*$, $k > 0$, such that $Ker(h) = D$. [15,16]

2. $\boldsymbol{MMI_3}$-algebra with a finite set of free generators.

2.1. Definition. If $c > 0$ is an arbitrary cardinal number, then we say that $\mathbb{L}(c)$ (or $\mathbb{ML}(c)$) is the MMI_3-algebra (or ML_3-algebra) with c free generators if:

(i) $\mathbb{L}(c)$ (or $\mathbb{ML}(c)$) has a set G of generators such that $|G| = c$,

(ii) any map f from G into an MMI_3-algebra (or ML_3-algebra) A can be extended to an MMI_3-homomorphism (ML_3-homomorphism) $h \colon \mathbb{L}(c) \to A$ ($h \colon \mathbb{ML}(c) \to A$) such that $h(g) = f(g)$ for all $g \in G$.

Since the notion of MMI_3-algebra (or ML_3-algebra) is equationally definable, we can state, by a G. Birkhoff theorem of universal algebra [1], that for any cardinal $c > 0$ there exists $\mathbb{L}(c)$ (or $\mathbb{ML}(c)$) and it is unique up to isomorphisms. Moreover the MMI_3-homomorphism (ML_3-homomorphism) h of 2.1 (i) is unique.

Let G be a set of free generators of $\mathbb{L}(c)$ and $K(G) = \{\forall\sigma_1 g \colon g \in G\}$.

2.2. Lemma. $K(G)$ is the set of minimal elements of $\mathbb{L}(c)$.

Proof. Let S be the set of all $x \in \mathbb{L}(c)$ for which there exists $g \in G$ such that $\forall\sigma_1 g \leq x$. It is clear that $G \subseteq S$. Furthermore S is a subalgebra of $\mathbb{L}(c)$. Indeed, if $x, y \in S$ there exists $g \in G$ such that $\forall\sigma_1 g \leq y$ and since $y \leq x \to y$ it results that $x \to y \in S$. On the other hand if $x \in S$ there exists $g \in G$ such that $\forall\sigma_1 g \leq x$, then $\forall\sigma_1 g \leq \forall x$, $\forall\sigma_1 g \leq \sigma_1 x$, $\forall\sigma_1 g \leq \sigma_2 x$ so $\forall x$, $\sigma_1 x$ and $\sigma_2 x$ belong to S. Therefore (1) $S = \mathbb{L}(c)$.

Let $z \in K(G)$, so $z = \forall\sigma_1 g_0$, $g_0 \in G$. If z is not a minimal element of $\mathbb{L}(c)$ there exists $x \in \mathbb{L}(c)$ such that $x < z$, then by (1) there exists $g' \in G$ such that $\forall\sigma_1 g' \leq x$, so $\forall\sigma_1 g' < z = \forall\sigma_1 g_0$. Let $f \colon G \to \mathfrak{B}_1^*$ defined by $f(g) = 0$ if $g = g_0$ and $f(g) = 1$, otherwise.

So there exists an homomorphism $h \colon \mathbb{L}(c) \to \mathfrak{B}_1^*$ which extends f. Since h is an increasing function, it follows that $h(\forall\sigma_1 g') \leq h(\forall\sigma_1 g_0) = \forall\sigma_1 h(g_0) = \forall\sigma_1 f(g_0) = \forall\sigma_1 0 = 0$ which is a contradiction.

So the set $\mu(\mathbb{L}(c))$ of all minimal elements of $\mathbb{L}(c)$ is non empty and $K(G) \subseteq \mu(\mathbb{L}(c))$.

Let $m \in \mu(\mathbb{L}(c))$, then there exists $g \in G$ such that $\forall\sigma_1 g \leq m$, hence $\forall\sigma_1 g = m$ and $m \in K(G)$. □

2.3. Lemma. If G' is a set of free generators of $\mathbb{ML}(c)$ then $[G']_{\boldsymbol{MMI_3}}$ and $\mathbb{L}(c)$ are isomorphic MMI_3-algebras.

2.4. Remark. By 2.3 identifying isomorphic algebras we can write $\mathbb{L}(c) \subseteq \mathbb{ML}(c)$.

From now on, n is a positive integer and $G = \{g_1, g_2, \ldots, g_n\}$ denotes a set of free generators of $\mathbb{ML}(n)$, so $G \subseteq \mathbb{L}(n) \subseteq \mathbb{ML}(n)$ and G is a set of free generators of $\mathbb{L}(n)$.

Furthermore as $\mathbb{ML}(n)$ is finite, for any positive integer n, then $\mathbb{L}(n)$ is also finite.

2.5. Lemma. (i) For all t, $1 \leq t \leq 2^n$ there exists an epimorphism from $\mathbb{L}(n)$ onto $\mathfrak{B}_t^*$.
(ii) For all k, $1 \leq k \leq 3^n$ there exists an epimorphism from $\mathbb{L}(n)$ onto $\mathfrak{L}_k^*$.

Proof. (i) If $t = 1$, let $f: G \rightarrow \mathfrak{B}_1^*$ be the function such that $f(G) = \{0\}$. Then the homomorphism h which extends f is onto.

If $1 < t \leq 2^n$, it is well known that there exists an ML_3-epimorphism h from $\mathbb{ML}(n)$ onto $\mathfrak{B}_t$ [16]. Let f be the restriction of h to G. It is clear that $f(G) \neq \{1\}$. Since $\mathbb{L}(n) = [G]_{\boldsymbol{MMI_3}} \subseteq \mathbb{ML}(n)$ then by 2.1(ii) there exists an homomorphism $h_1: \mathbb{L}(n) \rightarrow \mathfrak{B}_t^*$ which extends f. On the other hand as $h_1(\mathbb{L}(n))$ is a non trivial subalgebra of the simple algebra $\mathfrak{B}_t^*$ hence by 1.15(3) it is a simple ML_3-algebra. Therefore $h_1(\mathbb{L}(n)) = \mathfrak{B}_j$, $j \leq t$. Since $f(G) \subseteq B_j$ then $[f(G)]_{\boldsymbol{ML_3}} = h(\mathbb{ML}(n)) = \mathfrak{B}_t$. hence $\mathfrak{B}_t \subseteq \mathfrak{B}_j$ and so $\mathfrak{B}_j^* = \mathfrak{B}_t^*$. Then h_1 is onto.

(ii) If $k = 1$ then $f: G \rightarrow \mathfrak{L}_1^*$ be the function such that $f(G) = \{c\}$. where c is the center of $\mathfrak{L}_1^*$, then the homomorphism which extends f is onto.

If $1 \leq k \leq 3^n$ it follows as (i). □

2.6. Lemma. (i) For all t, $1 \leq t \leq 2^n$,

(a) if $t > 1$ then $|Epi(\mathbb{L}(n), \mathfrak{B}_t^*)| = |Epi_{\boldsymbol{ML_3}}(\mathbb{ML}(n), \mathfrak{B}_t)|$,

(b) if $t = 1$ then $|Epi(\mathbb{L}(n), \mathfrak{B}_t^*)| = |Epi_{\boldsymbol{ML_3}}(\mathbb{ML}(n), \mathfrak{B}_t)| - 1$.

(ii) For all k, $1 \leq k \leq 3^n$, $|Epi(\mathbb{L}(n), \mathfrak{L}_k^*)| = |Epi_{\boldsymbol{ML_3}}(\mathbb{ML}(n), \mathfrak{L}_k)|$.

Proof. (i) (a) If $h \in Epi(\mathbb{L}(n), \mathfrak{B}_t^*)$ and $f = h/G: G \to B_t$ then there exists an unique ML_3-homomorphism $h_1: \mathbb{ML}(n) \to \mathfrak{B}_t$ which extends f. Since h_1 is also an MMI_3-homomorphism we can consider $h_2 = h_1/\mathbb{L}(n): \mathbb{L}(n) \to \mathfrak{B}_t^*$. Since $h_2/G = f = h/G$ then $h_2 = h$ and so h_2 is onto. Therefore $h_1 \in Epi_{\boldsymbol{ML_3}}(\mathbb{ML}(n), \mathfrak{B}_t)$. The mapping $h \to h_1$ is a bijection between $Epi(\mathbb{L}(n), \mathfrak{B}_t^*)$ and $Epi_{\boldsymbol{ML_3}}(\mathbb{ML}(n), \mathfrak{B}_t)$.

(b) If $h_1 \in Epi_{\boldsymbol{ML_3}}(\mathbb{ML}(n), \mathfrak{B}_1)$ is such that $h_1(G) = \{1\}$, then the extension h to $\mathbb{L}(n)$ of $f = h_1/G$ is such that $h(x) = 1$, for all $x \in \mathbb{L}(n)$, hence $h \notin Epi(\mathbb{L}(n), \mathfrak{B}_1^*)$.

(ii) Let $h \in Epi(\mathbb{L}(n), \mathfrak{L}_k^*)$ and $f = h/G$. Then there exists an unique ML_3-homomorphism $h_1: \mathbb{ML}(n) \to \mathfrak{L}_t$ which extends f. Since h_1 is an MMI_3-homomorphism we can consider $h_2 = h_1/\mathbb{L}(n)$. Then $h_2/G = f = h/G$. Therefore $h_2 = h$ and h_1 is onto. The mapping $h \to h_1$ is a bijection between $Epi(\mathbb{L}(n), \mathfrak{L}_k^*)$ and $Epi_{\boldsymbol{ML_3}}(\mathbb{ML}(n), \mathfrak{L}_k)$. □

2.7. Lemma. [16] (i) $|Epi_{\boldsymbol{ML_3}}(\mathbb{ML}(n), \mathfrak{B}_t)| = V_{2^n, t}$, $1 \le t \le 2^n$.

(ii) $|Epi_{\boldsymbol{ML_3}}(\mathbb{ML}(n), \mathfrak{L}_k)| = V_{3^n, k} - V_{2^n, k}$, $1 \le k \le 3^n$, where $V_{m,n} = \dfrac{m!}{(m-n)!}$ if $m \ge n$ and $V_{m,n} = 0$ if $m < n$.

2.8. Let $[\forall\sigma_1 g_i) = \{x \in \mathbb{L}(n): \forall\sigma_1 g_i \le x\}$, $1 \le i \le n$. By lemma 2.2 we have $\mathbb{L}(n) = \bigcup_{i=1}^{n} [\forall\sigma_1 g_i)$, hence

$$|\mathbb{L}(n)| = \sum_{j=1}^{n} (-1)^{j+1} \alpha_j^{(n)},$$

where $\alpha_j^{(n)} = \sum_{1 \le i_1 < \ldots < i_j \le n} |\bigcap_{r=1}^{j} [\forall\sigma_1 g_{i_r})|$.

It easy to see that $\bigcap_{r=1}^{j} [\forall\sigma_1 g_{i_r}) = [\bigvee_{i=1}^{j} \forall\sigma_1 g_{i_r})$ and if G_1, G_2 are subsets of G such that $|G_1| = |G_2|$ then $|[\bigvee_{x \in G_1} \forall\sigma_1 x)| = |[\bigvee_{y \in G_2} \forall\sigma_1 y)|$, so we can write

$$|\mathbb{L}(n)| = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} |M_j^{(n)}|,$$

where $M_j^{(n)} = [\bigvee_{i=1}^{j} \forall\sigma_1 g_i)$, $1 \le j \le n$.

2.9. Lemma. For all j, $1 \le j \le n$, $|M_j^{(n)}| > 1$, $M_j^{(n)}$ is an MMI_3-subalgebra of $\mathbb{L}(n)$ and $M_j^{(n)} \in \boldsymbol{ML_3}$.

Proof. Assume $|M_j^{(n)}| = 1$, then (1) $\bigvee_{i=1}^{j} \forall\sigma_1 g_i = 1$. Let $f: G \to \mathfrak{B}_1^*$ be such that

$f(g_i) = 0,$
$1 \leq i \leq n$. Hence, there exists an homomorphism $h{:}\mathbb{L}(n) \to \mathfrak{B}_1^*$ such that $h/G = f$. Then (2) $h\left(\bigvee_{i=1}^{j} \forall\sigma_1 g_i\right) = \bigvee_{i=1}^{j} h\left(\forall\sigma_1 g_i\right) = \bigvee_{i=1}^{j} \forall\sigma_1 h(g_i) = \bigvee_{i=1}^{j} \forall\sigma_1 f(g_i) = 0$. By (1), $h\left(\bigvee_{i=1}^{j} \forall\sigma_1 g_i\right) = 1$ which contradicts (2).

It is easy to see that $M_j^{(n)}$ is an MMI_3-subalgebra of $\mathbb{L}(n)$ with first element $g_j^* = \bigvee_{i=1}^{j} \forall\sigma_1 g_i$. Therefore $M_j^{(n)} \in \boldsymbol{ML_3}$. □

2.10. Let $\mathbb{E}(M_j^{(n)})$ be the family of all maximal m.d.s. of $M_j^{(n)}$, or equivalently the family of all maximal ML_3-deductive systems of $M_j^{(n)}$, [16].

Since $M_j^{(n)}$ is a finite non trivial ML_3-algebra then

$$M_j^{(n)} \simeq \prod_{D \in \mathbb{E}(M_j^{(n)})} M_j^{(n)}/D. \quad [16]$$

Then to compute $|M_j^{(n)}|$ we need to know the number of maximal m.d.s. of $M_j^{(n)}$ and $|M_j^{(n)}/D|$ for each $D \in \mathbb{E}(M_j^{(n)})$.

Let $g_j^* = \bigvee_{i=1}^{j} \forall\sigma_1 g_i$. As $M_j^{(n)} = [g_j^*, 1]$ is an MMI_3-subalgebra of $\mathbb{L}(n)$ with more than one element, by 1.15(3) we have $D \in \mathbb{E}(M_j^{(n)})$ if and only if $D = M \cap M_j^{(n)}$, where $M \in \mathbb{E}(\mathbb{L}(n))$ and $M_j^{(n)} \not\subseteq M$. By 1.17(ii) if $M \in \mathbb{E}(\mathbb{L}(n))$, there exists an epimorphism (1) $h{:}\mathbb{L}(n)) \to \mathfrak{B}_t^*$, $1 \leq t \leq 2^n$ or (2) $h{:}\mathbb{L}(n)) \to \mathfrak{L}_k^*$, $1 \leq k \leq 3^n$. Let $G_j = \{g_1, g_2, \ldots, g_j\}$.

Observe that $h(g) \neq 1$, for all $g \in G_j$. Indeed, if there exists $g \in G_j$ such that $h(g) = 1$ then $h(\forall\sigma_1 g) = 1$. As $\forall\sigma_1 g \leq g_j^*$ we have $h(g_j^*) = 1$ so $M_j^{(n)} \subseteq Ker(h)$, contradiction. Let h' be the restriction of h to $M_j^{(n)}$. It is easy to see that $h'{:}M_j^{(n)} \to \mathfrak{B}_t$ is an MB-epimorphism if h verifies (1), or $h'{:}M_j^{(n)} \to \mathfrak{L}_k$ is an ML_3-epimorphism if h verifies (2). Furthermore $Ker(h') = D$ and $M_j^{(n)}/D = \mathfrak{B}_t$ or $M_j^{(n)}/D = \mathfrak{L}_k$ respectively.

From the above results and lemma 2.5

$$M_j^{(n)} = \prod_{t=1}^{2^n} (\mathfrak{B}_t)^{\alpha_{j,t}^{(n)}} \times \prod_{k=1}^{3^n} (\mathfrak{L}_k)^{\beta_{j,k}^{(n)}},$$

where $\alpha_{j,t}^{(n)}$ and $\beta_{j,k}^{(n)}$ are non negative integers to be determined.

2.11. Lemma. Let $f{:}G \to \mathfrak{B}_t^*$, $1 \leq t \leq 2^n$ (or $f{:}G \to \mathfrak{L}_k^*$, $1 \leq k \leq 3^n$) be a function and $h{:}\mathbb{L}(n) \to \mathfrak{B}_t^*$ (or $h{:}\mathbb{L}(n) \to \mathfrak{L}_k^*$) the homomorphism which extends f. For $1 \leq j \leq n$, the following conditions are equivalent:

(i) $M_j^{(n)} \cap Ker(h) \in \mathbb{E}(M_j^{(n)})$,

(ii) $1 \notin f(G_j)$.

Proof. (i)⇒(ii). If there exists $g \in G_j$ such that $f(g) = 1$, then $1 = \forall\sigma_1 1 = \forall\sigma_1 h(g) = h(\forall\sigma_1 g)$ hence $\forall\sigma_1 g \in Ker(h)$. Since $\forall\sigma_1 g \leq g_j^*$ we have $g_j^* \in Ker(h)$ and $M_j^{(n)} \subseteq Ker(h)$, which contradicts (i).

(ii)⇒(i). If $M_j^{(n)} \subseteq Ker(h)$ then $1 = h(g_j^*) = h(\bigvee_{i=1}^{j} \forall\sigma_1 g_i) = \bigvee_{i=1}^{j} \forall\sigma_1 h(g_i)$ (1). From the above results we know that $\forall\sigma_1 h(g_i) \in \{0,1\}$ (or $\forall\sigma_1 h(g_i) \in \{0,c,1\}$, c center of T^k).

Therefore, by (1) there exists $g_i \in G_j$ such that $1 = \forall\sigma_1 h(g_i)$. Since $\forall\sigma_1 h(g_i) \leq h(g_i)$ we obtain $h(g_i) = 1 = f(g_i)$ which contradicts (ii). □

2.12. Let $H = \{g_{\alpha_1}, g_{\alpha_2}, \ldots, g_{\alpha_k}\} \subseteq G$ and $A = [H]_{\boldsymbol{MMI_3}}$ be the subalgebra of $\mathbb{L}(n)$ generated by H. Then it is easy to see that $A = \mathbb{L}(k)$.

Let $R_{j,t}^{(n)} = \{h \in Epi(\mathbb{L}(n), \mathfrak{B}_t^*)\colon h(G_j) = 1\}$ and $Q_{j,s}^{(n)} = \{h \in Epi(\mathbb{L}(n), \mathfrak{L}_s^*)\colon h(G_j) = 1\}$.

2.13. Lemma. (i) If $1 \leq t \leq 2^{n-j}$, $1 \leq j \leq n$ then $Epi(\mathbb{L}(n-j), \mathfrak{B}_t^*)$ and $R_{j,t}^{(n)}$ are coordinable non empty sets. If $2^{n-j} < t \leq 2^n$, $R_{j,t}^{(n)}$ are empty sets.

(ii) If $1 \leq s \leq 3^{n-j}$, $1 \leq j < n$ then $Epi(\mathbb{L}(n-j), \mathfrak{L}_s^*)$ and $Q_{j,s}^{(n)}$ are coordinable non empty sets. If $3^{n-j} < s \leq 3^n$, $Q_{j,s}^{(n)}$ are empty sets.

Proof. (i) By hypothesis $n - j > 0$. Let $S = [G - G_j]_{\boldsymbol{MMI_3}}$, so by 2.11 $S = \mathbb{L}(n-j)$. As $1 \leq t \leq 2^{n-j}$, by 2.5 $Epi(\mathbb{L}(n-j), \mathfrak{B}_t^*) \neq \emptyset$. Let $h \in Epi(\mathbb{L}(n-j), \mathfrak{B}_t^*)$ and $f = h/(G - G_j)$. Let us consider the map $f'\colon G \to \mathfrak{B}_t^*$ defined by $f'(g) = 1$ if $g \in G_j$, $f'(g) = f(g)$ if $g \in G - G_j$. So f' can be extended to an MMI_3-homomorphism $h'\colon \mathbb{L}(n) \to \mathfrak{B}_t^*$.

Then $[h'(G)]_{\boldsymbol{MMI_3}} \supseteq [h'(G - G_j)]_{\boldsymbol{MMI_3}} = [f(G - G_j)]_{\boldsymbol{MMI_3}} = [h(G - G_j)]_{\boldsymbol{MMI_3}} = \mathfrak{B}_t^*$

and

by construction $h'(G_j) = \{1\}$, so $h' \in R_{j,t}^{(n)}$. The mapping $h \to h'$ is a bijection.

By 2.6 and 2.7 it is obvious that $R_{j,t}^{(n)} = \emptyset$ if $2^{n-j} < t \leq 2^n$.

(ii) It follows as (i). □

2.14. Let $r_{j,t}^{(n)} = |R_{j,t}^{(n)}|$ and $q_{j,s}^{(n)} = |Q_{j,s}^{(n)}|$ be. If $1 \leq j < n$, by lemmas 2.6, 2.7, 2.12 and [16] we have

$r_{j,1}^{(n)} = V_{2^{n-j},1} - 1$,

$r_{j,t}^{(n)} = V_{2^{n-j},t}$, if $1 < t \leq 2^{n-j}$,

$r_{j,t}^{(n)} = 0$, if $2^{n-j} < t \leq 2^n$,

$q_{j,s}^{(n)} = V_{3^{n-j},s} - V_{2^{n-j},s}$, if $1 \leq s \leq 2^{n-j}$,

$q_{j,s}^{(n)} = V_{3^{n-j},s}$, if $2^{n-j} < s \leq 3^{n-j}$,

$q_{j,s}^{(n)} = 0$, if $3^{n-j} < s \leq 3^n$.

Observe that $r_{n,t}^{(n)} = 0$, for all t, $1 \leq t \leq 2^n$ and $q_{n,s}^{(n)} = 0$, for all s, $1 \leq s \leq 3^n$.

2.15. Computation of $|\mathbb{L}(n)|$

Let $M_{j,t}^{(n)} = \{h \in Epi(\mathbb{L}(n), \mathfrak{B}_t^*): 1 \notin h(G_j)\}$, $1 \leq t \leq 2^n$ and $E_{j,s}^{(n)} = \{h \in Epi(\mathbb{L}(n), \mathfrak{L}_s^*): 1 \notin h(G_j)\}$, $1 \leq s \leq 3^n$.

By 2.11 is not difficult to see that

(1) $\alpha_{j,t}^{(n)} = \dfrac{|M_{j,t}^{(n)}|}{|Aut(\mathfrak{B}_t^*)|}$, (2) $\beta_{j,s}^{(n)} = \dfrac{|E_{j,s}^{(n)}|}{|Aut(\mathfrak{L}_s^*)|}$,

where $Aut(\mathfrak{B}_t^*)$ and $Aut(\mathfrak{L}_s^*)$ denote the sets of all automorphisms of $\mathfrak{B}_t^*$ and $\mathfrak{L}_s^*$ respectively. Furthermore $|Aut(\mathfrak{B}_t^*)| = |Aut(\mathfrak{B}_t)| = t!$ and $|Aut(\mathfrak{L}_s^*)| = |Aut(\mathfrak{L}_s)| = s!$ [16].

If $n = 1$, it is easy to see that $\alpha_{1,1}^{(1)} = \alpha_{1,2}^{(1)} = 1$ and $\beta_{1,1}^{(1)} = 1$, $\beta_{1,2}^{(1)} = 2$, $\beta_{1,3}^{(1)} = 1$.

If $n > 1$, we consider the sets

(3) $F_{j,t}^{(n)} = Epi(\mathbb{L}(n), \mathfrak{B}_t^*) - M_{j,t}^{(n)}$,

(4) $F_{k,j,t}^{(n)}(\mathbb{C}) = \{h \in F_{j,t}^{(n)}: \mathbb{C} \subseteq G_j, |\mathbb{C}| = k, h(\mathbb{C}) = \{1\}, 1 \notin h(G_j - \mathbb{C})\}$,

(5) $H_{j,s}^{(n)} = Epi(\mathbb{L}(n), \mathfrak{L}_s^*) - E_{j,s}^{(n)}$,

(6) $H_{k,j,s}^{(n)}(\mathbb{C}) = \{h \in H_{j,s}^{(n)}: \mathbb{C} \subseteq G_j, |\mathbb{C}| = k, h(\mathbb{C}) = \{1\}, 1 \notin h(G_j - \mathbb{C})\}$,

where $\{F_{k,j,t}^{(n)}(\mathbb{C}): \mathbb{C} \subseteq G_j, 1 \leq k \leq j\}$ and $\{H_{k,j,s}^{(n)}(\mathbb{C}): \mathbb{C} \subseteq G_j, 1 \leq k \leq j\}$ is a partition of $F_{j,t}^{(n)}$ and $H_{j,s}^{(n)}$ respectively. Therefore,

(7) $F_{j,t}^{(n)} = \bigcup\limits_{1 \leq k \leq j} \bigcup\limits_{\mathbb{C} \subseteq G_j, |\mathbb{C}| = k} F_{k,j,t}^{(n)}(\mathbb{C})$, $H_{j,s}^{(n)} = \bigcup\limits_{1 \leq k \leq j} \bigcup\limits_{\mathbb{C} \subseteq G_j, |\mathbb{C}| = k} H_{k,j,s}^{(n)}(\mathbb{C})$.

If $\mathbb{C} \subseteq G_j$, $\mathbb{C}' \subseteq G_j$ and $|\mathbb{C}| = |\mathbb{C}'|$ then it is easy to see that

(8) $|F_{k,j,t}^{(n)}(\mathbb{C})| = |F_{k,j,t}^{(n)}(\mathbb{C}')|$, $|H_{k,j,s}^{(n)}(\mathbb{C})| = |H_{k,j,s}^{(n)}(\mathbb{C}')|$.

From (7), (8) and taking into account that $\binom{n}{j}$ is the number of subsets of G_j with k elements, we have

(9) $\quad |F^{(n)}_{j,t}| = \sum_{k=1}^{j} \binom{n}{j} \, |F^{(n)}_{k,j,t}(G_k)|, \qquad |H^{(n)}_{j,s}| = \sum_{k=1}^{j} \binom{n}{j} \, |H^{(n)}_{k,j,s}(G_k)|.$

To compute $|F^{(n)}_{k,j,t}(G_k)| = f^{(n)}_{k,j,t}$ we consider the following cases:

CASE 1. $2^{n-k} < t$.

It follows that $F^{(n)}_{k,j,t}(G_k)$ is empty, so $f^{(n)}_{k,j,t} = 0$.

CASE 2. $t \leq 2^{n-k}$, $1 \leq k \leq j \leq n$.

In this case $F^{(n)}_{k,j,t}(G_k)$ is not empty.

(2.1) $k = j$.

We have $F^{(n)}_{j,j,t}(G_j) = \{h \in Epi(\mathbb{L}(n), \mathfrak{B}^*_t): h(G_j) = \{1\}\} = R^{(n)}_{j,t}$ and so $f^{(n)}_{j,j,t} = r^{(n)}_{j,t}$.

(2.2) $1 \leq k < j$.

If $S = [G_n - G_k]_{\boldsymbol{MMI_3}} \subseteq \mathbb{L}(n)$ then $S = \mathbb{L}(n-k)$. If $h \in F^{(n)}_{k,j,t}(G_k)$ then $h/S = h_1 \in Epi(\mathbb{L}(n-k), \mathfrak{B}^*_t)$ such that $1 \notin h_1(G_j - G_k)$. The mapping $h \to h_1$ is a bijection between $F^{(n)}_{k,j,t}(G_k)$ and $M^{(n-k)}_{j-k,t} = \{h_1 \in Epi(\mathbb{L}(n-k), \mathfrak{B}^*_t)$: $1 \notin h_1(G_j - G_k)\}$. If $m^{(n-k)}_{j-k,t} = |M^{(n-k)}_{j-k,t}|$ then $f^{(n)}_{k,j,t} = m^{(n-k)}_{j-k,t}$.

From cases 1 and 2 it results

$$(10) \quad f^{(n)}_{k,j,t} = \begin{cases} 0 & \text{if } 2^{n-k}, \\ m^{(n-k)}_{j-k,t} & \text{if } 1 \leq k < j \leq n, \ t \leq 2^{n-k}, \\ r^{(n)}_{j,t} & \text{if } k = j, \ 1 \leq j \leq n, \ t \leq 2^{n-j}. \end{cases}$$

From (3), (4), lemmas 2.6 and 2.7 we have

$$m^{(n)}_{j,t} = \begin{cases} V_{2^n,1} - 1 - f^{(n)}_{j,1} \,, & t = 1, \, 1 \leq j \leq n, \\ V_{2^n,t} - f^{(n)}_{j,t} \,, & 1 < t, \, 1 \leq j \leq n. \end{cases}$$

where $m^{(n)}_{j,t} = |M^{(n)}_{j,t}|$. So by (1) $\alpha^{(n)}_{j,t} = \dfrac{m^{(n)}_{j,t}}{t!}$.

To compute $|H^{(n)}_{k,j,s}(G_k)| = h^{(n)}_{k,j,s}$ we consider the following cases:

CASE 1. $3^{n-k} < s$.

It follows that $H^{(n)}_{k,j,s}(G_k)$ is empty, so $h^{(n)}_{k,j,s} = 0$.

CASE 2. $s \leq 3^{n-k}$, $1 \leq k \leq j \leq n$.

In this case $H^{(n)}_{k,j,s}(G_k)$ is not empty.

(2.1) $k = j$.

We have $H^{(n)}_{j,j,s}(G_j) = \{h \in Epi(\mathbb{L}(n), \mathcal{L}^*_s)\colon h(G_j) = \{1\}\} = Q^{(n)}_{j,s}$ and so $h^{(n)}_{j,j,s} = q^{(n)}_{j,s}$.

(2.2) $1 \leq k < j$.

If $S = [G_n - G_k]_{\boldsymbol{MMI_3}} \subseteq \mathbb{L}(n)$ then $S = \mathbb{L}(n-k)$. If $h \in H^{(n)}_{k,j,s}(G_k)$ then $h/S = h_1 \in Epi(\mathbb{L}(n-k), \mathcal{L}^*_s)$ such that $1 \notin h_1(G_j - G_k)$. The mapping $h \to h_1$ is a bijection between $H^{(n)}_{k,j,s}(G_k)$ and $E^{(n-k)}_{j-k,s} = \{h_1 \in Epi(\mathbb{L}(n-k), \mathcal{L}^*_s)\colon 1 \notin h_1(G_j - G_k)\}$. If $e^{(n-k)}_{j-k,s} = |E^{(n-k)}_{j-k,s}|$ then $h^{(n)}_{k,j,s} = e^{(n-k)}_{j-k,s}$.

From cases 1 and 2 it results

$$(11)\quad h^{(n)}_{k,j,s} = \begin{cases} 0 & \text{if } 3^{n-k} < s, \\ e^{(n-k)}_{j-k,s} & \text{if } 1 \leq k < j \leq n,\ s \leq 3^{n-k}, \\ q^{(n)}_{j,s} & \text{if } k = j,\ 1 \leq j \leq n,\ s \leq 3^{n-j}. \end{cases}$$

From (5),(6), lemmas 2.6 and 2.7 we have $e^{(n)}_{j,s} = V_{3^n,s} - V_{2^n,s} - h^{(n)}_{j,s}$, $1 \leq s \leq 3^n$, $1 \leq j \leq n$, where $h^{(n)}_{j,s} = |H^{(n)}_{j,s}|$. So by (2) $\beta^{(n)}_{j,s} = \dfrac{e^{(n)}_{j,s}}{s!}$.

Finally, if $n > 1$

$$(12)\quad |\mathbb{L}(n)| = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \prod_{t=1}^{2^n} (2^t)^{\alpha^{(n)}_{j,t}} \prod_{s=1}^{3^n} (3^s)^{\beta^{(n)}_{j,s}}.$$

It is easy to see that $|\mathbb{L}(1)| = 2^3 \cdot 3^8$.

From (12) it results $|\mathbb{L}(2)| = 2^{25} \cdot 3^{2096}(2^4 \cdot 3^{260} - 1)$.

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