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ON SPECTRAL DIRICHLET SERIES

A theorem of Åke Pleijel for curved polygons

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ABSTRACT

In his interesting paper,

"Can one hear the shape of a drum?", Am. Math. Monthly **73** (1966), no. 4, 1-23,

M. Kac called the attention to a theorem by Å. Pleijel,

"A study of certain Green's functions with applications in the theory of vibrating membranes", Arkiv för Matematik **2** (1954), nr. 29, 553-569.

The paper by the authors,

"Remarks on a theorem of Å. Pleijel and related topics, I, *Behavior of the eigenvalues of classical boundary problems in the plane*", Notas de Álgebra y Análisis #19, INMABB-CONICET, Universidad Nacional del Sur, Bahía Blanca, Argentina, 2005, MR2157976 (2006h:35196),

was an attempt to reduce the boundary requirements of the above-mentioned work. The following result was obtained:

Theorem. If the plane Jordan region D of area $|D|$ has a C^2 boundary J of length $\langle J \rangle$ and $\{\lambda_n\}$ is the set of eigenvalues of the Dirichlet problem for the Laplacian in D then it holds on Re $z > 1$: $\sum_1^\infty \lambda_n(D)^{-z} = \frac{|D|}{4\pi}\frac{1}{z-1} - \frac{\langle J \rangle}{8\pi}\frac{1}{z-1/2} + g(z)$, where $g(z)$ is a holomorphic function on Re $z > 0$.

In the present paper, we allow J to have a finite number of corners, that is, J is the boundary of a region D called "curved polygon", and obtain the same result. In our approach, it is crucial the use of a refinement of the implicit function theorem to which we devote all §**5**.

RESUMEN

Al sumario precedente podemos agregar que el teorema de Å. Pleijel en cuestión parece ser un resultado independiente de la regularidad del contorno de la región en el caso del problema de Dirichlet y podría conjeturarse que es válido para regiones de Jordan planas arbitrarias. En el trabajo presente vemos que es válido para regiones con contorno C^2 salvo por un número finito de puntos excepcionales de ese contorno, en particular, para polígonos convexos. Mediante un proceso límite descripto en el Apéndice 2 §**8**, puede demostrarse que el resultado es válido para otras regiones y es razonable pensar que entre ellas se encuentran los óvalos, que son regiones cuyo contorno puede tener infinitos puntos sin derivada. Quizá debamos agregar que el método al que hemos recurrido para demostrar el teorema, y que hemos utilizado en otras ocasiones, depende esencialmente de la parametrización del contorno.

ON SPECTRAL DIRICHLET SERIES

A theorem of Å. Pleijel for curved polygons

by

Agnes Benedek and Rafael Panzone

1. PRELIMINARIES. In most cases of this paper $D \subset R^2$ is a C^2 Jordan region or a C^2 curved polygon. That is, $J = \partial D$ is a Jordan curve defined by means of functions $y_1(s), y_2(s) \in C^2$ with a tangent of length one at each point in the first case or J has a finite number of exceptional points, the corners, in the second one. The precise definitions are in the next paragraphs. The following result holds (cf. [P], [K], [Z]):

THEOREM 1. If the Jordan region D of area $|D|$ has a C^2 boundary J of length $\langle J \rangle$ and $\{\lambda_n\}$ is the set of eigenvalues of the Dirichlet problem for the Laplacian in D then it holds on $\mathrm{Re}\, z > 1$: $\sum_1^\infty \lambda_n(D)^{-z} = \frac{|D|}{4\pi}\frac{1}{z-1} - \frac{\langle J \rangle}{8\pi}\frac{1}{z-1/2} + g(z)$, where $g(z)$ is a holomorphic function on $\mathrm{Re}\, z > 0$**.**

2. LOCAL COORDINATES. To deal with regular regions it is convenient to introduce local coordinates around the boundary in the following way where we follow Å. Pleijel. Let s be the parameter arc length of the C^2-Jordan curve J, (first case, see Fig. 3), starting at the origin O. We assume that J is positively oriented.

The points in J will be denoted by $y = y(s) = (y_1(s), y_2(s))$, $y(s) = y(s + \langle J \rangle)$, with $\langle J \rangle = \mathrm{length}\, J$. We assume that $y_i(s) \in C^2([0, \langle J \rangle])$, $y_i(0) = y_i(\langle J \rangle)$. Let $\vec{n} = \vec{n}(s) = (n_1(s), n_2(s)) = (-\dot{y}_2(s), \dot{y}_1(s)) = n_i(s)$ be the interior normal versor at a regular boundary point $y(s)$. If J is an oriented closed arc we suppose that it is contained in a C^2 open arc L and so we can include the extreme points among the regular ones.

Suppose $1 \geq \delta > 0$ and let I be an interval $I = [a \leq s \leq b] \subsetneqq [0, \langle J \rangle]$ such that $y(s) \in J$ is a regular point for $s \in I$.

Let us *define* the map T. $T: (s,t) \to x(s,t) := y(s) + t\vec{n}(s)$, from the rectangle $I \times (-\delta, \delta)$ to the strip $J_\delta := \{x : \mathrm{dist}(x, J) < \delta\}$, (Figs. 3 and 4 for the first case).

$\xi := (\xi_1, \xi_2)$ denotes a point of the rectangle $I \times (-\delta, \delta)$ and $x = (x_1, x_2)$ represents its image in the strip J_δ. Then $T: \xi \to x$ is written as

$$T: \xi = (\xi_1, \xi_2) \to x = (x_1, x_2) = \big(y_1(\xi_1), y_2(\xi_1)\big) + \xi_2(n_1(\xi_1), n_2(\xi_1)),$$

$0 \le \xi_1 = s < \langle I \rangle$, $|\xi_2| < \delta$. Thus, if $\xi_2 = 0$ then $x \in J$. Given $\eta = (\eta_1, 0)$ *its image will be also written as* $y = y(\eta_1)$ to underline the fact that it is in *J*. For δ sufficiently small the map T of the rectangle $I \times (-\delta, \delta) = \{(\xi_1, \xi_2): a \le \xi_1 \le b, |\xi_2| < \delta\}$ is a local homeomorphism onto its image.

In fact, since by hypothesis $y(s) \in C^2$ for $s \in [a, b]$, *T* is a C^1 map and ($\xi_1 = s$) it can be written as:

$$(1) \qquad T(\xi) = \begin{cases} x_1(\xi) = y_1(\xi_1) - \xi_2 \dot{y}_2(\xi_1) \\ x_2(\xi) = y_2(\xi_1) + \xi_2 \dot{y}_1(\xi_1) \end{cases}.$$

Since ξ_1 is the arclength, the interior normal $\vec{n} = \vec{n}(\xi_1) = \left(n_1(\xi_1), n_2(\xi_1)\right) = (-\dot{y}_2(\xi_1), \dot{y}_1(\xi_1))$ is such that $|\vec{n}| = 1$, (cf. [G], Ch. 2).

Its jacobian *B* is the absolute value of the determinant of the jacobian matrix,

$$(1') \qquad \frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)} = \begin{vmatrix} \dot{y}_1(\xi_1) - \xi_2 \ddot{y}_2(\xi_1) & -\dot{y}_2(\xi_1) \\ \dot{y}_2(\xi_1) + \xi_2 \ddot{y}_1(\xi_1) & \dot{y}_1(\xi_1) \end{vmatrix} = 1 - \xi_2 c(\xi_1).$$

$c(\xi_1) = \ddot{y}_2(\xi_1)\dot{y}_1(\xi_1) - \ddot{y}_1(\xi_1)\dot{y}_2(\xi_1)$ is the curvature of *J* at the point $T(\xi_1, 0)$.

For δ sufficiently small $1 - \xi_2 c(\xi_1) > 0$ whenever $|\xi_2| < \delta$. In this case

$0 < B = \frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)} = 1 - \xi_2 c(\xi_1)$ and *T* is *locally* a homeomorphism. Besides,

$$(1'') \qquad \left(\frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)}\right)^{-1} = \left(1 - \xi_2 c(\xi_1)\right)^{-1} =$$

$$= \begin{vmatrix} \dot{y}_1(\xi_1)/(1 - \xi_2 c(\xi_1)) & \dot{y}_2(\xi_1)/(1 - \xi_2 c(\xi_1)) \\ (-\dot{y}_2(\xi_1) - \xi_2 \ddot{y}_1(\xi_1))/(1 - \xi_2 c(\xi_1)) & (\dot{y}_1(\xi_1) - \xi_2 \ddot{y}_2(\xi_1))/(1 - \xi_2 c(\xi_1)) \end{vmatrix},$$

and $|c(\xi_1)| = |\ddot{y}(\xi_1)|$.

3. Irregular points at the boundary. See Figure 6. In this example the Jordan curve $J = \partial D$ is the union of three C^2 closed arcs. The continuous contour of the cap is C^2 with the exception of three singularities. Points 2 and 3 are of the same nature: the first derivative has a jump at them. At point 1 only the second derivative has a jump.

DEFINITION 1. A point $y(s) \in J$, a Jordan curve, will be called *regular* when its (continuous) components $y_i(\xi_1)$ have first and second continuous derivatives in a neighborhood of $\xi_1 = s$. Points which are not regular will be called *irregular***.**

Among them there are the *singular* ones which are isolated irregular points. Precisely,

DEFINITION 2. A point $y(s_0)$ is called *singular* if it is an isolated irregular point such that the function $\ddot{y}_2(\xi_1)$ does not exist at $\xi_1 = s_0$ but

(*) $\ddot{y}_2(\xi_1)$ has a finite limit at s_0+ and s_0- and there exist tangent vectors of length one at $s_0 \pm$.

If all the irregular points are singular then $y(s)$ describes a rectifiable Jordan curve union of a finite set of closed C^2 Jordan arcs. All ordinary polygons belong to this category.

DEFINITION 3. A Jordan domain D with a boundary J (positively oriented) will be called a proper curved polygon if J has a finite number of singular points with no reëntring angles at them. That is, the interior angles at singular points

(**) are positive and less than π.

In this case, if $y(s)$ is singular at s_1 we shall say that J has a (proper) corner at $y(s_1)$ and that s_1 is a singular parameter point. σ shall denote the finite set of singular parameter points. Assuming that 0 is a regular point of J, $\sigma = \{s_1, \dots, s_{|\sigma|}\}$, the values of the parameter s written according to their order of magnitude.

A_i shall denote the open parameter interval between s_i and s_{i+1} and J_i the closed side of J between $y(s_i)$ and $y(s_{i+1})$, (of course, $s_{|\sigma|+1} = s_1$).

We shall accept that for all i, (cf. [G], Th. 2.13),

(***) J_i can be continued as L_i, an open Jordan arc, and with the *same properties* of J_i, along an open interval $A_i{}' \supset \overline{A_i}$.

Ordinary convex polygons are proper curved polygons. In what follows we give a proof of a generalized version of Theorem 1 §1 that includes curved polygons.

DEFINITION 4. A curved polygon is a C^2 Jordan curve, ($|\sigma| = 0$), or a piecewise C^2 Jordan curve with $|\sigma|$ proper corners, $0 < |\sigma| < \infty$.

In case that there is only one singular parameter point, to avoid that $y(s_1) = y(s_2)$, we shall add any regular point as s_2 (i.e., we add a phony corner $y(s_2)$). Therefore, we may suppose that either $|\sigma| = 0$ or $|\sigma| > 1$.

4. Homeomorphisms. We write $J_n = T(\overline{A_n})$. Its *positive side* by definition will be the one in contact with D. Let us see, perhaps by using a smaller δ, (cf. §**2)**, that for all n, $n = 1, \dots, |\sigma|$, our T is *globally* a homeomorphism from

$$\overline{A_n} \times (-\delta, \delta) \ \text{ onto } \ J_n^\delta := T\big(\overline{A_n} \times (-\delta, \delta)\big).$$

DEFINITION 1. $c(J) = \sup |c(\xi_1)| = \sup |\ddot{y}(\xi_1)|$, $R(J) = \frac{1}{c(J)}$.

Then $c(J) \in [0, \infty)$ and $c(J) \in (0, \infty)$ if $|\sigma| = 0$. In general, $R \in (0,\infty]$.

Fig. 1

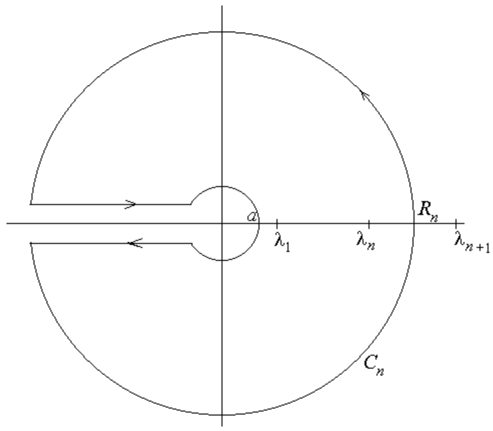


Fig.2

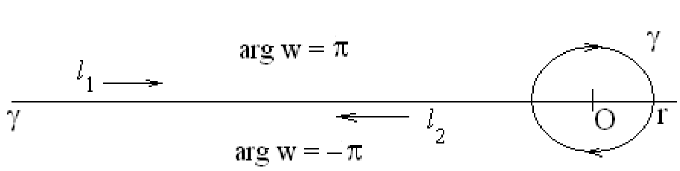
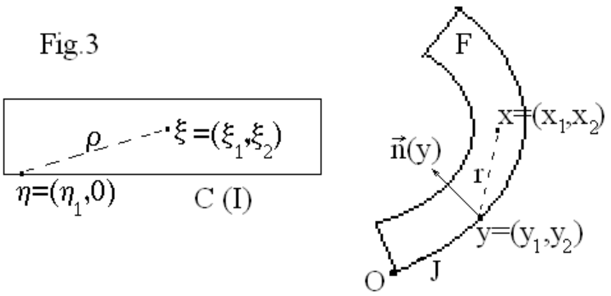


Fig.3



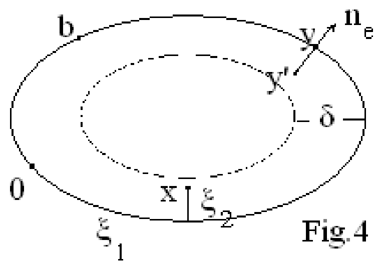


Fig.4

Fig.5

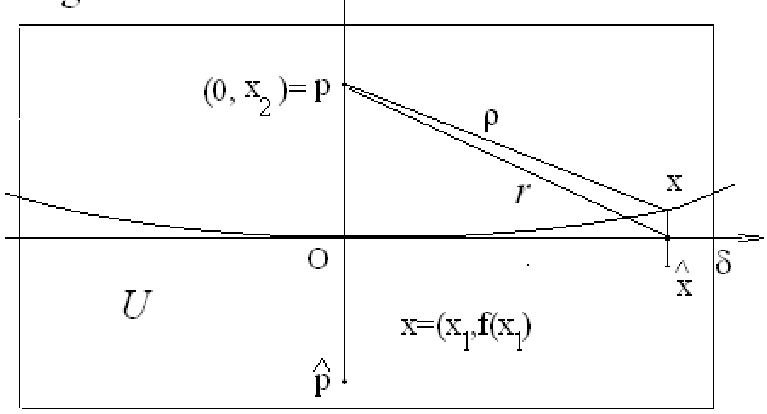
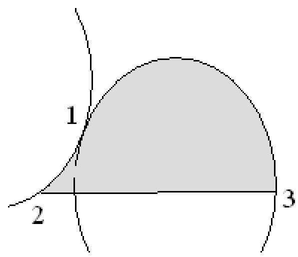


Fig. 6



LEMMA 1. There exists a $\delta_0(n) > 0$ such that for any $h \in (0, \delta_0(n))$, T is a homeomorphism from $\overline{A_n} \times (-h, h)$ onto its image J_n^h.

PROOF. Let $C = \{(t_i - \varepsilon_i, t_i + \varepsilon_i) : i = 1, \dots, N\}$, $t_i \in \overline{A_n}$, $\varepsilon_i > 0$, be a finite covering of $\overline{A_n}$ with the property that for some $\delta_i > 0$, $\delta_i < R/\sqrt{2}$, (cf. §**2**),

(#) T is a homeomorphism from $\big((t_i - 2\varepsilon_i, t_i + 2\varepsilon_i) \cap [s_n, s_{n+1}]\big) \times (-\delta_i, \delta_i)$ onto

$$U \coloneqq T\Big(\big((t_i - 2\varepsilon_i, t_i + 2\varepsilon_i) \cap [s_n, s_{n+1}]\big) \times (-\delta_i, \delta_i)\Big),$$

a relative neighborhood of $y(t_i)$.

Let $\varepsilon \coloneqq \min\{\varepsilon_i : i = 1, \dots, N\}$ and $\delta \coloneqq \min\{\delta_i : i = 1, \dots, N\}$.

Call $m \coloneqq \min_{|u-v| \geq \varepsilon} \{\, |y(u) - y(v)| : u, v \in \bar{A}_n \,\}$. Then $m > 0$.

We claim that the lemma holds for

(1) $$h < \delta_0(n) = \min(\delta, \tfrac{m}{2}) < R/\sqrt{2}.$$

In fact, suppose $T: \big(\overline{A_n} \times (-h, h)\big) \to T\big(\overline{A_n} \times (-h, h)\big)$ is not one to one. Then there exists $x \in J_n^h$ such that $x = T(u, t_u) = T(v, t_v)$. Now, if $u \in (t_i - \varepsilon_i, t_i + \varepsilon_i)$, then in view of (#) necessarily $v \,\overline{\in}\, (t_i - 2\varepsilon_i, t_i + 2\varepsilon_i)$. Therefore, $|u - v| \geq \varepsilon_i \geq \varepsilon$.

Using the definition of T, $0 = |x - x| \geq |y(u) - y(v)| - 2h \geq m - 2h > 0$, a contradiction, QED.

Lemma 1 holds for all n and for all $h \in (0, \delta_1)$ where

$$\delta_1 = \min \delta_0(n).$$

DEFINITION 2. Given $x = y(\xi_1) + \xi_2 n_i(\xi_1) \in J_n^h \cap D$, $\xi_1 \in A_n$, $\hat{x}$ denotes the *symmetric point of x with respect to J_n* : $\hat{x}(\xi_1, \xi_2) \coloneqq y(\xi_1) - \xi_2 n_i(\xi_1)$.

If $z = y(\tau_1) + \tau_2 n_i(\tau_1)$ also belongs to $J_n^h \cap D$ then $\hat{x}(\xi_1, \xi_2) = \hat{x}(\tau_1, \tau_2)$ implies $(\xi_1, \xi_2) = (\tau_1, \tau_2)$.

Let us put together some symbols we used and others that we shall use in what follows.

NOTATION. $J_n = T(\overline{A_n} \times \{0\}) = T(\overline{A_n})$. For $0 < h \leq \delta_1$, $J_{n,h} = \{x : \mathrm{dist}(x, J_n) < h\}$, $J_h = \{x : \mathrm{dist}(x, J) < h\}$, $F_h = J_h \cap D$, but $J_n^h \coloneqq T\big(\overline{A_n} \times (-h, h)\big)$. $F_n^h = J_n^h \cap D$ and $F_{n,h} = J_{n,h} \cap D$. If $|\sigma| = 0$, $J_0 = J$, $F_{0,h} \equiv F_h = J_h \cap D$ and $J_0^h = T\big([0, \langle J \rangle] \times (-h, h)\big)$.

(J_h is an open set containing J, sometimes called a Minkowski neighborhood of J.)

DEFINITION 3. $\tilde{F}_n^h \coloneqq \{x \in F_n^h : d(x, J_n) = d(x, J)\}$.

4.1. On corners. By definition of a curved polygon, if $|\sigma| \geq 2$, it is possible to choose a positive δ_1, let us call it $h = h(J)$, so small that for all n and all $x \in F_n^h = J_n^h \cap D$ it holds the

LEMMA 1. i) The following relation is verified for two indices n, m at most and in such a way that if this happens then n and m are consecutive,

$$x \in J_n^h \cap J_m^h \cap D,$$

ii) $$T\big(A_n \times (-h, 0)\big) \cap J = \emptyset.$$

But let us prove first the auxiliary

PROPOSITION 1. $J_{0,h} \supset J_0^h$. Idem for J_n: $J_{n,h} \supset J_n^h$.

In fact, the open disk with radius h and center $y(s)$ contains the segment $(y(s) - n_i(s)h, y(s) + n_i(s)h)$, QED.

PROOF OF LEMMA 1 AND NEXT FORMULA (1) (cf. § 2). Let

$$m \coloneqq \min\{|y(s_i) - y(s_j)| : i \neq j\}$$

be the minimum distance between two vertices of J. Then, the disks $\bar{B}_r\left(y(s_j)\right)$, $j = 1, \dots, |\sigma|$, are pairwise disjoint whenever $r < m/2$.

We suppose without loss of generality that (0,0) is the initial extreme of $J_1 \subset L_1$ corresponding to $\xi_1 = 0$. After a rotation of the coordinate system we can assume that

$\dot{y}(0) = \left(\dot{y}_1(\xi_1), \dot{y}_2(\xi_1)\right)\Big|_{\xi_1 = 0} = (1{,}0).$

Therefore, in a neighborhood of the origin y_1 is a strictly increasing function of ξ_1 because of $\frac{dy_1}{d\xi_1} > 0$. We consider this function only on $0 < \xi_1 < m/2$. There $\dot{y}_1$ and $\ddot{y}_1$ are continuous functions since there is no $s_i \in \sigma$ in this interval.

Because of $\frac{dy_1}{d\xi_1} > 0$, we can eliminate the parameter ξ_1 from the equation of the curve $y = y(\xi_1)$ obtaining $y_2 = f(y_1)$ with $f(0) = 0$, $f'(0) = 0$, $f'(y_1) = \frac{dy_2}{dy_1} = \frac{\dot{y}_2(\xi_1)}{\dot{y}_1(\xi_1)}$ and

$$f''(y_1) = \frac{d(\frac{dy_2}{dy_1})}{dy_1} = \frac{d\xi_1}{dy_1}\left(\frac{d}{d\xi_1}\left(\frac{\dot{y}_2(\xi_1)}{\dot{y}_1(\xi_1)}\right)\right) = \frac{c(\xi_1)}{\dot{y}_1(\xi_1)^2}\frac{d\xi_1}{dy_1} = \frac{c(\xi_1)}{\dot{y}_1(\xi_1)^3}.$$

Therefore, if $\dot{y}_1(\xi_1) > 1/2$ then $|f''(y_1)| < 8c(J)$ and $f(y_1) = O(1)y_1^2$, where $|O(1)| < 4c(J)$. We obtained:

if $\dot{y}_1(\xi_1) > \frac{1}{2}$ whenever $0 \leq \xi_1 < \varepsilon < m/2$ then

for $0 < y_1 < y_1(\varepsilon)$ it holds that $|f(y_1)| < 4c(J)y_1^2$.

Where does $\dot{y}_1(\xi_1) > 1/2$ hold? From $\dot{y}_1(\xi_1) = 1 + \int_0^{\xi_1} \ddot{y}_1(t)\,dt > 1 - c(J)\xi_1$ we see that it is enough to ask that $0 < \xi_1 < 1/(2c(J))$ together with $0 < \xi_1 < m/2$ to make sure that $\dot{y}_1$ is a continuous function greater than ½.

That is, if $y_1 < y_1(\varepsilon)$, $\varepsilon := \min(1/(2c(J)), m/2)$ then

(1) $$|y_2| = |f(y_1)| < 4y_1^2 c(J).$$

On the other hand, we have, $2[y_1\dot{y}_1 + y_2\dot{y}_2] = \frac{d(|y|^2)}{d\xi_1} \equiv (\dot{|y|^2}) = 2\,(y \times \dot{y})$,

$$\frac{d^2(|y|^2)}{d{\xi_1}^2} \equiv (\ddot{|y|^2}) = 2[1 + y_1\ddot{y}_1 + y_2\ddot{y}_2] = 2[1 + y \times \ddot{y}] \geq 2(1 - |y|c(J)).$$

In consequence, $(\ddot{|y|^2})$ is positive and $(\dot{|y|^2})$ is increasing if $|y| < 1/c(J)$.

Since $(\dot{|y|^2})(0) = 0$, $(\dot{|y|^2})$ is positive as long as the continuous curve $y(s)$ that starts at 0 does not reach $\partial B_{1/c(J)}$. This is the case for $0 \leq s < \varepsilon = \min(1/(2c(J)), m/2)$, since $|y(s)| \leq s < 1/(2c(J))$. Therefore,

(2) $|y(s)|$ is strictly increasing as long as $0 \leq s < \varepsilon$.

Thus, the curve

(†) $\{y(\xi) : 0 < \xi \leq \varepsilon\}$ exits *any* disk $\bar{B}_r(0,0)$ if $0 < r \leq y_1(\varepsilon)$ and does not re-enter in it.

However, $\{y(\xi) : 0 < \xi\}$ could re-enter for $\xi > \varepsilon$. Analogously, the curve $\{y(\xi) : -\varepsilon \leq \xi < 0\}$ enters in any $\bar{B}_r(0,0)$ if $0 < r \leq y_1(\varepsilon)$ and does not exit.

Let $0 < r < \varepsilon$ and $P(i)$ be the first point where the curve exits $\bar{B}_r(y(s_i))$ and $Q(i)$ the last one where the curve $J = \{y(s_i + \xi) : 0 \leq \xi < \langle J \rangle\}$ enters $\bar{B}_r(y(s_i))$. Let $K(i)$ be the piece of the curve between $P(i+1)$ and $Q(i)$. Then $\overline{K(i)} \cap \overline{J_i} = \overline{K(i)} \cap J_i = \emptyset$.

Defining $h_i := \text{dist}\,(K(i), J_i)$, any number h such that $0 < h < h_i/2$ for all i satisfies $J_{i,h} \cap \bar{K}_{i,h} = \emptyset$. Then, i) follows from Proposition 1.

ii) Since the angles of a curved polygon are by definition smaller than π, one can choose the points $P(i+1)$ and $Q(i)$ so near to their respective corners, that the arcs of J from $Q(i)$ to $y(s_i)$ and from $y(s_{i+1})$ to $P(i+1)$ do not intersect $T\big(A_i \times (-\delta_1, 0)\big)$. Then the previous choice of h will also satisfy ii), QED.

We shall assume that h is such that Lemma 1 holds. This requirement and new ones will be satisfied with a smaller h. We shall keep track of the parameter h, the "width" of J_n^h, as soon as changes are produced. If D is a curved polygon with boundary J and $h = h(J)$ is *small enough* we have the following Propositions together with the preceding results.

PROPOSITION 2. $p = y(s) + n_i(s)v \in J_0^h \Rightarrow |v| = \mathrm{dist}(p, J_0)$ and s is uniquely determined. Moreover, if $x \in J_m,\ m \neq i, i \pm 1$, then $|x - p|, |x - \hat{p}| \geq h$**.**

PROOF. Suppose that $p \in D$. If $\mathrm{dist}(p, J_0) = \beta \leq v\ (< h)$ then the distance is realized at a point $y(\alpha) \in J_0$ and $y(\alpha) + n_i(\alpha)t$ is orthogonal to J_0. Thus, $p = y(\alpha) + n_i(\alpha)\beta$. From $y(s) + n_i(s)t = y(\alpha) + n_i(\alpha)\beta$ and Lemma 1, §**4**, we obtain $s = \alpha, t = \beta$, QED.

PROPOSITION 3. For $J = J_0$ it holds that $J^h = J_h$**.**

It remains to prove that $J^h \supset J_h$. But this follows from Proposition 2.

PROPOSITION 4. In case of a proper curved polygon D,

$$\cup\, \tilde{F}_n^h = F_h = \{x \in D : \mathrm{dist}(x, J) < h\} = J_h \cap D.$$

PROOF. Since $J_n^h \subset J_{n,h}$, we have $\cup\, \tilde{F}_n^h \subset F_h$. If $x \in F_h$ and $\mathrm{dist}(x, J) = |x - y|$ with $y \in J$ there are two possibilities: 1) y is an interior point of some J_i or 2) y is a corner of J, say $y = J_i \cap J_{i+1}$. In case 1), as in Prop. 3, $x \in F_i^h$. Therefore, $x \in \tilde{F}_i^h$. In case 2), the interior angle at y must be $\geq \pi$, in contradiction with the hypothesis that D is a proper curved polygon, so this case cannot occur, QED.

4.2. The curve $\boldsymbol{P}$. Let us fix our attention on the corner $\mathbf{O} = y(s_n) = T(\bar{A}_{n-1}) \cap T(\bar{A}_n)$.

Inside a neighborhood $B_r(\mathbf{O})$, the curves, defined by the functions of $\tau_1, \sigma \in [0, \varepsilon]$,

$$T(s_n - \tau_1, \tau_2) \quad \text{and} \quad T(s_n + \sigma, \tau_2),$$

intersect in only one point $P(\tau_1)$ for any $\tau_2 \in [0, \delta]$ whenever ε, δ are sufficiently small and $\delta < h$. This point verifies $\mathrm{dist}(P(\tau_1), J_{n-1}) = \mathrm{dist}(P(\tau_1), J_n) = \tau_2$.

If $\tilde{F}_{n-1}^\delta \cap \tilde{F}_n^\delta \ni X$ then $X = P(\tau_1)$ for a certain τ_1.

Now, the last argument in the proof of Lemma 1 §**4.1** can be used again; from (†), §**4.1,** and the fact that $T(s_n - \cdot, \tau_2) \xrightarrow[\tau_2 \to 0]{\cdot} J_{n-1}$, it is possible to show that if δ and ε are sufficiently small then $P([0, \varepsilon]) = \tilde{F}_{n-1}^\delta \cap \tilde{F}_n^\delta$. Besides, $P(\tau), \tau \in (0, \varepsilon)$, is a rectifiable Jordan arc since $|P(\tau_1) - P(\tau_2)| \leq k|\tau_1 - \tau_2|$, (cf. §**5.1**. We return to the properties of the curve P in Appendix 1 §**7**.) Then, its plane Lebesgue measure is zero.

As a matter of fact, $\varepsilon = \varepsilon(n),\ \delta = \delta(n)$. Let us define h as a positive number less than the minimum of the previous h and $\min_{n=1,\dots,|\sigma|} \delta(n)$.

4.3. The points $\hat{\boldsymbol{p}}$, $\boldsymbol{p} \in \boldsymbol{F_n^h}$. If h is chosen as in Lemma 1 **§4.1** then ii) of that lemma implies that $\hat{p} \notin \overline{D}$ for $p \in \tilde{F}_n^h$.

4.4. Minkowski neighborhoods. Suppose $A = \{y(s) : 0 \le s \le L\}$ is a Jordan arc with $\dot{y}(s), \ddot{y}(s)$ continuous, $|\dot{y}(s)| = 1$, $|\ddot{y}(s)| \le c$ in $0 \le s \le L$.

Let $n(s) \coloneqq (-\dot{y}_2(s), \dot{y}_1(s))$ be the normal vector to A at $y(s)$. Then there exists a $\delta > 0$ such that the map

$T(s,t) \coloneqq y(s) + tn(s)$ is a homeomorphism from $[0, L] \times (-\delta, \delta)$ onto

$A^\delta \coloneqq T\big([0, L] \times (-\delta, \delta)\big)$ as we have seen in §**4**. If $A_\delta \coloneqq \{x : \mathrm{dist}(x, A) < \delta\}$, then

(9) $$A_\delta = A^\delta \cup B_\delta(y(0)) \cup B_\delta(y(L)).$$

PROOF of (9). Since obviously $A_\delta \supset A^\delta$ we get the inclusion $\supset$. To prove $\subset$ observe that if $x \in A_\delta$ and $t \coloneqq \mathrm{dist}(x, A) = |x - y(s_0)| < \delta$, then there are two possibilities.

1) $0 < s_o < L$, 2) s_0 is equal to 0 or L.

In case 1) $(x - y(s_0))$ is perpendicular to A, therefore $x = y(s_0) \pm tn(s_0) \in A^\delta$.

In case 2) $x \in B_\delta(y(0)) \cup B_\delta(y(L))$, QED.

Therefore, $J_{n,h} = J_n^h \cup B_h(y(s_n)) \cup B_h(y(s_{n+1}))$.

NB. For a $\delta \le h$ we have $\forall n$, $J_{n,\delta} \subset L_n^h$. Therefore, $J_{n,\delta} \subset L_n^\delta$. Thus, we can always suppose that h is so small that for any n, $J_{n,h} \subset L_n^h$. (Recall that we supposed that $h \le c(J)^{-1}$).

4.5. $|\boldsymbol{f(x_1(s))}| \le \boldsymbol{4c(J)x_1(s)^2}$**.** Given $p \in F_n^h$, let U be a neighborhood of $O \in J_n$, O the mid-point of the segment $p\hat{p}$ (see Fig. 5 and **NOTATION**). Let us take $x \in J_n \cap U$ and U and h such that for W a bounded interval around 0, T is a homeomorphism from $W \times (-h, h)$ onto

(1) $$T\big(W \times (-h, h)\big) \supset U,$$

(see (***) §**3**). After a translation and a rotation, the old coordinates x_1, x_2 are transformed in such a way that in the new coordinates the equations of J_n, assuming that $s(0) = 0$, are of the form:

(2) $$\begin{cases} x_1(s) = \big(y_1(s) - y_1(0)\big)\dot{y}_1(0) + \big(y_2(s) - y_2(0)\big)\dot{y}_2(0) \\ x_2(s) = -\big(y_1(s) - y_1(0)\big)\dot{y}_2(0) + \big(y_2(s) - y_2(0)\big)\dot{y}_1(0) \end{cases}$$

and $p = (0, x_2)$, $\hat{p} = (0, -x_2)$. x_2 is a positive number since p is in D.

This system of coordinates is similar to the one used in **§4.1** when proving inequality (1) except for the fact that now 0 is not a corner point but the regular point $(p + \hat{p})/2$.

Let us define as in §**4.1**: $f(x_1) = x_2(s(x_1))$, in a neighborhood of 0. Then, the same reasoning that led to (1) §**4.1** now yields the next Lemma.

LEMMA 1. $|f(x_1)| \leq 4c(J)x_1^2$ whenever $x_1(m_1) < x_1 < x_1(m_2)$, where

(3) $m_1 := \max\{s_n, -1/(2c(J))\}$ and $m_2 := \min\{s_{n+1}, 1/(2c(J))\}$**.**

Let us call $\rho := \inf(|x-p|, |x-\hat{p}|)$, $r := \sqrt{x_1^2 + x_2^2}$, (Fig.5). Since $p = (0, x_2)$ we have

$$\rho^2 \geq x_1^2 + (x_2 - |f(x_1)|)^2 \geq x_1^2 + \frac{x_2^2}{2} - f^2(x_1).$$

It follows that $\rho^2 \geq x_1^2 + \frac{x_2^2}{2} - 16c(J)^2 x_1^4$. Then, if $|x_1| \leq 1/\left(4\sqrt{2}c(J)\right)$,

(4) $$\rho^2 \geq (x_1^2 + x_2^2)/2.$$

That is, $\rho \geq r/\sqrt{2}$ and we have the following lemma.

LEMMA 2. Let $x(s) \in J_n$, $|x_1(s)| \leq 1/\left(4\sqrt{2}c(J)\right)$, $s_n < 0 < s_{n+1}$. Then,

$$\rho/r \geq 1/\sqrt{2} \text{ and } |f(x_1(s))| \leq 4c(J)x_1(s)^2.$$

NB. We used in this paragraph the same h as in the preceding Note because of (1) §**5.1.**

5. On the implicit function theorem. Lemma 1 §**4** is an illuminating result but it is difficult to think that a bound for the value of h could depend on m, that is, upon the auxiliary covering C. We shall get rid of this limitation.

The implicit function theorem for $x, y \in R^n$, $F(x,y) \equiv x - f(y)$, $f \in C^1$ on an open neighborhood E of 0, $f(0) = 0$, $\det\frac{\partial f}{\partial y} \neq 0$ on E, says that there exist ε and γ such that $x = f(y)$ has on $|x| < \gamma$ a unique continuous solution $y = g(x)$ such that $|y| < \varepsilon$. Moreover, $g(0) = 0$, $g \in C^1$ on $|x| < \gamma$ and $g(|x| < \gamma)$ is a neighborhood of $y = 0$.

This result can be rewritten in the following more precise way, (see Ważewski's Th., [H] Ch. X).

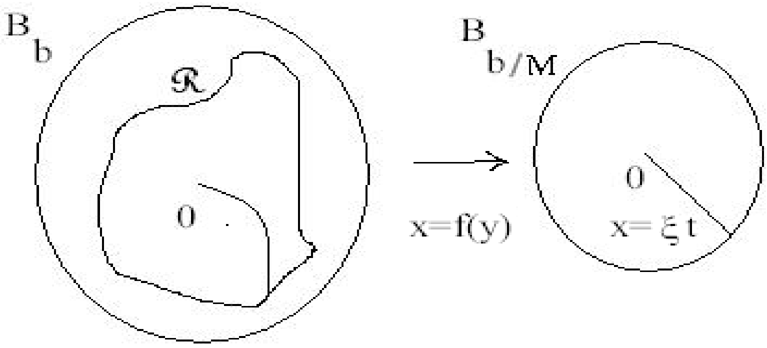
THEOREM 1. Let $x, y \in R^n$, E an open neighborhood of $y = 0$ and for $y \in E$, $x = f(y) \in C^1, f(0) = 0$. Let $E \supset \bar{B}_b(0) = \{y : |y| \leq b\}$ and $\left\|\frac{\partial f}{\partial y}\right\|$ be the norm as operator of the jacobian matrix on E.

Assume that there exist constants M, M_1 such that on E,

$$M > \left\|\left(\frac{\partial f}{\partial y}\right)^{-1}\right\|, \quad M_1 > \left\|\frac{\partial f}{\partial y}\right\|.$$

Then there exists an open set $\mathcal{R}$ such that $B_{\frac{b}{MM_1}}(y=0) \subset \mathcal{R} \subset B_b(y=0)$, where $x = f(y)$ is a homeomorphism from $\mathcal{R}$ onto $B_{\frac{b}{M}}(x=0)$**.**

PROOF. First, we observe that $MM_1 > 1$. Let ξ be a non null vector, t real and $\xi t = f(y)$. A solution $y = y(t)$ of this equation should verify in the sense of Fréchet



(1) $\frac{dy}{dt} = y' = \left(\frac{\partial f}{\partial y}\right)^{-1} . \xi.$

If we add the condition $y(0) = 0$, we have an initial value problem.

Peano's Theorem asserts that there exists a solution (may be not unique) $y = Y(t, \xi)$ defined on $\{t : |t| < b/M|\xi|\}$, with values in $\{y : |y| < b\}$, ($|y(t)| \leq \left\|\left(\frac{\partial f}{\partial y}\right)^{-1} . \xi\right\| \frac{b}{M|\xi|} < b$). But

$$h(t) \coloneqq f(Y(t, \xi))$$

verifies $\frac{dh}{dt} = \frac{\partial f}{\partial y} . \frac{dy}{dt} = \frac{\partial f}{\partial y} . \left(\frac{\partial f}{\partial y}\right)^{-1} \cdot \xi = \xi.$

Then, $h = \xi t + \eta$. Since $h(0) = f\big(Y(0, \xi)\big) = f(0) = 0$, it follows that $\eta = 0$ and we have

(2) $$h(t) = f\big(Y(t, \xi)\big) = \xi t.$$

Because of $f \in C^1$ and the implicit function theorem, $Y(t, \xi)$ is locally the unique solution of $f(y) = \xi t$. Since $Y(0, \xi) = 0$ it follows that $Y(t, \xi)$ is the unique solution of our differential equation.

Thus, $$\frac{dY(t, \xi)}{dt} = \left(\frac{\partial f}{\partial y}\right)^{-1} . \xi, \quad |t| < b/M|\xi|.$$

If we replace ξ by $c\xi,\ c > 0$, then $y(t) = Y(t, c\xi)$ satisfies the equation

(3) $$y' = \left(\frac{\partial f}{\partial y}\right)^{-1} . c\xi, \ \ y(0) = 0 \quad \text{if } |t| < \frac{b}{Mc|\xi|}.$$

But in this interval, it holds that $\frac{dY(ct, \xi)}{dt} = c\frac{dY(ct, \xi)}{dct} = \left(\frac{\partial f}{\partial y}\right)^{-1} (Y(ct, \xi)). c\xi,$

i.e., $Y(ct, \xi)$ satisfies (3).

Because of the unicity of the solution we have for $|t| < \frac{b}{Mc|\xi|}$

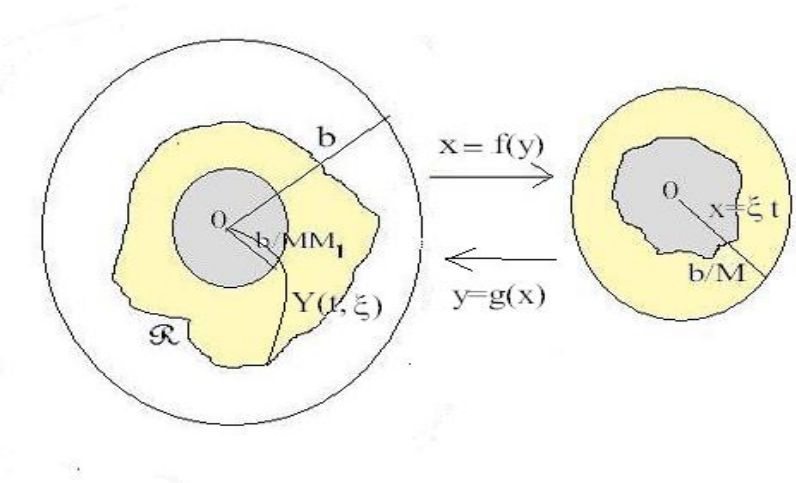
(4) $$Y(t, c\xi) = Y(ct, \xi).$$

Assume now that ξ is a versor. In this case $Y(t, \xi)$ exists on $|t| < \frac{b}{M}$ and

$Y(t, c\xi) = Y(ct, \xi)$ holds on $|t| < \frac{b}{Mc}$. Then, $Y(1, c\xi) = Y(c, \xi)$ whenever $0 < c < \frac{b}{M}$.

Thus, $Y(1, t\xi) = Y(t, \xi)$ for $0 < t < \frac{b}{M}$.

Since $Y(1,0) \coloneqq Y(0, \xi) = 0$, we have $Y(1, t\xi) = Y(t, \xi)$ for $0 \le t < \frac{b}{M}$ and an arbitrary versor ξ.



Then, from (2) we get

$$f\big(Y(1, t\xi)\big) = t\xi$$

for $|t| < \frac{b}{M}$ and $|\xi| = 1$.

From this we obtain

$f(Y(1, x)) = x$ if $|x| < b/M$.

Then, because of

$$\frac{\partial Y(1,x)}{\partial x} = \left(\frac{\partial f}{\partial y}\right)^{-1},$$

there we have $\det \frac{\partial Y(1,x)}{\partial x} \neq 0$.

Thus, if $x \in B_{\frac{b}{M}}(0)$ then $g(x) \coloneqq Y(1, x) \in B_b(0)$ and $g(x) \in C^1\big(B_{b/M}\big)$.

Using the implicit function theorem (or Brower's theorem) we deduce that

$$\mathcal{R} \coloneqq Y(1, B_{b/M})$$

is a (simply connected) open set which by construction is included in B_b, as we have seen. Therefore, we have proved the

PROPOSITION 1. $x = f(y)$ defines a bijective application from $\mathcal{R}$ onto $B_{b/M}$ and therefore a C^1-homeomorphism with inverse $y = g(x) = Y(1, x)$**.**

$f(E)$ is an open set that contains the compact set $f(\bar{B}_b) \supset \bar{B}_{b/M}$ and $f(\bar{\mathcal{R}}) \supset \bar{B}_{b/M}$. If one starts from $\bar{B}_{b+\varepsilon}$ one proves that f is a homeomorphism from $\bar{\mathcal{R}}$ onto $\bar{B}_{b/M}$.

Applying the Proposition to the map $y = g(x)$ on $\bar{B}_{b/M}$, we find an open set $\mathcal{R}_1 \subset B_{\frac{b}{M}}(x = 0)$ such that $y = g(x)$ defines a homeomorphism from $\mathcal{R}_1$ onto $B_{\left(\frac{b}{M}\right)\frac{1}{M_1}}(y = 0) \subset \mathcal{R}$, QED.

5.1. Some more precise statements about the proof of Lemma 1 §4. To fix ideas *we assume that* $|\sigma| = 0$. We shall use the following corollary of the preceding theorem.

THEOREM 0. Let E be an open plane set, $E \supset \bar{B}_b(0)$, and assume that there are two numbers M, M_1 such that the homeomorphism T defned in §**2**, $T(0) = 0$,

$$x = T(\xi) = \begin{cases} x_1(\xi) = y_1(\xi_1) - \xi_2 \dot{y}_2(\xi_1) \\ x_2(\xi) = y_2(\xi_1) + \xi_2 \dot{y}_1(\xi_1) \end{cases},$$

verifies on E: $M_1 > \left\|\frac{\partial x}{\partial \xi}\right\|_{op}$, $M > \left\|\left(\frac{\partial x}{\partial \xi}\right)^{-1}\right\|_{op}$. Then, there exists a region $\mathcal{R}$ such that $B_{\frac{b}{MM_1}}(0) \subset \mathcal{R} \subset B_b(0)$, where $x = T(\xi)$ is a homeomorphism from $\mathcal{R}$ onto $B_{\frac{b}{M}}(0)$.

In our case a *uniform* upper bound $M_1{}'$ for $\left\|\frac{\partial x}{\partial \xi}\right\|_{op}$ can be obtained from (1′) of §**2** in a certain cylindrical open set E, $E \supset I \times [-1,1]$, (the end points of I are identified in case $|\sigma| = 0$). In fact, the norm of a matrix $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ as an operator verifies $\|A\|_{op} \le \left(\sum_{i,j=1}^{2} |a_{ij}|^2\right)^{1/2}$. From (1″) §**2**, it follows that if

(1) $$|\xi_2| \le h < \min(1, c(J)^{-1})$$

there exists $\left(\frac{\partial x}{\partial \xi}\right)^{-1}$ and there is a uniform upper bound M' for $\left\|\left(\frac{\partial x}{\partial \xi}\right)^{-1}\right\|_{op}$ on $I \times [-h, h]$. But we shall use the following greater constants,

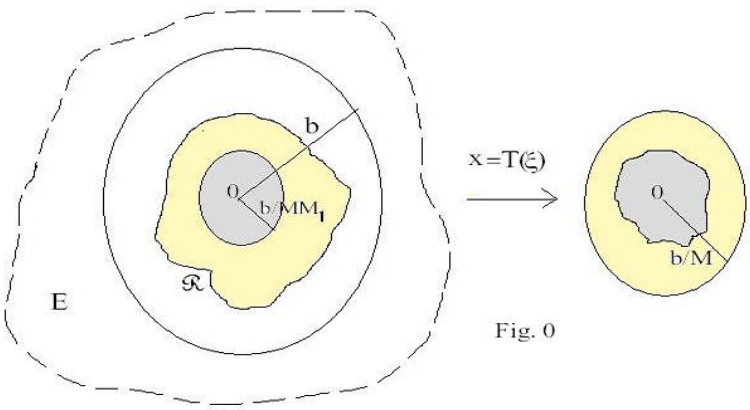


Fig. 0

$$g = \max\left(\sup\left\|\frac{\partial x}{\partial \xi}\right\|_{op}, \sup\left\|\left(\frac{\partial x}{\partial \xi}\right)^{-1}\right\|_{op}\right), \quad M_1 = 1 + 2g, \quad M = 1 + 2g.$$

$g \ge 1$ is a number such that for $\xi, \eta \in I \times (-h, h)$ it holds that

(2) $$\frac{1}{g}|\xi - \eta| \le |T(\xi) - T(\eta)| \le g|\xi - \eta|.$$

In fact,

LEMMA 1. 1) If $T(\xi) = \begin{vmatrix} x_1(\xi_1, \xi_2) \\ x_2(\xi_1, \xi_2) \end{vmatrix}$ is a C^1-homeomorphism from $U = I \times (-\delta, \delta)$ onto $V = T(U)$, then $|T(\xi) - T(\eta)| \le \sup\left\|\frac{\partial x}{\partial \xi}\right\|_{op} |\xi - \eta|$ for $\xi, \eta \in U$,

2) $$|\xi - \eta| \le \sup\left\|\frac{\partial \xi}{\partial x}\right\|_{op} |T(\xi) - T(\eta)|,$$

3) If $T(\xi) = \begin{vmatrix} x_1(\xi_1, \xi_2) \\ x_2(\xi_1, \xi_2) \end{vmatrix}$ is the C^1-homeomorphism from $U = I \times (-\delta, \delta)$ onto $V = T(U) = J_\delta$ defined by (1) §**2** and (1) holds (*i.e.*, $\delta c(J) < 1$), then

$\left\| \frac{\partial x}{\partial \xi} \right\|_{op} \leq 2 + |\xi_2||\ddot{y}(\xi_1)|,$ $\quad$ $\sup \left\| \frac{\partial x}{\partial \xi} \right\|_{op} \leq 2 + \delta . c(J),$

$\left\| \frac{\partial \xi}{\partial x} \right\|_{op} \leq (2 + \xi_2 |\ddot{y}(\xi_1)|)/(1 - \xi_2 |\ddot{y}(\xi_1)|),$ $\quad$ $\sup \left\| \frac{\partial \xi}{\partial x} \right\|_{op} \leq (2 + \delta c(J))/(1 - \delta c(J)).$

PROOF. 1) $T(\xi) - T(\eta) = \int_0^1 \frac{dT(\eta + t(\xi - \eta))}{dt} dt$. The integrand is:

$\begin{vmatrix} \partial x_1/\partial \xi_1 & \partial x_1/\partial \xi_2 \\ \partial x_2/\partial \xi_1 & \partial x_2/\partial \xi_2 \end{vmatrix} \begin{vmatrix} \xi_1 - \eta_1 \\ \xi_2 - \eta_2 \end{vmatrix} = \frac{\partial x}{\partial \xi}(\eta + t(\xi - \eta)) \cdot (\xi - \eta)$. Therefore, taking norms we get

$$|T(\xi) - T(\eta)| \leq \int_0^1 \sup \left\| \frac{\partial x}{\partial \xi} \right\|_{op} |\xi - \eta| \, dt = \sup \left\| \frac{\partial x}{\partial \xi} \right\|_{op} |\xi - \eta|.$$

2) Applying 1) to T^{-1} and taking into account that the jacobian matrix of T^{-1} is $\frac{\partial \xi}{\partial x} = \left(\frac{\partial x}{\partial \xi} \right)^{-1}$, we get $|\xi - \eta| \leq \sup \left\| \frac{\partial \xi}{\partial x} \right\|_{op} |T(\xi) - T(\eta)|$. Hence,

$$\left(\sup \left\| \frac{\partial \xi}{\partial x} \right\|_{op} \right)^{-1} |\xi - \eta| \leq |T(\xi) - T(\eta)| \leq \sup \left\| \frac{\partial x}{\partial \xi} \right\|_{op} |\xi - \eta|.$$

3) The norm of a matrix $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ as an operator verifies:

$$\|A\|_{op} \leq \left(\sum_{i,j=1}^2 |a_{ij}|^2 \right)^{1/2}$$

Applying this to the jacobian matrices $\frac{\partial x}{\partial \xi} = \begin{vmatrix} \dot{y}_1(\xi_1) - \xi_2(\ddot{y}_2(\xi_1)) & -\dot{y}_2(\xi_1) \\ \dot{y}_2(\xi_1) + \xi_2(\ddot{y}_1(\xi_1)) & \dot{y}_1(\xi_1) \end{vmatrix}$ and

$$\frac{\partial \xi}{\partial x} = \begin{vmatrix} \dot{y}_1(\xi_1) & \dot{y}_2(\xi_1) \\ -\dot{y}_2(\xi_1) - \xi_2(\ddot{y}_1(\xi_1)) & \dot{y}_1(\xi_1) - \xi_2(\ddot{y}_2(\xi_1)) \end{vmatrix} / (1 - \xi_2 c(\xi_1)),$$

and using the fact that $|c(\xi_1)| = |\ddot{y}(\xi_1)|$, one gets

$$\left\| \frac{\partial x}{\partial \xi} \right\|_{op}^2 \leq 1 - 2\xi_2 \big(\dot{y}_1(\xi_1)\ddot{y}_2(\xi_1) - \dot{y}_2(\xi_1)\ddot{y}_1(\xi_1) \big) + \xi_2^2 |\ddot{y}(\xi_1)|^2 + 1 =$$

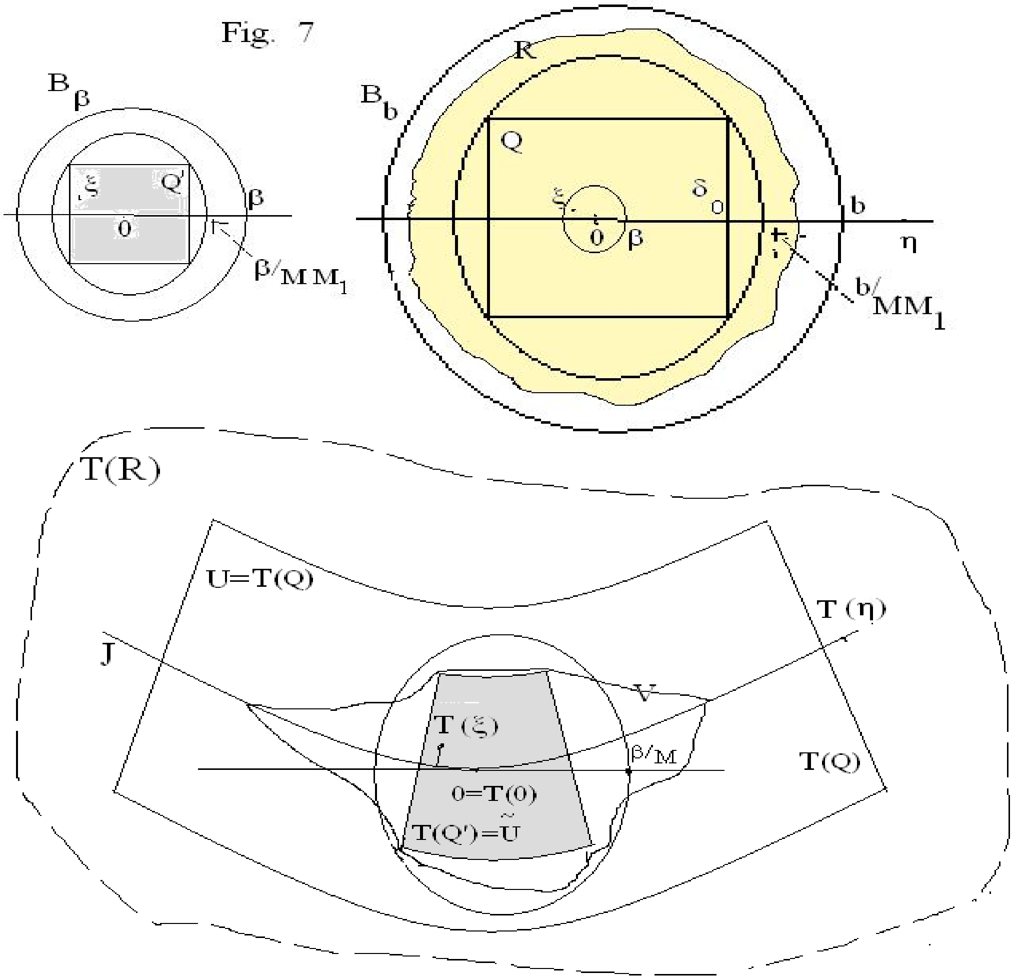
$= (1 - \xi_2 c(\xi_1))^2 + 1$. Therefore, these inequalities hold:

(*) $\quad \left\| \frac{\partial x}{\partial \xi} \right\|_{op} \leq (1 - \xi_2 c(\xi_1)) + 1$ and similarly

(**) $\quad \left\| \frac{\partial \xi}{\partial x} \right\|_{op} \leq \big((1 - \xi_2 c(\xi_1)) + 1 \big) / (1 - \xi_2 c(\xi_1))$

and 3) follows, QED.

Fig. 7



Let us define $\delta := \frac{h}{2g} = \min\ (1, R(J))/2g$. Then, we have

$$|\xi_2| < \delta \Rightarrow T(\xi) \in J_{\delta g} \subset J_{h/2} \quad \text{and} \quad \frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)} = 1 - \xi_2 c(\xi_1) > 1/2.$$

Using Theorem 0 with $b = \delta$ we get that T maps homeomorphically $B_{\frac{\delta}{MM_1}}(\xi = 0)$ onto an open neighborhood of $x = 0$, $\mathcal{R}_1 \subset B_{\frac{b}{M}}(x = 0)$. Then, it maps the square Q inscribed in $B_{\frac{\delta}{MM_1}}(\xi = 0)$, that is, the square $Q = (-\delta_0, \delta_0) \times (\delta_0, \delta_0)$ with $\delta_0 = \frac{\delta}{\sqrt{2}MM_1}$, onto $U := T(Q)$ $(\subset \mathcal{R}_1 \subset B_{\frac{b}{M}})$.

Let $\beta = \frac{\delta_0}{4}$. The disk $B_\beta(0)$ is such that the distance from any of its points to points outside Q is greater than β.

Then, for $\xi \in B_\beta(0)$ and $\eta = (\eta_1, \eta_2)$, $\eta \in I \times (-\delta, \delta) \setminus Q$, we have

(3) $$|T(\xi) - T(\eta)| \geq \frac{|\xi - \eta|}{g} > \frac{\beta}{g} \quad \left(= \frac{\delta_0}{4g} = \frac{\delta}{4\sqrt{2}MM_1 g}\right).$$

In particular, for $\eta = (\eta_1, 0)$, $T(\eta) \in J \setminus T(Q)$.

Therefore, the disk of radius β/M centered at $x = 0 = T(0)$ verifies for $T(\xi) \in B_{\frac{\beta}{M}}(0)$

and $T(\eta) \in J_\delta \setminus T(Q)$ that $\xi \in B_\beta(0), \eta \in I \times (-\delta, \delta) \setminus Q$.

Because of (3) and $M > g$ we get

(4) $$|T(\xi) - T(\eta)| > \frac{\beta}{g} > \frac{\beta}{M}.$$

Then, $|T(0) - T(\eta)| > \frac{\beta}{M}$ and $B_{\frac{\beta}{M}}(0) \subset T(Q) \subset J^\delta$.

The square Q' inscribed in B_{β/MM_1} of half side

$$\tilde{\delta} = \frac{\beta}{\sqrt{2}MM_1}$$

is mapped by T homeomorphically onto $\widetilde{U} \coloneqq T(Q')$.

Because of (4),

(5) if $T(\xi) \in T(Q')$ and $T(\eta) \in J_\delta \setminus T(Q)$ then $|T(\xi) - T(\eta)| > \frac{\beta}{M}$.

Because of $M_1 > 2g$,

(6) $$\operatorname{diam} T(Q') \leq g \operatorname{diam} Q' \leq g.2\beta/MM_1 < \beta/M.$$

Then we have,

(7) $$|T(\xi) - T(\eta)| > \beta/M > \operatorname{diam} T(Q') \geq \operatorname{dist}(T(\xi), J).$$

We proved the

THEOREM 1. Assume that Q is the square with center s and sides parallel to the axes of half side equal to $\delta_0 = \frac{\delta}{\sqrt{2}MM_1}$ and Q' a similar square with the same center and of half side $\tilde{\delta} = \frac{\delta}{2(2MM_1)^2}$.

If $T(\xi) \in T(Q')$ and $T(\eta) \in J_\delta \setminus T(Q)$, it holds that

(8) $$|T(\xi) - T(\eta)| > \frac{\beta}{M} = \frac{\delta}{4\sqrt{2}M^2M_1} > \operatorname{dist}(T(\xi), J).$$

6. The spectral Dirichlet series. We focus on the functions

$F(p;\lambda) = \lambda \sum_{n=1}^{\infty} \varphi_n^2(p)/\lambda_n(\lambda_n - \lambda), -\lambda = t = \chi^2, \chi \geq 1$ and

$\int_D F(p;\lambda)dp = \lambda \sum_{n=1}^{\infty} 1/\lambda_n(\lambda_n - \lambda)$ (cf. §**4.14** and §**4.17** of BPII).

The eigenvalues and the normalized eigenfunctions are those of the Dirichlet problem $-\Delta u = \lambda u$, (cf. BPII, §**4.4**). We know that, (BPII, §**4.14**, §**4.15** for $k(p) \equiv 1, w = 1$)

$$F(p,q;\lambda) = G(p,q;\lambda) - G(p,q) =$$
$$= (K_0(\chi|p-q|)/2\pi - H(p,q;\lambda)) - ((1/2\pi)\log(M/|p-q|) - H(p,q)),$$

where $M = \operatorname{diam} D$, $H(p,\cdot)$ is a harmonic function and $H(p,\cdot;\lambda)$ is a χ-harmonic function (a metaharmonic function[1]) such that $H(p,\tilde{q};-\chi^2) = K_0(\chi|p-\tilde{q}|)/2\pi$ if $(p,\tilde{q}) \in D \times J,\ J = \partial D$. Using the fact[2] that the Kelvin function of order 0,

$K_0(r) = \int_1^\infty e^{-rt}(t^2-1)^{-1/2}dt = \log(1/r) + (I_0(r)-1)\log(1/r) + P(r)$, where $P(r)$ and the modified Bessel function $I_0(r)$ are entire functions in r^2, $I_0(0) = 1$, (BPII, §**4.15**), we get[3] $K_0(\chi|p-q|) = \log\frac{1}{\chi|p-q|} + Q(\chi|p-q|)$, with $Q(|x|) \in C^1(R^2)$. Therefore,

$$F(p,q;\lambda) = -H(p,q;\lambda) - (\log\chi)/2\pi + H(p,q) + Q(\chi|p-q|)/2\pi - (\log M)/2\pi).$$

Hence, for $q \to p$,

(1) $\quad F(p;\lambda) := \lim_{q\to p} F(p,q;\lambda) =$

$$= H(p,p) - H(p,p;\lambda) + (Q(0) - \log\chi - \log M)/2\pi$$

where

(1') $\quad H(q,b;\lambda) = K_0(\chi|b-q|)/2\pi > 0$, for $q \in D$, $b \in J$.

Since $K_0(r)$ is a decreasing function, if $l(q) := \operatorname{dist}(q,J), q \in D$, we have

$$\sup_{b\in J} K_0(\chi|b-q|) = K_0\big(\chi l(q)\big).$$

Thus, by (1') and the maximum principle for χ-harmonic functions, (BPIII, p. 45), if $p \in \overline{D}$ and $q \in D$ we have

(2) $\quad 0 < H(p,q;-\chi^2) \leq K_0(\chi l(q))/2\pi.$

To estimate $H(p,p;\lambda)$ for $\chi \to \infty$ observe that if $l(q) \geq \delta > 0$ then, because of (2), (cf. BPII, §**4.15**, p. 42),

(3) $\quad H(q,q;-\chi^2) \leq K_0(\frac{\delta\chi}{2})e^{-\frac{\delta\chi}{2}}/2\pi.$

Then, we have

LEMMA 1. $0 < H(q,q;-\chi^2) < K_0(\frac{\delta\chi}{2})e^{-\delta\chi/2}$ for $l(q) \geq \delta > 0,\ q \in D$**.**

Thus, to estimate $H(q,q;-\chi^2)$ on the whole of D it will suffice to estimate $H(p,p;\lambda)$ on F_δ = the interior δ-neighborhood of J.

[1] Informe Técnico Interno n° 79, INMABB, UNS-CONICET, 2002, Ch. 4-5.

[2] Magnus, W., Oberhettinger, F., Formulas and Theorems for the Special Functions of Mathematical Physics, 1954, p. 27.

[3] Schwartz, Laurent, Théorie des distributions, II, 1951, formula (VII, 10:15).

6.1. MAIN RESULT. Our objective is to prove Th. 1 of §**1** for curved polygons. For this we use in an essential way the next Lemma 1 that we shall prove in §**6.4**, (cf. BPII, §**4.15**).

LEMMA 1. Let $I(\chi^2) := \int_D H(x,x;\,-\chi^2)\,dx$. $I(t) - \langle J\rangle/8\sqrt{t}$ is measurable and bounded on any closed subinterval of $(0,\infty)$ and if $\chi \gg 1$ then

(4) $$I(\chi^2) - \frac{\langle J\rangle}{8\chi} = L(\chi^2)\left(\frac{\log \chi^2}{\chi^2}\right), \quad L(\chi^2) = O(1).$$

From (1) §**6** and what we have said above we deduce that for $w = \chi^2$ it holds that

(5) $$\int_D F(x;-w)\,dx = -\frac{|D|}{4\pi}\log w + C - I(w) = -w\sum_1^\infty \frac{1}{\lambda_n(\lambda_n+w)},$$

where $C = \int_D H(p,p)dp + \frac{(Q(0)-\log M)|D|}{2\pi}$ is a finite constant.

In fact, $\int_D |H(p,p)|dp = \int_D H(p,p)dp < \infty$, (cf. BPII, Th. 2 §**4.13** and Th. 1 §**4.19**).

NOTATION. We wrote in (5) w in place of t to indicate that it may take complex values: $w \in C$, $t = \chi^2 \in R^+$. We shall *write* $f(-w)$ *instead* of $-w\sum_1^\infty \frac{1}{\lambda_n(\lambda_n+w)}$.

Then, for $w > 0$ and assuming (4), we obtain from (5),

(5') $$f(-w) = -\frac{|D|}{4\pi}\log w + C - \left[\frac{\langle J\rangle}{8}w^{-1/2} + L(w)w^{-1}\log w\right].$$

THEOREM 1. If D is a curved polygon and $\operatorname{Re} z > 1$ then

(6) $$\sum_1^\infty \lambda_n^{-z} = \int_r^\infty \lambda^{-z}dN(\lambda) = \frac{|D|}{4\pi}\frac{1}{z-1} - \frac{\langle J\rangle}{8\pi}\frac{1}{z-\frac{1}{2}} + g(z),$$

where $g(z)$ is holomorphic on $\operatorname{Re} z > 0$, $N(\lambda) := \#\{\lambda_j : \lambda_j \le \lambda\}$ = the counting function of the eigenvalues $\lambda_n(D)$ of the Dirichlet problem and r is such that $0 < 2r < \inf(\lambda_1, 1)$.

PROOF. (Cf. [BPII] § **4.20-21**). $f(w) = w\sum_1^\infty \frac{1}{\lambda_n(\lambda_n-w)}$ is a meromorphic function on the plane with simple poles at the points λ_n, a non-decreasing family.

Let $a = -|D|/4\pi$, $b = -\langle J\rangle/8$, $q(t) := -L(t)(t^{-1}\log t)$. Then, in view of (5'),

(7) $$f(-t) = a\log t + bt^{-1/2} + q(t) + C, \quad t > 0.$$

Because of Lemma 1, we have

(7') $q(t)$ is a measurable bounded function on $t \in (1/N, \infty)$, for all $N > 1$, such that $\frac{q(t)}{t^s} \in L^1(\frac{1}{N},\infty)$ for any $s > 0$.

Let Γ be the region at the left of the contour γ shown in Fig. 2. We have, for $s > 1$,

(8) $$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^s}dw = \frac{-1}{2\pi i}\int_\gamma \sum_1^\infty \frac{w^{1-s}}{\lambda_n(w-\lambda_n)}dw = \frac{-1}{2\pi i}\sum_1^\infty \int_\gamma \frac{w^{1-s}}{\lambda_n(w-\lambda_n)}dw = -\sum_1^\infty \lambda_n^{-s}.$$

In fact, let $t=|w|$. Because of $s>1$ and H. Weyl's theorem $(\lambda_n \approx n)$ we have $\sum \int \left|\frac{t^{1-s}}{\lambda_n(t+\lambda_n)}\right| dt < \infty$. This allows to prove the second equality in (8).

For the last equality observe that the region Γ is the limit, as $R_n \to \infty$, of the region contained between the circumferences of radii r and R_n except for the negative real axis, (see Fig. 1).

Since $s>1$, the integral over the circumference of radius R tends to zero for $R\to\infty$ and the last equality in (8) follows from Cauchy's theorem of residues, (cf. BPII, §**4.18**).

Let $\gamma' = \gamma \cap \{w: |w|<1\}$ and $l = \gamma \setminus \gamma' = l_1 + l_2$, (Fig. 2), with l_1 going from $-\infty$ to -1 where $\arg w = \pi$ and with l_2 going from -1 to $-\infty$ where $\arg w = -\pi$.

Then we get (arguing as in BPII, §**4.18**):

(9) $\quad -\sum_1^\infty \lambda_n^{-s} = \frac{1}{2\pi i}\int_l f(w)\, w^{-s} dw + f_0(s), \qquad f_0(s) := \frac{1}{2\pi i}\int_{\gamma'} f(w)\, w^{-s} dw,$

(10) $$\frac{1}{2\pi i}\int_l f(w)\, w^{-s} dw = \frac{1}{2\pi i}\left[-\int_{-l_1} f(w)\, w^{-s} dw + \int_{l_2} f(w)\, w^{-s} dw\right] =$$

$$= \frac{1}{2\pi i}\int_1^\infty f(-t)\, t^{-s}\left(e^{-i\pi s} - e^{i\pi s}\right) dt = \left(\text{cf. } (7)\right) =$$

$$= -\frac{\sin \pi s}{\pi}\int_1^\infty f(-t) t^{-s} dt =$$

$$= -\frac{\sin \pi s}{\pi}\int_1^\infty \left(a \log t + C + b t^{-\frac{1}{2}}\right) t^{-s} dt + \left(-\frac{\sin \pi s}{\pi}\right)\int_1^\infty q(t) t^{-s} dt.$$

Because of (7'), the last term defines a holomorphic function $g_0(s)$ on $\operatorname{Re} s > 0$. Thus,

(11) $\quad \sum_1^\infty \lambda_n^{-s} = \frac{\sin \pi s}{\pi} \int_1^\infty \left(a \log t + b t^{-1/2} + C\right) t^{-s} dt - [f_0(s) + g_0(s)],\ s>1,$

where the square brackets define a holomorphic function on $\operatorname{Re} s > 0$.

The integral in (11) is equal to

$$\int_1^\infty \left(a \log t + b t^{-1/2} + C\right) t^{-s}\, dt = a(s-1)^{-2} + b(s-\tfrac{1}{2})^{-1} + C(s-1)^{-1}.$$

Thus, from $\frac{\sin s\pi}{\pi(s-1)} = -1 + (s-1)[\cdots]$, we obtain on $\{s>1\}$,

(12) $$\frac{\sin s\pi}{\pi}\left\{a(s-1)^{-2} + b(s-\tfrac{1}{2})^{-1} + C(s-1)^{-1}\right\} = -\frac{a}{s-1} + \frac{b/\pi}{\left(s-\frac{1}{2}\right)} + g_2(s),$$

where $g_2(s)$ is holomorphic on $\operatorname{Re} s > 0$.

Thus, on $\operatorname{Re} z > 1$ it holds that

(13) $\quad \sum_1^\infty \lambda_n{}^{-z} = -\frac{a}{z-1} + \frac{b/\pi}{z-\frac{1}{2}} + g(z), \qquad g = g_2(z) - [f_0(z) + g_0(z)].$

$g(z)$ is a holomorphic function on $\operatorname{Re} z > 0$ and (6) follows, QED.

THEOREM 2. The residues and the function g are uniquely determined**.**

PROOF. In fact, $\frac{A}{z-1} + \frac{B}{z-\frac{1}{2}} + G(z) \equiv 0$ on $\operatorname{Re} z > 1$ implies $A = B = 0,\ G \equiv 0$.

That is, *if (4) of Lemma 1 holds then we have proved that Theorems 1 and 2 are valid* for convex polygons. For $|\sigma| = 0$ we would have proved Theorem 1 of §**1**.

6.2. About formula (4) §6.1 for D a curved polygon:

$$I(\chi^2) = \int_D H(p,p;-\chi^2)dp = \frac{\langle J\rangle}{8\chi} + O\left(\frac{\log \chi^2}{\chi^2}\right).$$

Because of Prop. 4 §**4.1** and (3) §**6**, we have for $F_h = D \cap J_h$, (h as in §**4.5**),

(14) $\quad I = \int_D H(x,x;-\chi^2)dx = \int_{F_h} H(x,x;-\chi^2)dx + \int_{D\setminus F_h} H(x,x;-\chi^2)dx =$

$$= \int_{\cup \tilde{F}_n^h} H(x,x;-\chi^2)dx + O(1)|D|K_0(\chi h/2)e^{-\chi h/2},$$

where $|O(1)| \le 1$, $\tilde{F}_n^h = \{x \in J_n^h \cap D : \operatorname{dist}(x,J) = \operatorname{dist}(x,J_n)\}$.

If $E = \sum_n \int_{\tilde{F}_n^h} H(x,x;-\chi^2)dx - \int_{\cup \tilde{F}_n^h} H(x,x;-\chi^2)dx$ then $E \ge 0$ and

(14') $\quad I = \sum_n \int_{\tilde{F}_n^h} H(x,x;-\chi^2)dx - E + O(|D|)K_0(\chi h/2)e^{-\chi h/2}.$

We know from §**4.2** that if h is small enough then $E = 0$. In consequence, we have,

(15) $\quad I = \sum_n \int_{\tilde{F}_n^h} H(x,x;-\chi^2)dx + o\left(e^{-\chi h/2}\right), \qquad 1 \le \chi \to \infty.$

The function $R(.,.;.)$. We wish that $K_0(\chi|p-\hat{p}|)/2\pi$ replaces $H(p,p;-\chi^2)$ in (15). We will consider, for this purpose, the difference,

(16) $\quad R(x,p;-\chi^2) := H(x,p;-\chi^2) - K_0(\chi|x-\hat{p}|)/2\pi, \qquad p \in \tilde{F}_n^h,\ x \in J.$

Observe that, whenever h is small enough, for each $p \in \tilde{F}_n^h$, R is χ-harmonic in D as a function of x since $\hat{p} \,\overline{\in}\, \overline{D}$, (cf. §**4.3**).

That is, $(\Delta_x - \chi^2)R = 0$, $x \in D$, (cf. (2) and Th. 1, [BPII], §**4.15**).

If $x \in J$ then $H(x,p;-\chi^2) = \frac{1}{2\pi}K_0(\chi|p-x|)$ and

(17) $\quad R(x,p;-\chi^2) = \{K_0(\chi|p-x|) - K_0(\chi|\hat{p}-x|)\}/2\pi.$

For χ large enough we have the next lemma where

$$O_i = y(s_i),\ \ \{O_i\} = J_{i-1} \cap J_i,\ \ l(p) := \operatorname{dist}(p,J).$$

LEMMA 1. 1) If h is sufficiently small and $p \in \tilde{F}_n^{h/3} \setminus \{J_{n+1,h} \cup J_{n-1,h}\}$, $x \in J$, then there exists $M = M(h) \in (0,\infty)$ such that

(18) $\quad |R(x,p;-\chi^2)| < M.\inf\left(1/\chi,\ e^{-\chi\, l(p)/T}\right).$

T is a positive integer ≥ 2. In particular, if $|\sigma| = 0$, then (18) holds for $p \in F^{\frac{h}{3}}$, $x \in J$.

2) If $|\sigma| \geq 2$ and $p \in \tilde{F}_n^{h/3} \cap J_{n+1,h}$, then (18) holds except possibly for $x \in J_{n+1} \cap J_{n,h}$. Similarly, if $p \in \tilde{F}_n^{h/3} \cap J_{n-1,h}$, then (18) holds with a certain $T \geq 2$, except possibly for $x \in J_{n-1} \cap J_{n,h}$.

Then, the exceptional pairs $\{p, x\}$ where (18) does not hold, if any, are contained in the sets:

(19) $\quad p \in \tilde{F}_i^{h/3} \cap J_{i+1,h}$ and $x \in J_{i+1} \cap J_{i,h}, \quad i = 1, \dots, |\sigma|$,

(19') $\quad p \in \tilde{F}_i^{h/3} \cap J_{i-1,h}$ and $x \in J_{i-1} \cap J_{i,h}, \quad i = 1, \dots, |\sigma|$.

PROOF. If $\hat{x} = (x_1, -f(x_1)$ represents the symmetric point of x with respect to the first coordinate axis, (see Fig. 5, $p = (0, |p|)$, where the curve is L_n), we have:

$|x - \hat{p}| = |\hat{x} - p|$ and $\big||x - p| - |x - \hat{p}|\big| \leq |x - \hat{x}| = 2|f(x_1)|$.

If $|x_1| < \varepsilon$ then we obtain for

$$x \in U \cap L_n, \;\; U = T(Q) = T\left(Q \cap \left(I \times (-h, h)\right)\right), \;\; p \in \tilde{F}_n^h \cap T(Q'),$$

(cf. §**5.1** and Fig. 7, $x = T(\eta)$; $p = T(\xi)$),

(20) $\quad d := |K_0(\chi|p - x|) - K_0(\chi|p - \hat{x}|)| \leq \chi 2|f(x_1)||K_0{}'(\chi\tilde{\rho})|$,

where $\tilde{\rho}$ is a number between $|x - p|$ and $|x - \hat{p}|$.

By Lemma 2, §**4.5**, we have $\rho > r/\sqrt{2}$ and therefore

$$\tilde{\rho} > r/\sqrt{2} \geq \mathrm{dist}(p, L_n)/\sqrt{2}.$$

Since

(20') $\quad |p| \geq \mathrm{dist}(p, J_n) \geq \mathrm{dist}(p, L_n) = |p|$,

we also have

$$\tilde{\rho} > r/\sqrt{2} \geq \mathrm{dist}(p, J_n)/\sqrt{2}\,.$$

The right-hand side of (20) is equal to, $(1 \leq \chi)$,

(20") $\quad \frac{2|f(x_1)|}{\tilde{\rho}} |\chi\tilde{\rho}K_0{}'(\chi\tilde{\rho})| \leq \frac{2|f(x_1)|}{\tilde{\rho}} C_1 e^{-\chi\tilde{\rho}/2} \leq C_1 2\sqrt{2}\, \frac{|f(x_1)|}{\sqrt{x_1^2 + x_2^2}}\, e^{-\chi\tilde{\rho}/2}$,

because of $rK_0'(r) \leq C_1 e^{-r/2}$ for $r > 0$, (C_1 a constant, cf. [BPII]§**4.15**).

Because of Lemma 2, §**4.5**, the last term is not greater than $(c' = C_1 2^{\frac{7}{2}} c(J))$,

$$\leq C_1 8\sqrt{2} c(J)\, \frac{x_1^2}{\sqrt{x_1^2 + x_2^2}}\, e^{-\frac{\chi r}{2\sqrt{2}}} \leq c'|x_1| e^{-\chi r/2\sqrt{2}} \leq c' r e^{-\chi r/4} \leq$$

$$\leq c''\inf(1/\chi, (\mathrm{diam}\, D) e^{-\chi\,\mathrm{dist}(p, J_n)/4}).$$

Thus, $d \leq M \inf\left(\frac{1}{\chi}, e^{-\chi\,\mathrm{dist}(p, J_n)/4}\right)$ for $x \in L_n \cap U$.

If $L_n \ni x \,\overline{\in}\, U = T(Q)$ then $|x-p|,\ |x-\hat{p}| \geq \vartheta h,\ 0 < \vartheta = \vartheta(J) < 1$, because of (8) Th. 1, §**5.2**. In this case, the argument to obtain

$$|R(x,p;-\chi^2)| < M_1.\inf\left(1/\chi,\ e^{-\chi\frac{l(p)}{2/\vartheta}}\right)$$

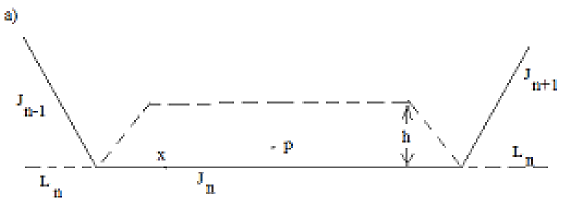
is similar to the one we use in next point **b)** since we can write

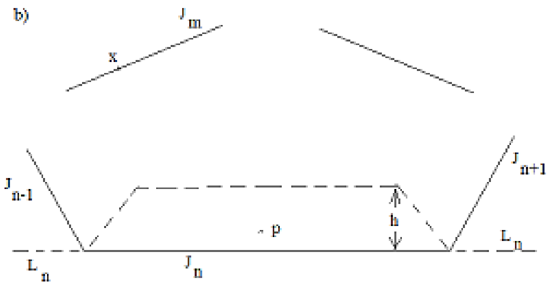
$$2\pi|R(x,p;-\chi^2)| \leq K_0(\chi|p-x|) + K_0(\chi|\hat{p}-x|) \leq 2K_0(\chi\vartheta h).$$

We obtained,

a) For $x \in L_n,\ p \in \tilde{F}_n^h$, a constant $T = T(J) \in [2,\infty),\ M_a = M_a(h) \in (0,\infty)$, we have

(21) $$|R(x,p;-\chi^2)| = d/2\pi < M_a \inf\left(\frac{1}{\chi}, e^{-\chi l(p)/T}\right).$$





NB. In particular, when $J = J_0$ we get (18) for $p \in F_h,\ x \in J$, with $M = M(h),\ T = 4$**.** The cases **a)-f)** are schematically depicted in the figures at the left.

Let us prove part 2).

b) If $m \neq n, n \pm 1,\ x \in J_m,\ p \in \tilde{F}_n^h$, then from $|x-p|,\ |x-\hat{p}| \geq h$ if $h \ll 1$, (§4.1), and

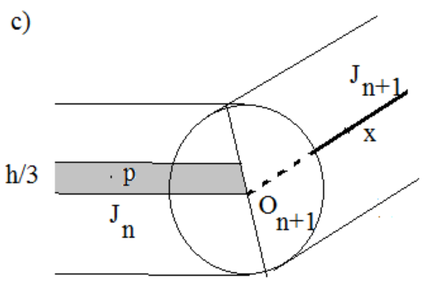
$$2\pi|R(x,p;-\chi^2)| \leq$$
$$\leq K_0(\chi|p-x|) + K_0(\chi|\hat{p}-x|)$$
$$\leq 2K_0(\chi h),$$

we obtain

$$\pi|R(x,p;-\chi^2)| \leq e^{-\chi h/2}K_0(\chi h/2) \leq e^{-\chi l(p)/2}K_0(\chi h/2),$$

$$\pi|R(x,p;-\chi^2)| \leq \frac{1}{\chi}\left(\chi h e^{-\frac{\chi h}{2}}\right)\left(\frac{1}{h}K_0\left(\frac{\chi h}{2}\right)\right) = O\left(\frac{1}{\chi}\right).$$ Then,

(22) $$|R(x,p;-\chi^2)| \leq M_b \inf\left(\frac{1}{\chi}, e^{-\frac{\chi l(p)}{2}}\right).$$



c) For $p \in \tilde{F}_n^{h/3}$ we have $p, \hat{p} \in J_n^{h/3}$.

If $x \in J_{n+1}\backslash J_{n,h}$ or $x \in J_{n-1}\backslash J_{n,h}$ then,

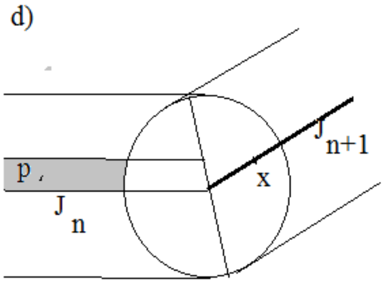
$$|p-x|, |\hat{p}-x| \geq 2h/3.$$

In fact, $|p-x| \geq \operatorname{dist}(x,J_n) - \operatorname{dist}(p,J_n) \geq h - h/3$.

Now we can argue as in **b)** obtaining

(23) $$|R(x,p;-\chi^2)| \leq M_c \inf\left(\frac{1}{\chi}, e^{-\frac{\chi l(p)}{3}}\right).$$

d)



d) If $p \in \tilde{F}_n^{h/3} \backslash J_{n+1,h}$, $x \in J_{n+1}$ then $|p-x| \geq h$ and

$|\hat{p}-x| \geq |p-x| - |\hat{p}-p| \geq h - \frac{2h}{3} = h/3.$

Therefore,

(24) $2\pi|R(x,p;-\chi^2)| \leq$

$$\leq K_0(\chi h) + K_0(\tfrac{\chi h}{3}) \leq M_d \inf\left(\frac{1}{\chi}, e^{-\frac{\chi l(p)}{6}}\right).$$

e) If $p \in \tilde{F}_n^{h/3} \backslash J_{n-1,h}$, $x \in J_{n-1}$ then, as in d), $|p-x| \geq h$ and

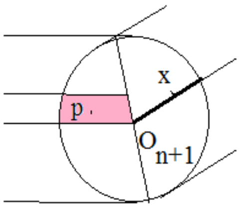
$$|\hat{p}-x| \geq |p-x| - |\hat{p}-p| > h - \frac{2h}{3} = h/3.$$

Therefore, for $M_e = M_e(h)$, we obtain

(24') $2\pi|R(x,p;-\chi^2)| \leq 2K_0(\frac{\chi h}{3}) \leq M_e \inf\left(\frac{1}{\chi}, e^{-\frac{\chi l(p)}{6}}\right).$

f) From c), d) and e) it is clear that the sets $\left\{(p,x): p \in \tilde{F}_n^{\frac{h}{3}} \cap F_{n+1,h},\ x \in J_{n+1} \cap J_{n,h}\right\}$

f)



and

$\left\{(p,x): p \in \tilde{F}_n^{\frac{h}{3}} \cap F_{n-1,h},\ x \in J_{n-1} \cap J_{n,h}\right\}$ contain all the cases that were not considered in **a)-e)**.

Lemma 1 follows after collecting results, QED

Now we shall find a bound for $R(x,p;-\chi^2)$ in the exceptional cases, where the proof of (18) in Lemma 1 does not work.

Without loss of generality we consider the case

(25) $$p \in \tilde{F}_n^{\frac{h}{3}} \cap F_{n+1,h},\ x \in J_{n+1} \cap J_{n,h}.$$

In this case, for h small enough, we have

LEMMA 2. $2\pi|R(x,p;-\chi^2)| \equiv |K_0(\chi|p-x|) - K_0(\chi|\hat{p}-x|)| \leq$

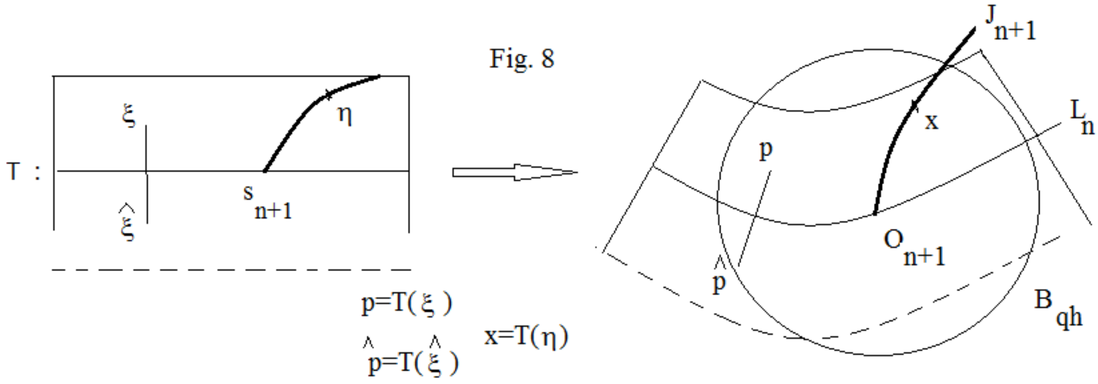
$$\leq K_0((\chi l_{n+1}(p))g^{-2}),$$

where $l_{n+1}(p) = \mathrm{dist}(p, J_{n+1})$ and $g = g(J) \in (1,\infty)$, (cf. (2) **§5.1).**

PROOF. We shall use the homeomorphism T treated extensively in **§5**, applied to L_n, in a neighborhood of the corner O_{n+1}, (see Fig. 8). We suppose h is so small that Theorem 0 §**5.1** applies to $B_b(O_{n+1}) \supset J_{n+1} \cap J_{n,h}$, $(b = qh)$. This is possible because of (†) §**4.1**. If $p \in \tilde{F}_n^{h/3} \cap F_{n+1,h}$ and $p = T(\xi)$ then by the construction of T, $\hat{p} = T(\hat{\xi})$.

Since the angle of the curve J at O_{n+1} is less than π, if h is small enough, the arc $J_{n+1} \cap J_{n,h}$

Fig. 8



does not cross L_n and so, if $x = T(\eta)$, we have $|\xi - \eta| \leq |\hat{\xi} - \eta|$. Using formula (2) **§5** we get $|p - x| \leq g|\xi - \eta| \leq g|\hat{\xi} - \eta| \leq g^2|\hat{p} - x|$.

Then,

(26) $$l_{n+1}(p) \leq |p - x| \leq g^2|\hat{p} - x|.$$

But $|K_0(\chi|p - x|) - K_0(\chi|\hat{p} - x|)| \leq \sup\big(K_0(\chi|p - x|), K_0(\chi|\hat{p} - x|)\big)$.

Since K_0 is a decreasing function, using (26),

$|K_0(\chi|p - x|) - K_0(\chi|\hat{p} - x|)| \leq \sup\Big(K_0\big(\chi l_{n+1}(p)\big), K_0\big(\chi l_{n+1}(p)/g^2\big)\Big) \leq$

$\leq K_0(\chi l_{n+1}(p)/g^2)$, QED.

COROLLARY 1. Let $z \in J$ and $1 \leq n \leq |\sigma|$.

We have, for a certain constant $T = T(J) \in [2,\infty)$:

a) for $p \in \tilde{F}_n^{h/3} \backslash \{J_{n+1,h} \cup J_{n-1,h}\}$, $|R(z,p;-\chi^2)| \leq M \inf\left(\frac{1}{\chi}, e^{-\frac{\chi l(p)}{T}}\right)$,

b) for $p \in \tilde{F}_n^{h/3} \cap F_{n\pm1,h}$, $|R(z,p;-\chi^2)| \leq M \inf\left(\frac{1}{\chi}, e^{-\frac{\chi l(p)}{T}}\right) + K_0(\chi g^{-2} l_{n\pm1}(p))$.

In fact, b) follows from Lemmas 1 and 2.

Since $R(z,p;-\chi^2) := H(z,p;-\chi^2) - K_0(\chi|z - \hat{p}|)/2\pi$ is a χ-harmonic function of $z \in D$, the maximum-minimum principle for these meta-harmonic functions implies that the next bounds hold, for $z \in D$, $T = T(J) \in [2,\infty)$, $1 \leq n \leq |\sigma|$,

$$|R(z,p;-\chi^2)| \leq M \inf\left(\frac{1}{\chi}, e^{-\frac{\chi l(p)}{T}}\right), \quad p \in \tilde{F}_n^{h/3} \backslash \{J_{n+1,h} \cup J_{n-1,h}\}, \quad (|\sigma| + 1 \equiv 1)$$

$$|R(z,p;-\chi^2)| \leq M \inf\left(\frac{1}{\chi}, e^{-\frac{\chi l(p)}{T}}\right) + K_0(\chi g^{-2} l_{n+1}(p)), \quad p \in \tilde{F}_n^{h/3} \cap F_{n+1,h}.$$

In particular, for $z = p$ we get

THEOREM 1. 1) $|H(p,p;-\chi^2)-K_0(\chi|p-\hat{p}|)/2\pi|\leq M.\inf\big(1/\chi, e^{-\chi\, l(p)/T}\big)$,

for $p\in\tilde{F}_n^{h/3}\backslash\{J_{n+1,h}\cup J_{n-1,h}\}$,

2) $|H(p,p;-\chi^2)-K_0(\chi|p-\hat{p}|)/2\pi|\leq M.\inf\left(1/\chi, e^{-\chi\frac{l(p)}{T}}\right)+K_0(\chi g^{-2}l_{n\pm1}(p))$,

for $p\in\tilde{F}_n^{\frac{h}{3}}\cap F_{n\pm1,h}$,

3) If $|\sigma|=0$ then 1) is valid for $p\in F_h$**.**

Observe that all $p\in F_{\frac{h}{3}}$ were considered whenever h is sufficiently small. And only once, except for those of a set of measure zero, (cf.§**4.2**), that were considered twice.

Next, we shall use the bounds of Theorem 1 writing h instead of $h/3$.

6.3. A bound for $\int_{F_h}\boldsymbol{R(q,q;-\chi^2)\,dq}$.

The absolute value of the integral is not greater than

(27) $$\int_{F_h}|R(p,p;-\chi^2)|\,dp\leq\int_{F_h}M.\inf\left(1/\chi, e^{-\chi\frac{l(p)}{T}}\right)dp+$$

$$+\sum\int_{F_{n+1,3h}\cap\tilde{F}_n^h}K_0(\chi g^{-2}l_{n+1}(p))\,dp+\sum\int_{F_{n-1,3h}\cap\tilde{F}_n^h}K_0(\chi g^{-2}l_{n-1}(p))\,dp.$$

Using Steiner's theorem, $|\{p: l(p)<t\}|\leq(\langle J\rangle+\pi)t$, we have

$$\int_{F_h}|R(p,p;-\chi^2)|\,dp\leq\int_{F_h}Me^{-\frac{\chi l(p)}{T}}I_{\{l(p)\geq t\}}\,dp+\int_{F_h}\frac{M}{\chi}I_{\{l(p)<t\}}\,dp+$$

$$+\left[\sum_n\left(\int_{F_{n+1,3h}\cap\tilde{F}_n^h}K_0(\chi g^{-2}l_{n+1}(p))\,dp+\int_{F_{n-1,3h}\cap\tilde{F}_n^h}K_0(\chi g^{-2}l_{n-1}(p))\,dp\right)\right]\leq$$

$$\leq M|D|e^{-\frac{\chi t}{T}}+\frac{M}{\chi}\,|\{p: l(p)<t\}|+[\cdots]\leq M\left\{|D|e^{-\frac{\chi t}{T}}+\frac{1}{\chi}(\langle J\rangle+\pi)t\right\}+\left[\sum_n\cdots\right].$$

Taking $t=\frac{T}{\chi}\log\chi^2$, $\chi\gg1$, we get for $M=M(\square)$,

(28) $$\int_{F_h}|R(p,p;-\chi^2)|\,dp\leq M\left\{\frac{|D|}{\chi^2}+\frac{\langle J\rangle T}{\chi^2}\log\chi^2+\frac{\pi T}{\chi^2}\log\chi^2\right\}+\left[\sum_n\cdots\right].$$

If $|\sigma|>0$, the square brackets contain the sum of $2|\sigma|$ integrals like

$$I_n=\int_{F_{n+1,3h}\cap\tilde{F}_n^h}K_0(\chi g^{-2}l_{n+1}(p))\,dp.$$

But, for $p\in F_{n+1,3h}\cap\tilde{F}_n^h$, $\liminf_{p\to O_{n+1}}\frac{l_{n+1}(p)}{|p-O_{n+1}|}\geq\sin(\frac{\gamma}{2})$, (see Appendix 1). Then, given c such that $0<c<\sin(\frac{\gamma}{2})$, we can choose h so small that for $p\in F_{n+1,3h}\cap\tilde{F}_n^h$,

(29) $$\frac{l_{n+1}(p)}{|p|}\geq c\ .$$

Therefore,

(30) $I_n < \int_{R^2} K_0(\chi g^{-2} l_{n+1}(p))\, dp \leq \int_{R^2} K_0(\chi g^{-2} c|p|)\, dp \leq$

$$\leq 2\pi \int_0^\infty K_0\big((c\chi g^{-2})\tau\big)\tau\, d\tau = \kappa/\chi^2 < \infty.$$

From (28) and (30) we get

$$\int_{F_h} |R(p,p;-\chi^2)|\, dp < M\left\{\frac{|D|}{\chi^2} + \frac{\langle J\rangle T}{\chi^2}\log\chi^2 + \frac{\pi T}{\chi^2}\log\chi^2\right\} + \frac{2\kappa|\sigma|}{\chi^2}.$$

In consequence, $\int_{F_h} |R(p,p;-\chi^2)| dp = O\left(\frac{\log\chi^2}{\chi^2}\right)$. Then, for χ large enough we obtain the first of the next inequalities for curved polygons. The second one follows from (3) §**6**.

THEOREM 1. For C a constant and $\chi \gg 1$, we have

(31) $\int_{F_h} |R(p,p;-\chi^2)|\, dp \leq C\,\frac{\log\chi^2}{\chi^2}$,

(32) $\int_{D\setminus F_h} |H(q,q;-\chi^2)|\, dq \leq |D| K_0\left(\chi\frac{h}{2}\right) e^{-\frac{\chi h}{2}}$.

(32) implies that $\int_{D\setminus F_h} |H| dq = o(1) e^{-\frac{\chi h}{2}}, \chi \uparrow \infty$, as we have already observed.

6.4. Last steps of the proof of Lemma 1, §6.1. Next, we evaluate an approximation of $\int_{F_h} H(p,p;-\chi^2) dp$ for χ large enough.

PROPOSITION 1. $\frac{1}{2\pi}\int_{F_n^h} K_0(\chi|p-\hat{p}|)\, dp = \frac{1}{8\chi}\langle J_n\rangle + O(1)\frac{1}{\chi^2}$.

PROOF. $\int_{F_n^h} \frac{K_0(\chi|p-\hat{p}|)}{2\pi} dp = (\S 2) = \frac{1}{2\pi}\int_0^{\langle J_n\rangle} d\xi_1 \int_0^h K_0(\chi 2\xi_2)\big[1 - c(\xi_1)\xi_2\big]\, d\xi_2 =$

$$= \frac{1}{2\pi}\int_0^{\langle J_n\rangle}\left\{\frac{1}{2\chi}\int_0^{2h\chi} K_0(t) dt - \frac{c(\xi_1)}{(2\chi)^2}\int_0^{2h\chi} K_0(t) t\, dt\right\} d\xi_1$$

Since $K_0(r) = \int_1^\infty \frac{e^{-rt}}{\sqrt{t-1}\sqrt{t+1}}\, dt$ and $\chi h \geq 1$, the last term is equal to

$$\frac{1}{2\pi}\int_0^{\langle J_n\rangle}\left\{\frac{1}{2\chi}\int_0^\infty K_0(t) dt - \frac{1}{2\chi}\int_{2h\chi}^\infty K_0(t)\, dt\right\} d\xi_1 + O(1/\chi^2) =$$

$$= \frac{1}{2\pi}\int_0^{\langle J_n\rangle}\left\{\frac{1}{2\chi}\int_0^\infty K_0(t) dt + O(1)\, e^{-\chi h}/\chi\right\} d\xi_1 + O(1/\chi^2) =$$

$= $ (because of $\int_0^\infty K_0(r)\, dr = \frac{\pi}{2}$ and for $\chi \gg 1$) $=$

$$= \int_0^{\langle J_n\rangle} \frac{1}{8\chi} d\xi_1 + O(\chi^{-2}) =$$

$$= \frac{\langle J_n\rangle}{8\chi} + O(\chi^{-2}),$$ QED

PROPOSITION 2. i) For $p \in F_n^h \setminus \tilde{F}_n^h$ we have

(33) $\quad \mathrm{dist}\,(p, J_n) = |p-\hat{p}|/2 \geq c|p - O_{n+1}|,\ c > 0.$

ii) $\sum_{n=1}^{|\sigma|} \int_{\tilde{F}_n^h} K_0(\chi|p-\hat{p}|)\, dp = \sum_1^{|\sigma|} \int_{F_n^h} K_0(\chi|p-\hat{p}|)\, dp + O(\chi^{-2})$.

PROOF. i) can be proved as (29) §**6.3**, (cf. App. 1).

ii) In view of (33), $\int_{\tilde{F}_n^h} K_0(\chi|p-\hat{p}|)\, dp = \int_{F_n^h} K_0(\chi|p-\hat{p}|)\, dp + E_n$ with

(34) $\quad |E_n| \le 2\int_{|p|<\infty} K_0(\chi 2c|p|)\, dp = 2\chi^{-2}\int_{|p|<\infty} K_0(2c|p|)\, dp = O(\chi^{-2}),$ QED.

Thus, we have for $\chi \gg 1$, (cf. Prop. 4, §**4.1** and (32)),

(35) $\quad I = I(\chi^2) := \int_D \; H(x,x;-\chi^2)\, dx =$

$= \int_{F_h} H(x,x;-\chi^2)\, dx + \int_{D\setminus F_h} H(x,x;-\chi^2)\, dx =$

$= \int_{\cup \tilde{F}_n^h} H(x,x;-\chi^2)\, dx + \int_{D\setminus F_h} H(x,x;-\chi^2)\, dx =$(14') §**6.2** =

$= \sum_n \int_{\tilde{F}_n^h} H(x,x;-\chi^2)\, dx + o\left(e^{-\chi h/2}\right)$ = (16) §**6.2** =

$= \sum_n \int_{\tilde{F}_n^h} \left\{R(p,p;-\chi^2) + \frac{K_0(\chi|p-\hat{p}|)}{2\pi}\right\} dp + o\left(e^{-\chi h/2}\right)$ = §**4.2** =

$= \int_{\cup \tilde{F}_n^h} R(p,p;-\chi^2)\, dp + 0 + \sum_n \int_{\tilde{F}_n^h} \frac{K_0(\chi|p-\hat{p}|)}{2\pi}\, dp +$
$o\left(e^{-\chi h/2}\right) =$

$= \int_{F_h} R(p,p;-\chi^2)\, dp + \sum_n \int_{\tilde{F}_n^h} \frac{K_0(\chi|p-\hat{p}|)}{2\pi}\, dp + o\left(e^{-\chi h/2}\right) =$
(31) =

$= \sum_n \int_{\tilde{F}_n^h} \frac{K_0(\chi|p-\hat{p}|)}{2\pi}\, dp + O\left(\frac{\log\chi^2}{\chi^2}\right) + o\left(e^{-\chi h/2}\right)$ = ii) Prop. 2,
§**6.4** =

$= \frac{1}{2\pi}\sum_n \int_{F_n^h} K_0(\chi|p-\hat{p}|)\, dp + O\left(\frac{1}{\chi^2}\right) + O\left(\frac{\log\chi^2}{\chi^2}\right)$ = Prop.1,
§**6.4** =

$= \frac{\langle J\rangle}{8\chi} + O(\frac{\log\chi^2}{\chi^2})$.

We obtained the

PROPOSITION 3. If $\chi \ge \chi_0 \gg 1$ then there exists a constant C independent of χ such that $|L(\chi^2)| = \left|\frac{\langle J\rangle}{8\chi} - I(\chi^2)\right| \frac{\chi^2}{\log\chi^2} \le C < \infty$.

Therefore, the proof of Lemma 1, §**6.1,** and those of Theorems 1, 2, §**6.1**, are finally accomplished.

7. APPENDIX 1. PROPOSITION. Let γ be the measure of the interior angle at O_{n+1}, $\widehat{J_n J_{n+1}}$. Then, $P \coloneqq \{x \in F_n^h; \text{dist}(x, J_{n)} = \text{dist}(x, J_{n+1)}\}$ is a C^1 curve verifying that the measure of the angle at the vertex O_{n+1} between P and J_{n+1} is equal to $\gamma/2$**.**

Let us verify this assertion using an implicit function theorem.

(1) $\quad P \coloneqq \{x : x = y(s) + \tau n_y(s) = z(u) + \tau n_z(u)\}$

where $\{y(s): s \le 0\}$ is the curve J_n and $\{z(u): u \ge 0\}$ is the curve J_{n+1} and $y(0) = z(0) = O_{n+1}$.

Here, $n_y(s) = (-\dot{y}_2(s), \dot{y}_1(s))$, $n_z(u) = (-\dot{z}_2(u), \dot{z}_1(u))$.

The equation (1) can be written as

(2) $$\begin{cases} F(s,u,\tau) \equiv y_1(s) - \tau\dot{y}_2(s) - z_1(u) + \tau\dot{z}_2(u) = 0 \\ G(s,u,\tau) \equiv y_2(s) + \tau\dot{y}_1(s) - z_2(u) - \tau\dot{z}_1(u) = 0 \end{cases}.$$

Then,

(3) $$\frac{\partial(F,G)}{\partial(s,u)} = \begin{vmatrix} \dot{y}_1(s) - \tau\ddot{y}_2(s) & -\dot{z}_1(u) + \tau\ddot{z}_2(u) \\ \dot{y}_2(s) + \tau\ddot{y}_1(s) & -\dot{z}_2(u) - \tau\ddot{z}_1(u) \end{vmatrix} =$$

$$= \left(-\dot{y}_1(s)\dot{z}_2(u) + \dot{y}_2(s)\dot{z}_1(u)\right) + \tau[\cdots],$$

where

$$-\dot{y}_1(0)\dot{z}_2(0) + \dot{y}_2(0)\dot{z}_1(0) = -\sin(\pi - \gamma) = -\sin\gamma < 0$$

and $[\cdots] = O(1)$ because of

(4) $$[\cdots] = \ddot{y}_2(s)\dot{z}_2(u) + \ddot{y}_1(s)\dot{z}_1(u) - \dot{y}_1(s)\ddot{z}_1(u) - \dot{y}_2(s)\ddot{z}_2(u) + \\ +\tau\left(\ddot{y}_2(s)\ddot{z}_1(u) - \ddot{y}_1(s)\ddot{z}_2(u)\right).$$

Applying Schwarz's inequality, we obtain

(5) $$[\cdots] \le |\ddot{y}(s)| + |\ddot{z}(u)| + \tau|\ddot{y}(s)||\ddot{z}(u)| \le 2c(J) + \tau c(J)^2 = O(1).$$

Thus $\frac{\partial(F,G)}{\partial(s,u)}(0,0,0) \ne 0$ and s and u are C^1 functions of τ in a small neighborhood of O_{n+1}, ([F]).

We can assume without loss of generality that $(\dot{z}_1(0), \dot{z}_2(0)) = (1,0)$. Then,

$$\dot{y}(0) = (-\cos\gamma, -\sin\gamma), \quad n_y(0) = (\sin\gamma - \cos\gamma).$$

From (1) we see that the equation of P is $x(\tau) = y(s(\tau)) + \tau n_y(s(\tau))$ and we have, if $s = u = \tau = 0$,

(6) $$\begin{cases} \frac{dF(s,u,\tau)}{d\tau} \equiv \dot{y}_1(0)\frac{ds}{d\tau}(0) - \dot{y}_2(0) - \dot{z}_1(0)\frac{du}{d\tau}(0) + \dot{z}_2(0) = 0 \\ \frac{dG(s,u,\tau)}{d\tau} \equiv \dot{y}_2(0)\frac{ds}{d\tau}(0) + \dot{y}_1(0) - \dot{z}_2(0)\frac{du}{d\tau}(0) - \dot{z}_1(0) = 0 \end{cases}.$$

That is, $\begin{cases} \dot{y}_1(0)\frac{ds}{d\tau}(0) - \dot{y}_2(0) - \frac{du}{d\tau}(0) = 0 \\ \dot{y}_2(0)\frac{ds}{d\tau}(0) + \dot{y}_1(0) - 1 = 0 \end{cases}$. Equivalently,

(7) $\begin{cases} -\cos\gamma \frac{ds}{d\tau}(0) + \sin\gamma - \frac{du}{d\tau}(0) = 0 \\ -\sin\gamma \frac{ds}{d\tau}(0) - \cos\gamma - 1 = 0 \end{cases}$ $\Rightarrow$

(8) $\frac{ds}{d\tau}(0) = -\frac{1+\cos\gamma}{\sin\gamma} = -\frac{\cos\gamma/2}{\sin\gamma/2}, \qquad \frac{du}{d\tau}(0) = \cos\gamma \frac{\cos\gamma/2}{\sin\gamma/2} + \sin\gamma = \frac{\cos\gamma/2}{\sin\gamma/2}.$

But,

$$\left.\frac{dx}{d\tau}\right|_{\tau=0} = \dot{y}(0)\frac{ds}{d\tau}(0) + n_y(0) = \frac{\cos\gamma/2}{\sin\gamma/2}(\cos\gamma, \sin\gamma) + (\sin\gamma, -\cos\gamma) =$$

$$= \frac{1}{\sin(\gamma/2)}\left(\cos\frac{\gamma}{2}, \sin\frac{\gamma}{2}\right).$$

In consequence, the measure of the angle $\widehat{P\,J_{n+1}}$ is $\frac{\gamma}{2}$, QED.

8. Appendix 2. In this paragraph, we consider regions obtained as an increasing limit of sequences of special regions.

Definition 0. We will say that a region D has property $\boldsymbol{P}$ if there exists a positive constant V such that its family of eigenvalues $\lambda_n(D)$ of the Dirichlet problem verifies

(1) $\sum_1^\infty \lambda_n^{-z} = \frac{|D|}{4\pi}\frac{1}{z-1} - \frac{V}{8\pi}\frac{1}{z-\frac{1}{2}} + g(z)$, where $g(z)$ is holomorphic on $\operatorname{Re} z > 0$.

Curved polygons $\in \boldsymbol{P}$ with $V =$ length of ∂D. The problem is:

Does the property $\boldsymbol{P}$ remain valid under limits of sequences of increasing regions that have this property?

Definition 1. A region S will be called a *feasible* region if it is the increasing limit of a *feasible* sequence $\boldsymbol{S}$ of regions:

$\boldsymbol{S} = \{D_k\}$, $D_k \uparrow S$, $D_k \subset D_{k+1}$, D_k a curved polygon, such that $\sup_k \langle J_k \rangle =: \iota(\boldsymbol{S}) < \infty$.

Since by definition a region is an open bounded connected set we have $|S| = \lim_k |D_k| = \sup_k |D_k| < \infty$.

Definition 2. For $\boldsymbol{S}$ a feasible sequence, $\kappa(\boldsymbol{S}) := \sup_k \sup_{s \neq \text{corner}} |\ddot{y}_k(s)|$.

Example. An *oval* is a plane convex body, *i.e.*, the closure of a convex region. It is possible to construct a rectangle T consisting of support lines, that is, a rectangle circumscribed about the oval. Around any corner of T the contour of the oval is a curve that, except for some segments on the sides, can be described as a convex monotone function. Then S, the interior of the oval, is such that ∂S is a rectifiable curve and it is the increasing limit of a set $\boldsymbol{S}$ of inscribed convex polygonal regions $D_k \uparrow S$, $D_k \subset D_{k+1}$. The boundary J of S verifies $\langle J_k \rangle \to \langle J \rangle$. Thus, the oval S is a feasible region that admits a feasible sequence $\boldsymbol{S}$ such that $\kappa(\boldsymbol{S}) = 0$, $\iota(\boldsymbol{S}) = \langle \partial S \rangle$.

Feasible sequences. Assume S is a feasible region. We will use the following assertions (see, for example, [BPII] Th. 3, Appendix part II; Th. 7, §**4.1**; Ths. 1 and 3, §**4.3**; Th. 1, §**4.4**; Th. 1, §**4.8**. Cf. also [BP III] Th. 3 i) §**6.1.1**.)

1) For each n it holds that $\lambda_n(D_k) \downarrow$ if $k \uparrow \infty$ and that $\lambda_n(D_k) \geq \lambda_n(S)$.

2) For the regions D_k and S the variational and classical eigenvalues and eigenfunctions coincide.

3) For each k, $0 < \lambda_n(D_k) \uparrow \infty$ as $n \uparrow \infty$ but in such a way that $\sum_1^\infty \lambda_n(D_k)^{-2} < \infty$.

4) The Green functions verify: $G_{D_k} \uparrow G_S$, $\left\|G_{D_k}\right\|_2^2 = \sum \lambda_n(D_k)^{-2}$ and

$\|G_S\|_2^2 = \sum \lambda_n(S)^{-2}$.

1)-4) imply that

(2) $$\left\|G_{D_k}\right\|_2^2 \uparrow \|G_S\|_2^2 \quad \text{and} \quad \left(\frac{1}{\lambda_n(D_k)}\right)^2 \uparrow \left(\frac{1}{\lambda_n(S)}\right)^2.$$

Therefore, we get the

PROPOSITION 1. If $k \to \infty$, it holds for each n,

(3) $$\lambda_n(D_k) \downarrow \lambda_n(S).$$

Then, for $s > 1$, $\sum \lambda_n(D_k)^{-s} \uparrow \sum \lambda_n(S)^{-s} < \infty$, since $\lambda_n(S) \sim cn$, (H. Weyl's theorem, [BPII], Th. 2. §**4.10**). In consequence, we obtain the

PROPOSITION 2. The following limit holds almost uniformly on Re $z > 1$, $z = s + it$:

(4) $$\sum_{n=1}^\infty \lambda_n(D_k)^{-z} \underset{k\to\infty}{\longrightarrow} \sum_{n=1}^\infty \lambda_n(S)^{-z}.$$

In fact, for fixed N, $\sum_1^N \lambda_n(D_k)^{-z} - \sum_1^N \lambda_n(S)^{-z} \to 0$ uniformly on compact sets of Re $z > 1$.

For N sufficiently large we have for certain $\varepsilon > 0$,

$$\left|\sum_{N+1}^\infty \lambda_n(D_k)^{-z} - \sum_{N+1}^\infty \lambda_n(S)^{-z}\right| \le 2\sum_{N+1}^\infty \lambda_n(S)^{-s} \le 2\sum_{N+1}^\infty \lambda_n(S)^{-(1+\varepsilon)} \to 0$$

for $N \to \infty$, QED.

Therefore, $M_j(z) := \left(\sum_{n=1}^\infty \frac{1}{\lambda_n^z(D_j)}\right) - \frac{|D_j|}{z-1}$ is almost uniformly convergent on Re $z > 1$ to $M(z) := \left(\sum_{n=1}^\infty \frac{1}{\lambda_n^z(S)}\right) - \frac{|S|}{z-1}$. Because of Pleijel's theorem, $-\frac{\langle J_j\rangle}{8\pi\left(z-\frac{1}{2}\right)} + g_j(z)$ is almost uniformly convergent on Re $z > 1$ to $M(z)$ where $g_j(z)$ are holomorphic functions on Re $z > 0$.

In particular, $\{g_j(z)\}$ is an almost uniformly bounded family on $\{\text{Re}\, z > 1\}$.

THEOREM 1. Assume S is a feasible region and that $\{g_j(z)\}$ is an almost uniformly bounded family on $\{\text{Re}\, z > 0\}$. Then, there is a number V, the "virtual length" of ∂S, such that $\langle J_k\rangle \to V$ and

(5) $$\sum_1^\infty \lambda_n^{-z}(S) = \frac{|S|}{4\pi(z-1)} - \frac{V}{8\pi\left(z-\frac{1}{2}\right)} + g(z), \quad g(z) \text{ holomorphic on Re } z > 0.$$

PROOF. Let $\{D_{j_i}\}$ be a subsequence of the sequence $\{D_j\}$. Then, for each sequence $y_0 = \{j_i\}$, it holds on $\operatorname{Re} z > 1$ that

(6) $\quad \sum_1^\infty \lambda_n^{-z}(D_{j_i}) - \frac{|D_{j_i}|}{4\pi(z-1)} = -\frac{\langle J_{j_i}\rangle}{8\pi\left(z-\frac{1}{2}\right)} + g_{j_i}(z),\ \ g_{j_i}$ holomorphic on $\operatorname{Re} z > 0$,

and there is a subsequence of y_0, $y_1 = \{j_{i_k}\}$, such that $\{\langle J_{j_{i_k}}\rangle\}$ converges to a certain positive number V because of the definition of a feasible region.

(If S is an oval it is possible to choose $\boldsymbol{S}$ in such a way that $V = \langle J\rangle = \lim \uparrow \langle J_k\rangle$.)

Because of the Stieltjes-Osgood Theorem, ([SZ]), $\{g_{j_{i_k}}(z)\}$ converges through a subsequence of y_1, $y_2 = \{j_{i_{k_l}}\}$, almost uniformly to $g(z)$, a holomorphic function on $\operatorname{Re} z > 0$: $\qquad g(z) = \lim_{a.u.} g_{j_{i_{k_l}}}(z).$

(If a meromorphic function on $\operatorname{Re} z > 1$ is of the form $\frac{A}{z-\frac{1}{2}} + B(z)$, A a constant and $B(z)$ a holomorphic function on $\operatorname{Re} z > 0$, then A and B are uniquely determined. From this we know that V and g do not depend on the subsequences y_1, y_2 chosen.)

In consequence, the right-hand side of (6) converges almost uniformly to

(7) $$\frac{-V}{8\pi\left(z-\frac{1}{2}\right)} + g(z) \ \text{ on } \operatorname{Re} z > 1/2.$$

Therefore,

$$\sum_{n=1}^\infty \lambda_n^{-z}(D_j) - \frac{|D_j|}{4\pi(z-1)} \to \frac{-V}{8\pi\left(z-\frac{1}{2}\right)} + g(z) \quad \text{almost uniformly on } \operatorname{Re} z > 1.$$

Then, on this half plane,

(8) $$M(z) + \frac{V}{8\pi\left(z-\frac{1}{2}\right)} = \sum_{n=1}^\infty \lambda_n^{-z}(S) - \frac{|S|}{4\pi(z-1)} + \frac{V}{8\pi\left(z-\frac{1}{2}\right)} = g(z),$$

with $g(z)$ holomorphic on $\operatorname{Re} z > 0$, QED.

9. Appendix 3. If D is a curved polygon, then $S(z) := \sum_1^\infty \lambda_n^{-z}$, where $\{\lambda_n^{-z}(D) : n = 1,2,\dots\}$ is the family of eigenvalues for the Dirichlet problem, is well defined on $\operatorname{Re} z > 1$.

If $z = s + it$ then $S(\bar{z}) = \overline{S(z)}$ and $S(s)$ is real.

The meromorphic function $M(z) = \frac{|D|}{4\pi}\frac{1}{z-1} - \frac{\langle J\rangle}{8\pi}\frac{1}{z-\frac{1}{2}}$ verifies these same properties. Since $S = M + g$ on $\operatorname{Re} z > 1$, ($g(z)$ is holomorphic on $\operatorname{Re} z > 0$), g also verifies $g(\bar{z}) = \overline{g(z)}$ on $\operatorname{Re} z > 1$. It can be proved that its continuation g to $\operatorname{Re} z > 0$ must have the same property.

But, $\Sigma(z) := M(z) + g(z)$ on $\operatorname{Re} z > 0$ is the analytic continuation of S to $\operatorname{Re} z > 0$. Then, $\Sigma(z)$ also verifies this property and $\Sigma(s)$ is a real number except for $s = \frac{1}{2}, 1$.

If we consider Neumann's problem[4] for he Laplacian in a plane Jordan region D with a C^2 regular boundary J and if $\{l_n = l_n(D) : n = 1,2, \dots\}$ is the family of eigenvalues then on $\operatorname{Re} z > 1$ it holds that $S_N(z) := \sum_1^\infty \frac{1}{l_n^z} = \frac{|D|}{4\pi} \frac{1}{z-1} + \frac{\langle J \rangle}{8\pi} \frac{1}{z-\frac{1}{2}} + f(z) = M_n(z) + f(z)$.

Here, again, $f(z)$ is a holomorphic function on the right half plane and $\Sigma_N(z) := M_N(z) + f(z)$ on $\operatorname{Re} z > 0$, is the analytic continuation of S_N to $\operatorname{Re} z > 0$. Because of the signs of the residues in M_N, from Darboux's theorem it follows that there is an $x \in \left(\frac{1}{2}, 1\right)$ such that $\Sigma_N(x) = 0$.

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SYMBOLS and DEFINITIONS

λ_n eigenvalue	§6
$\tilde{F}_n^h$ partial strip	Def. 3, §4
$\int_{F_h} R(q,q;-\chi^2)\,dq$	§6.3
A_i , A'_i intervals	§3
$B_{qh}(O_i)$	§6.2
F_h interior strip, $F_{n,h}$, F_n^h	Notation §4
$\langle J\rangle$	§1
$\langle J\rangle$ length	§2
J_0 closed arc, J_n open arc, $J_n^h, J_{n,h}$	Notation §4
K_0	§6
L_n open arc	§3
M, M_1	§5
$\lvert T(\xi) - T(\eta)\rvert \cong \lvert\xi - \eta\rvert$	§5
$\hat{x}, \hat{p}$, symmetric points	Def. 2, §4
$y_i, \dot{y}_i, \ddot{y}_i$	§2
Curved polygon	Def. 4, §3
h	§4, §4.1, NB §4.4, etc.
Homeomorphism	§4
Irregular boundary point	Def. 1, §3
Local coordinates	§2
Minkowski neighborhoods	§4.4
Positive side of J_n	§4
Proper cuved polygon	Def. 3, §3

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