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Integrable Systems in Finite and Infinite Dimension

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INTEGRABLE SYSTEMS IN FINITE AND INFINITE DIMENSION

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ABSTRACT. The purpose of this paper is to present some advances into considering a.c.i. systems as pieces of infinite dimensional systems such as the **KP** hierarchy or Multicomponent **KP** hierarchy. This is to be done in such a way that the affine invariant manifolds of the a.c.i. systems embedd into the Universal Grassmann Manifold **UGM** and the Hamiltonian flows coincide with some flows of the infinite dimensional systems. As an example one considers the case of the rigid body motion in $SO(3)$.

1. Introduction.

Finite dimensional integrable systems are pretty well understood. A good description of these systems is given by the Arnold-Liouville theorem [Ar]:

THEOREM 1. *Let $F_1, \dots, F_n, F_{n+1}, \dots, F_{n+k} : \mathbb{R}^{2n+k} \to \mathbb{R}$ be $(n+k)$ independent functions in involution (i.e. $\{F_i, F_j\} = 0$ with respect to a Poisson bracket $\{\cdot,\cdot\}$), where $F_{n+1}, \dots, F_{n+k}$ are trivial invariants and F_i, $i = 1, \dots, n$ generate nontrivial vector fields on the manifolds $M_c = \{x \in \mathbb{R}^{2n+k} : F_i(x) = c_i, \quad i = 1, \dots, n+k\}$. The hamiltonian vector fields X_{F_i}, $i = 1, \dots, n$, span the tangent space to M_c at each point and the compact and connected components of M_c are tori. Moreover, if we consider the map $F : \mathbb{R}^{2n+k} \to \mathbb{R}^{n+k}$, $F = (F_1, \dots, F_{n+k})$ generically submersive, then, around regular points $u \in \mathbb{R}^{n+k}$ we have principal fibrations by tori (or cylinders). Also, on these fibrations we can pick a globally defined coordinate system: the action-angle variables.*

An example of such systems is given by the Euler top, which describes the rigid body motion around the fixed center of gravity. In the angular moment coordinates,

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it reduces to the equations

$$(1) \qquad \begin{cases} \dot{z_1} = (\lambda_2 - \lambda_3) z_2 z_3 \\ \dot{z_2} = (\lambda_3 - \lambda_1) z_3 z_1 \\ \dot{z_3} = (\lambda_1 - \lambda_2) z_1 z_2 \end{cases}$$

It has two independent integrals

$$(2) \qquad \begin{aligned} Q_1 &= z_1^2 + z_2^2 + z_3^2 \\ Q_2 &= \lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 \end{aligned}$$

which commute with respect to the Poisson bracket. Q_1 being the trivial invariant and Q_2 the nontrivial Hamiltonian.

Although the real geometry of integrable systems is described, to some degree, by the Arnold-Liouville theorem, their complex geometry is not so well understood. The nature of the solutions to integrable systems depends heavily on the complex geometry. If we require the solutions to be expressible in terms of theta functions related to some abelian variety (i.e. a complex torus in projective space), then, we call such systems algebraic complete integrable (a.c.i.). Many of these systems were known classically in Mechanics and studied in detail by Adler and Van Moerbeke [A-VM 1,2] and many others [Du], [Mo], [Mu 1,3].

In the Adler and Van Moerbeke picture, the real phase space $\mathbb{R}^{2n+k}$ is complexified, and the integrals are polynomials. The complexified invariant manifolds $\tilde{M}_c = \{z \in \mathbb{C}^{2n+k},\ F_i(z) = c_i,\quad i = 1, \dots, n+k\}$ are affine varieties in $\mathbb{C}^{2n+k}$. They are affine pieces of abelian varieties A_c in such a way that the coordinates z_i become nontrivial abelian functions on A_c. Thus $z_i \in L(\mathcal{D})$ = functions on A_c that blow up on a divisor $\mathcal{D}$ of A_c, and moreover $\tilde{M}_c = A_c \backslash \{$ the reduced divisor $\mathcal{D}\}$.

For instance, in the Euler top case, one obtains (by setting Q_1 and Q_2 to constants) the affine part of an elliptic curve in $\mathbb{P}^3 = \mathbb{P}(L(\mathcal{D}))$ with $\mathcal{D}$ = divisor at infinity = 4 points.

In order to treat infinite dimensional systems we will consider the K.P. hierarchy and the Universal Grassmannian manifold **UGM**. (See Sato [Sa]).

The K.P. hierarchy can be obtained by considering the pseudodifferential operator in one variable x:

$$Q = \partial + u_{-1}(x, t\!t)\partial^{-1} + u_{-2}(x, t\!t)\partial^{-2} + \cdots, \tag{3}$$

where $\partial = \frac{d}{dx}$ and $t\!t = (t_1, \ldots, t_n, \ldots)$ is an infinite collection of time variables. Then, one has the Lax equations

$$\frac{\partial Q}{\partial t_n} = [(Q^n)_+, Q], \quad n = 1, 2, \ldots, \tag{4}$$

where $(Q^n)_+$ is the differential operator part of the order n pseudodifferential operator Q^n.

The systems (4) can be shown to possess an infinite number of integrals: that is, functionals $H_i : \partial + \mathcal{G}_- \to \mathbb{C}$, $\mathcal{G}_-$ = "space of elements of the form $\sum_{i=-\infty}^{-1} u_i(x, t\!t)\partial^i$", such that $\frac{\partial H_i}{\partial t_j} = 0 \ \forall i, j$. Also, the vector fields (4), thought of as vector fields on the manifold $\partial + \mathcal{G}_-$, commute, i.e.

$$\frac{\partial^2 Q}{\partial t_n \partial t_m} = \frac{\partial^2 Q}{\partial t_m \partial t_n}. \tag{5}$$

The K.P. hierarchy can be also described by the pseudodifferential operator in the Lie group $\mathbf{G} = 1 + \mathcal{G}_-$

$$W = 1 + w_{-1}(x, t\!t)\partial^{-1} + w_{-2}(x, t\!t)\partial^{-2} + \cdots, \tag{6}$$

which is related to Q by the formula

$$Q = W \frac{d}{dx} W^{-1}. \tag{7}$$

Describing the evolution of Q is equivalent to describing the evolution of W. Thus, the equations (4) can be written as

$$\frac{\partial W}{\partial t_n} = B_n W - W \left(\frac{d}{dx}\right)^n, \quad B_n = \left(W \left(\frac{d}{dx}\right)^n W^{-1}\right)_+ . \tag{8}$$

This is saying that $\dfrac{\partial W}{\partial t_n} + W\left(\dfrac{d}{dx}\right)^n$ belongs to the cyclic $\mathcal{D}$-module $\mathcal{D}W$ ($\mathcal{D} = \{\sum_{i=0}^n a_i(x,t)\partial^i\}$ = space of differential operators).

In order to define the universal Grassmannian manifold we introduce the space of pseudodifferential operators $\mathcal{P} = \{\sum_{-\infty<i\ll\infty} a_i\partial^i : a_i \in$ ring $\mathcal{O}$ with derivation $\partial : \mathcal{O} \to \mathcal{O}\}$ (e.g. $\mathcal{O}$ = formal power series in x) and the space of differential operators $\mathcal{D} = \{\sum_{0\le i\ll\infty} a_i\partial^i : a_i \in \mathcal{O}\}$. $\mathcal{P}$ has a natural filtration $(\cdots \subset \mathcal{P}^{(-1)} \subset \mathcal{P}^{(0)} \subset \mathcal{P}^{(1)} \subset \cdots)$ by the order of the pseudodifferential operators, and we can split $\mathcal{P} = \mathcal{D} \oplus \mathcal{G}_- = \mathcal{D} \oplus \mathcal{P}^{(-1)}$. We consider the infinite dimensional vector space $V = \mathcal{P}/\mathcal{P}x = \{\sum_{-\infty<i\ll\infty} a_i\partial^i : a_i \in \mathbb{C}\}$ of pseudodifferential operators with constant coefficients. This is a $\mathcal{P}$ and $\mathcal{D}$-module on which is to be modelled the universal Grassmannian manifold **UGM**.

We define

$$\mathbf{UGM} := \left\{\text{vector subspaces } U \subset V : \dim(U \cap V^{(-1)}) = \dim\left(\frac{V}{U + V^{(-1)}}\right) < \infty\right\}.$$

Here $V^{(-1)} = \{\sum_{i<0} a_i\partial^i : a_i \in \mathbb{C}\}$.

That is, **UGM** consists of subspaces of V which are comparable to $U_0 = \mathcal{D} \cap V$.

Now, any W as in (6) has a point $W^{-1}U_0 \in \mathbf{UGM}$ associated to it. Thus, we can view the KP flows as moving in the universal Grassmann manifold.

In section **2** we present a construction of the Baker-Akhiezer functions for a data depending on divisors on generic abelian varieties. The main result of this paper is Theorem 2. Next (In **3**), we apply this Theorem to the case of elliptic curves. In particular, to elliptic curves in $\mathbb{P}^3$; which are related to the Euler Top. In **4** we explore the possibility of viewing holomorphic flows on abelian varieties as flows in **UGM**. We define an appropriate map of the universal covering of the abelian varieties into **UGM** through solutions of the **KP** hierarchy. Section **5** presents some basics about Multicomponent **KP** hierarcy. Finally, the main purpose of section **6** is to identify the Euler Top flow with a Multicomponent **KP** flow under a suitable embedding. The Lemmas and Propositions there describe this identification.

2. Baker-Akhiezer functions for certain data on abelian varieties.

We consider here the construction of Baker-Akhiezer functions for the data related to an abelian variety A, an ample divisor $\mathcal{D}$ (posibly reducible) on A, a (finite) group G of automorphisms of $(A, \mathcal{D})$ and a line bundle $\mathcal{F}$ on A such that $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$. This data will be called Mumford data [Mu 1] and we want to give a generalization of the dictionary between these data and commutative rings of differential operators satisfying certain conditions (e.g. $\dim R_n/R_{n-1} \leq 1$ and $= 1$ for n large).

We will give as examples the case of an elliptic curve in $\mathbb{P}^3$ with $\mathcal{D}$ the divisor cut out by an odd section (i.e. four points at infinity), and the case of an abelian surface with a smooth curve $\mathcal{D}$ as divisor at infinity. The bundle $\mathcal{F}$ will be of the form $\mathcal{F} = [\tau_x^{-1}\mathcal{D} - \mathcal{D}]$ for some $x \in A$, i.e. $\mathcal{F} \in \mathrm{Pic}^0(A)$ will indicate a direction in the Picard variety.

Let us start with

Data A: an abelian variety A, an ample divisor $\mathcal{D}$ on A. A group G of automorphisms of $(A, \mathcal{D})$ and a line bundle $\mathcal{F}$ such that $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$.

We want to recover

Data B: A ring R of differential operators such that $\dim R_n/R_{n-1} \leq c$ and $= c$ for n large.

We want to construct a line bundle $\mathcal{F}^*$ on $A \times \mathbb{C}^\infty$ in the following way. Take the covering formed by $(U - \mathcal{D}) \times \mathbb{C}^\infty = \mathcal{U}_0$ and neighborhoods around $\mathcal{D} \times \mathbb{C}^\infty$ $U_\alpha \times \mathbb{C}^\infty = \mathcal{U}_\alpha$, and let $\mathcal{F}^*$ be obtained by the transition functions $g_{0,\alpha}(u, x_\alpha, t\!\!t) = \exp\left(\sum_{i\geq 1} t_i \int_{x_\alpha}^{u} \Omega_i\right)$, where $t\!\!t = (t_1, t_2, \ldots, t_n, \ldots) \in \mathbb{C}^\infty$ and Ω_i are differential forms of 2^{nd} kind on A such that their expansions around smooth points of $\mathcal{D}$ is $(-1)^i \frac{dz}{z^{i+1}} c_i(x) + O(z^{-(i-1)})\, dz$ with x moving along $\mathcal{D}$. An arsenal of such forms is gotten by taking the differential of derivatives of $log\, \vartheta$, where ϑ is the theta function vanishing on $\mathcal{D}$. Of course, one makes sure that the $g_{0,\alpha}(u, x_\alpha, t\!\!t)$ are compatible transition functions. This can be done by picking a covering of $\mathcal{D}$ by algebraic charts $\mathcal{U}_\alpha$ around points x_α of A belonging to some finite translation group (e.g.

some half periods). Thus, the transition functions will satisfy a cocycle relation $g_{0,\alpha}(u,t\!\!t) = g_{\beta,\alpha}(t\!\!t)g_{0,\beta}(u,t\!\!t)$ with $g_{\beta,\alpha}(t\!\!t)$ depending on some fractional periods of the Ω_i's.

On $A \times \mathbb{C}^\infty$ we can also define the line bundle $\underline{\mathcal{F}} = \pi_1^*(\mathcal{F}) \otimes \pi_2^*(\mathcal{O}_{\mathbb{C}^\infty})$ where π_1 and π_2 are the projections onto the 1, 2-coordinates. Thus, we also have the bundle $\underline{\mathcal{F}}^* = \underline{\mathcal{F}} \otimes \mathcal{F}^*$ with transition functions $\tilde{g}(u,t\!\!t) = g_{\alpha,\beta}(u)\, g_{\gamma,0}(u,t\!\!t)$ where $g_{\alpha,\beta}$ is a set of transition functions for $\mathcal{F}$.

Notice that $\int_{x_\alpha}^{x} \Omega_1 = \dfrac{a(x)}{z} + c(x) + O(z)$ where z is the local parameter about $x \in \mathcal{D}$ such that $\dfrac{\partial}{\partial z}$ is a holomorphic vector field on A. We want to define a differential operator $\nabla : \underline{\mathcal{F}}^* \to \underline{\mathcal{F}}^*(*\mathcal{D})$ such that

$$\nabla(s) = \frac{a(x)\, s}{z} + \text{ section of } \underline{\mathcal{F}}^* \tag{9}$$

for a section s of $\underline{\mathcal{F}}^*$.

Take $\nabla := \dfrac{\partial}{\partial t_1}$, then $\nabla \tilde{g}_{\alpha\beta} = \dfrac{a(x)}{z}\tilde{g}_{\alpha\beta} + (c(x)+O(z))\tilde{g}_{\alpha\beta}$, and for a holomorphic section s of $\mathcal{F}^*$ we have $\nabla(s) = \nabla s_\alpha = \nabla \tilde{g}_{\alpha\beta} s_\beta = \dfrac{a(x)}{z} s_\alpha + \tilde{g}_{\alpha\beta}(\overbrace{\nabla s_\beta + c(x) + O(z)}^{t_\beta})$ on $U_{\alpha\beta}$, since there are holomorphic functions s_α such that $s_\alpha = \tilde{g}_{\alpha\beta} s_\beta$ on $U_{\alpha\beta}$.

On $U_{\alpha\beta\gamma}$ we have the relation $\dfrac{a(x)}{z} s + \tilde{g}_{\alpha\beta} t_\beta = \dfrac{a(x)}{z} s + \tilde{g}_{\alpha\gamma} t_\gamma$. Thus, $t_\beta = \tilde{g}_{\beta\gamma} t_\gamma$ over $U_{\alpha\beta\gamma}$, i.e. t is a section of $\underline{\mathcal{F}}^*$.

Now, notice that

$$H^i(A \times \mathbb{C}^\infty, \underline{\mathcal{F}}) = \bigoplus_{j=0}^{i} H^j(A, \mathcal{F}) \otimes H^{i-j}(\mathbb{C}^\infty, \mathcal{O}_{\mathbb{C}^\infty}) = 0 \qquad \text{for } i = 0,\, 1,$$

implies that $H^i(A \times \mathbb{C}^\infty, \underline{\mathcal{F}}^*) = 0$, $i = 0, 1$. So, we have an isomorphism

$$H^0(A \times \mathbb{C}^\infty, \underline{\mathcal{F}}^*(\mathcal{D})) \cong H^0(A \times \mathbb{C}^\infty, \underline{\mathcal{F}}^*(\mathcal{D})/\underline{\mathcal{F}}^*). \tag{10}$$

Note 1. In dimensions higher than one it will be important to consider the behaviour of the above homology groups under desingularization of $\mathcal{D}$ embedded in A, i.e. $(\tilde{A}, \tilde{\mathcal{D}}) \to (A, \mathcal{D})$. One should look for birational invariants in case the hypothesis are destroyed after desingularization.

Definition 1: A section of the line bundle $\mathcal{F}^*$ which is meromorphic on A and holomorphic on $\mathbb{C}^\infty$ will be called a Baker-Akhiezer section (or Baker-Akhiezer function). In practice, one looks at Baker-Akhiezer sections that have pole divisors on A of the form $n\mathcal{E}$, with $\mathcal{E}$ an ample divisor different from $\mathcal{D}$. The vector space of sections of $\mathcal{F}^*$ which blow up at most once at $\mathcal{E}$ is denoted by $\Lambda(\mathcal{E})$.

Given the divisor $\mathcal{E}$, one considers the ϑ-function Θ associated to it ([We], [Ig]), i.e. Θ vanishes once on $\mathcal{E}$. Let $\mathcal{A}_{x_\alpha} : A \to H^0(A,\Omega^1)^*/H_1(A,\mathbb{Z})$ be a set of Albanese's maps, $\mathcal{A}_{x_\alpha}(x) = \left(\int_{x_\alpha}^x \omega_1, \ldots, \int_{x_\alpha}^x \omega_n\right)$, for some conveniently chosen $x_\alpha \in A$. Here, the integrals are along a path γ joining x_α and x. For abelian varieties these maps are isomorphisms and any two of them differ by a translation on A.

There is a basis of holomorphic differentials $\omega_1, \ldots, \omega_n$, and basis of homology cycles $\{a_i, b_i\}$, $i = 1, \ldots, n$, such that the period matrix has the form $\left(\int_{a_i} \omega_j, \int_{b_i} \omega_j\right) = (I, \tau)$. According to Igusa [Ig] any ϑ-function Θ can be written as a linear combination of ϑ-series of the form

$$\Theta_m(\tau, z) = \sum_{p \in \mathbb{Z}^n} \mathrm{e}(\tfrac{1}{2}(p+m')\tau^t(p+m') + (p+m')^t(z+m'')) \tag{11}$$

where $m = (m'm'')$ and m', m'' in $\mathbb{R}^n$ and $\mathrm{e}(x) = \exp(2\pi i x)$. Such a ϑ-series satisfies

$$\Theta_m(\tau, z + n'\tau + n'') = \Theta_m(\tau, z)\ \mathrm{e}(-\tfrac{1}{2}n'\tau^t n' - n'^t z)\ \mathrm{e}(m'^t n'' - n'^t m'') \tag{12}$$

for any element $n'\tau + n''$, $n', n'' \in \mathbb{Z}^n$, belonging to the lattice of the abelian variety.

Moreover, if $\delta = (\delta_1, \ldots, \delta_n)$, δ_i/δ_j are the integers defining the polarization type of $\mathcal{E}$, then there exist real vectors $m', m'' \in \mathbb{R}^n$ such that

$$\Theta(z) = \sum_{r \bmod \mathbb{Z}^n} \text{constant} \cdot \Theta_{(r+m'\Delta_\delta^{-1}, m'')}(\tau, z), \tag{13}$$

where r runs over a complete set of representatives of $\left(\frac{1}{\delta_1}\mathbb{Z} \times \cdots \times \frac{1}{\delta_n}\mathbb{Z}\right)/\mathbb{Z}^n$ and $\Delta_\delta = \mathrm{diag}(\delta_1, \ldots, \delta_n)$.

Following [Du], [Sh] and [Ma–Ka] we define the Baker-Akhiezer function associated to the divisors $\mathcal{D}$ and $\mathcal{E}$ as follows.

$$\psi(u, t\!\!t, x_\alpha, x), \quad u \in \mathbb{C}^n, \quad \Theta(u) \neq 0, \quad t\!\!t \in \mathbb{C}^\infty, \quad x \in A - \mathcal{D},$$

$$z_\alpha \text{ a local parameter of } \mathcal{D} \text{ in } U_\alpha.$$

$$\psi(u,\mathbb{t},x_\alpha,x) = f_\alpha(\mathbb{t})\,\mathrm{e}\left(\sum t_i \int_{x_\alpha}^{x} \eta_i\right) \frac{\Theta\left(\mathbb{B}^t\mathbb{t} + u - \mathcal{A}_{x_\alpha}(x)\right)}{\Theta\left(u - \mathcal{A}_{x_\alpha}(x)\right)}, \tag{14}$$

where η_i are normalized 2^{nd} kind differentials and $\mathbb{B}$ its matrix of b-periods: $\mathbb{B} = \left(\int_{b_i}\eta_j\right)$. As before, η_i have local expansions around $\mathcal{D}\cap U_\alpha$, $\eta_i \sim (-1)^i c_i \dfrac{dz_\alpha}{z_\alpha^{i+1}} + O(z_\alpha^{-i})\,dz_\alpha$. The functions $f_\alpha(\mathbb{t})$ are holomorphic functions of the variable $\mathbb{t}$ to be chosen later.

As we increase w by the period $n'\tau + n''$ we get the change

$$\begin{aligned}\Theta(w+n'\tau+n'') &= \sum_{r \bmod \mathbb{Z}^n} c_r\Theta_{(r+m'\Delta_\delta^{-1},m'')}(\tau, w + \,'\!n\tau + n'')\\ &= \left[\sum_{r \bmod \mathbb{Z}^n} c_r\Theta_{(r+m'\Delta_\delta^{-1},m'')}(\tau,w)\,\mathrm{e}\big((r+m'\Delta_\delta^{-1})^t n'' - \,'\!n^t m''\big)\right]\cdot\\ &\quad\cdot \mathrm{e}(-\tfrac{1}{2}n'\tau^t n' - \,'\!n^t w).\end{aligned}$$

Thus $\dfrac{\Theta(w+w_0)}{\Theta(w)}$ is changed by the factor $\mathrm{e}(-\,'\!n^t w_0)$.

Now, changing $\sum t_i \int_{x_\alpha}^{x}\eta_i$ by the homology cycle $n'^t b + n''^t a$ produces the extra factor $\mathrm{e}\left(\sum_i t_i\,'\!n^t\left(\int_{b_j}\eta_i\right)\right)$ in ψ, which cancels with the contribution of the term $\mathrm{e}\left(-\,'\!n\left(\sum_i t_i{}^t\left(\int_{b_j}\eta_i\right)\right)\right) = \mathrm{e}(-\,'\!n\mathbb{B}^t\mathbb{t})$ due to the theta functions.

This shows that the function (13) extends to a well defined meromorphic function on the open set $\mathcal{U}_0$ that blows up once at $\mathcal{E}$. The quotient $\frac{\Theta\left(\mathbb{B}^t\mathbb{t}+u-\mathcal{A}_{x_\alpha}(x)\right)}{\Theta(u-\mathcal{A}_{x_\alpha}(x))}$ extends on $\mathcal{U}_\alpha$ to a function that blows up at the divisor $\mathcal{E} = \{x \in A : \Theta\left(u - \mathcal{A}_{x_\alpha}(x)\right) = 0\}$.

Let $s_0 \in L(\mathcal{U}_0 \cap \mathcal{E}) = \{$ space of meromorphic functions on $\mathcal{U}_0$ that blow up once on $\mathcal{E}\cap\mathcal{U}_0$ $\}$ denote the function (13), and $s_\alpha \in L(\mathcal{U}_\alpha \cap \mathcal{E})$ the function $s_\alpha = f_\alpha(\mathbb{t})h_\alpha(u,x,\mathbb{t}) = f_\alpha(\mathbb{t})\frac{\Theta\left(\mathbb{B}^t\mathbb{t}+u-\mathcal{A}_{x_\alpha}(x)\right)}{\Theta(u-\mathcal{A}_{x_\alpha}(x))}$. Then, we can write (13) in the form $s_0 = g_{0,\alpha}s_\alpha$, which says that $\{s := s_\alpha$ on $\mathcal{U}_\alpha\}$ is a section of the bundle $\mathcal{F}^*$. The compatibility at the intersection $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ requires that $s_\alpha = g_{\alpha,\beta}\,s_\beta$, where $g_{\alpha,\beta} = p_\alpha/p_\beta$ and $p_\alpha = \exp\left(\sum_{i\geq 1} t_i \int_0^{x_\alpha}\Omega_i\right)$, $0 =$ an origin of A. These conditions are satisfied if we choose $f_\alpha = p_\alpha$, $f_\beta = p_\beta$ and pick $x_\alpha \in$ group of translates t such that $\Theta(x+t) = c_t\Theta(x)$. (There are at most $(\delta_1\cdots\delta_n)$ points of this kind on an abelian variety).

Now, we can state the

THEOREM 2. *There is a Baker-Akhiezer function ψ on A with expansion*

$$\psi(t\!\!t, u, x_\alpha; x) = f_\alpha(t\!\!t) g_{0,\alpha}(x, x_\alpha, t\!\!t)(1 + a_1(u, t\!\!t)z + \cdots) \tag{15}$$

about the divisor $\mathcal{D}$ on A , which has pole divisor $\mathcal{E}$, for some divisor $\mathcal{E} \neq \mathcal{D}$ and is holomorphic everywhere else. This function is associated with a global section of $\mathcal{F}^*$. Moreover, the space of global sections of $\mathcal{F}^*$ which blow up at most once at $\mathcal{E}$ is a rank r $\mathbb{C}[[t\!\!t]]$-module, where r is the dimension of the space of meromorphic functions on A that blow up at most once at $\mathcal{E}$.

Proof. As seen before, (14) gives such a section of $\mathcal{F}^*$. If s and s' are sections of $\mathcal{F}^*$, $\frac{s}{s'}$ is a function on A holomorphic everywhere and having pole divisor at a translate $w_0 + \mathcal{E}$ of $\mathcal{E}$. The space of such functions has dimension $r = \delta'_1 \cdots \delta'_n$ where $\delta' = (\delta'_1, \ldots, \delta'_n)$ is the polarization type of $\mathcal{E}$.

Finally, we notice that if x_α is fixed, we get an expansion around $\mathcal{D}$

$$\frac{\Theta(\mathbb{B}^t t\!\!t + u - \mathcal{A}_{x_\alpha}(x))}{\Theta(u - \mathcal{A}_{x_\alpha}(x))} = (1 + a_1(u, t\!\!t)z + \cdots), \quad |t\!\!t| \text{ small.}$$

where z is a local parameter around $\mathcal{D}$. ☐

3. The case of elliptic curves.

We consider the situation of A an elliptic curve and $\mathcal{D}$ = sum of points with positive coefficients $= \Sigma n_i p_i$. Then $H^0(A \times \mathbb{C}^\infty, \underline{\mathcal{F}}^*(\mathcal{D})/\underline{\mathcal{F}}^*) \simeq \mathbb{C}[[t\!\!t]]^{\deg \mathcal{D}}$. This follows from the skycraper sheaf $\mathbb{C}(\mathcal{D})$ in $0 \to \mathcal{F}_A \to \mathcal{F}_A(\mathcal{D}) \to \mathbb{C}(\mathcal{D}) \to 0$; which implies $0 \to \underline{\mathcal{F}}^* \to \underline{\mathcal{F}}^*(\mathcal{D}) \to \pi_1^*(\mathbb{C}(\mathcal{D})) \otimes \pi_2^*(\mathcal{O}_{\mathbb{C}^\infty}) \otimes \mathcal{F}^* \to 0$ and

$$\begin{aligned} \Gamma(\pi_1^*(\mathbb{C}(\mathcal{D}) \otimes \pi_2^*(\mathcal{O}_{\mathbb{C}^\infty}) \otimes \mathcal{F}^*) &\cong \Gamma(\mathbb{C}(\mathcal{D})) \otimes \mathbb{C}[[t\!\!t]] \otimes \Gamma(\mathcal{F}^*) \\ &\cong \mathbb{C}[[t\!\!t]]^{\deg \mathcal{D}} \otimes \Gamma(\mathcal{F}^*). \end{aligned} \tag{16}$$

Now, we wish to show the representability of the affine ring R as a ring of differential operators. Let $\mathcal{D} = \Sigma n_i p_i$, $R = \Gamma(A - \mathcal{D}, \mathcal{O}_A) \hookrightarrow \oplus_n \Gamma(A, \mathcal{L}(\mathcal{D})^{\otimes n}) =$ homogeneous coordinate ring, and take $\nabla := \frac{\partial}{\partial t_1}$. Then one uses the sequence

$$0 \to \underline{\mathcal{F}}^*(n\mathcal{D}) \to \underline{\mathcal{F}}^*((n+1)\mathcal{D}) \to \pi_1^*(\mathbb{C}(\mathcal{D})) \otimes \pi_2^*(\mathcal{O}_{\mathbb{C}^\infty}) \otimes \mathcal{F}^* \to 0 \tag{17}$$

to deduce that if $s_1, \ldots, s_k$ ($k = \deg \mathcal{D}$) is a $\mathbb{C}[[t]]$-basis of $\Gamma(\underline{\mathcal{F}}^*(\mathcal{D}))$, then $\{\nabla^r s_1, \ldots, \nabla^r s_k;\quad r = 0, 1, \ldots, n\}$ is a $\mathbb{C}[[t]]$-basis of $\Gamma(A \times \mathbb{C}^\infty, \underline{\mathcal{F}}^*((n+1)\mathcal{D})$. Indeed,this is because $H^1(\pi_1^*\mathbb{C}(\mathcal{D}) \otimes \pi_2^*(\mathcal{O}_{\mathbb{C}^\infty}) \otimes \mathcal{F}^*) \simeq H^1(\underline{\mathcal{F}}^*((n+1)\mathcal{D})/\underline{\mathcal{F}}^*(n\mathcal{D})) = 0$, which implies $H^1(\mathcal{F}^*(n\mathcal{D})) = 0$ for all n, and consequently the exact sequence of $\mathbb{C}[[t]]$-modules:

$$\begin{aligned} &0 \to \Gamma(A \times \mathbb{C}^\infty, \underline{\mathcal{F}}^*(n\mathcal{D})) \to \\ &\to \Gamma(A \times \mathbb{C}^\infty, \mathcal{F}^*((n+1)\mathcal{D})) \to \Gamma(\pi_1^*(\mathbb{C}(\mathcal{D})) \otimes \pi_2^*(\mathcal{O}_{\mathbb{C}^\infty}) \otimes \mathcal{F}^*) \to 0. \end{aligned} \tag{18}$$

Also, there is a mapping

$$\Gamma(A, \mathcal{L}(\mathcal{D})^n) \otimes \Gamma(A \times \mathbb{C}, \underline{\mathcal{F}}^*(k\mathcal{D})) \to \Gamma(A \times \mathbb{C}, \underline{\mathcal{F}}^*((n+k)\mathcal{D})), \tag{19}$$

and, if $\alpha \in R$, $\alpha = \sum \dfrac{\alpha_n(x_1)}{(z - z(x_1))^n} +$ lower terms $\in \Gamma(A, \mathcal{L}(\mathcal{D})^n)$.

So that we have

$$\alpha . s_i = \sum a_{ir}^j(t) \nabla^r s_j = {}^t\!\left(\sum a_{ir}^j(t) \nabla^r\right) \begin{pmatrix} s_1 \\ \vdots \\ s_k \end{pmatrix}, \tag{20}$$

i.e.

$$\alpha \begin{pmatrix} s_1 \\ \vdots \\ s_k \end{pmatrix} = \sum_{r=0}^{n} \begin{pmatrix} a_{1r}^1(t) & \ldots & a_{1r}^k(t) \\ \vdots & & \vdots \\ a_{kr}^1(t) & \ldots & a_{kr}^k(t) \end{pmatrix} \nabla^r \begin{pmatrix} s_1 \\ \vdots \\ s_k \end{pmatrix}. \tag{21}$$

We define an immersion ring map $\Phi : R \hookrightarrow M_k(\mathbb{C}[[t]])[\nabla]$ by

$$\Phi(\alpha) = \sum_{r=0}^{n} \left(a_{ir}^j(t)\right) \nabla^r.$$

Thus we can state the

PROPOSITION 3. *There is an injection $R \hookrightarrow M_k(\mathbb{C}[[t]])[\nabla]$ in case the space $\Gamma(\underline{\mathcal{F}}^*(\mathcal{D})/\underline{\mathcal{F}}^*)$ has a finite basis of k elements, (k =deg$\mathcal{D}$).*

Let us consider now the case where A is an elliptic curve in $\mathbb{P}^3$ and $\mathcal{D} = \sum_{i=1}^4 p_i$ (tipically the section cut out by an odd theta function). Then $\Gamma(\underline{\mathcal{F}}^*((n+1)\mathcal{D}))$ has generators $\{s_1, s_2, s_3, s_4, \ldots, \nabla^n s_1, \nabla^n s_2, \nabla^n s_3, \nabla^n s_4\}$, where $\{s_i\}$ is a $\mathbb{C}[[t]]$-basis of $\Gamma(\underline{\mathcal{F}}^*(\mathcal{D}))$.

For an elliptic curve and a divisor ε on it, $h^0(\varepsilon) = h^1(\varepsilon) = 0$ if $\deg \varepsilon = 0$. Conversely, if $\varepsilon \in$ Jacobian of $A = \{\varepsilon,\ \deg \varepsilon = 0\}$, then $\varepsilon \sim_\ell p - p_0$ and therefore $h^0(\varepsilon) = h^1(\varepsilon) = 0$ unless $\varepsilon \sim_\ell 0$ (e.g. Prop. 4.1.2, [Ha]).

Thus, there is an embedding of $R = \Gamma(A - \mathcal{D}, \mathcal{O}_A)$ into $M_4(\mathbb{C}[[t]])[\nabla]$.

4. The one dimensional case and KP hierarchy.

Let us consider the case similarly treated by Mumford in his Tata Lectures [Mu 3]. If Γ is a an elliptic curve and $p = \infty$ a distinguished point then we have the $\wp$-Wierstrass function, which together with 1 form a basis of the linear system $L(2p)$ (whose dimension for elliptic curves is $\dim L(np) = n$). Analogously $\{1, \wp, \wp', \ldots, \wp^{(n-1)}\}$ form a basis of $L((n+1)p)$.

Let $\psi(x, z(x), t)$ be the Baker-Akhiezer function associated to the divisor $\mathcal{D} = p$. Then it is easy to see the existence of a function $q(x, t)$ such that $\wp(z)\psi = (D^2 + q)\psi$, $D = \frac{\partial}{\partial x}$, and one checks that $\wp'(z)\psi = (D^3 + \frac{1}{4}(qD + Dq))\psi$.

In other words the value $\wp(z)$ can be thought of as an eigenvalue of the operator $D^2 + q$ with eigenfunction ψ. See [Mu 1] and [S-W].

On the other hand, one can consider the KP flows for a given ψ. This is the set of equations

$$\frac{\partial}{\partial t_i}\psi = K_i \psi, \tag{22}$$

where K_n is the differential operator part of the pseudodifferential operator Q^n, where $Q = D + \sum_{i>0} w_i D^{-i} = W^{-1} D W$. In our case Q has to satisfy $Q^2 = D^2 + q = Q^2_+$ and $D^3 + \frac{1}{4}(qD + Dq)$ equals Q^3_+.

The equations (22) can be written as (23) below, if we put $\psi = W^{-1}\phi(t)$, $\phi(t) = \exp\left(\frac{t_1}{z} + \frac{t_2}{z^2} + \ldots\right)$ and $W = 1 + \sum_{i>0} a_i D^{-i}$:

$$\frac{\partial W^{-1}}{\partial t_i} = K_i W^{-1} - W^{-1} D^i, \tag{23}$$

where $K_i = [W^{-1} D^i W]_+ = Q^n_+$.

As it is shown in the theory of Mumford [Mu 1], and also follows from the theorem in 2, the affine ring $B = \mathbb{C}[\wp, \wp'] = \mathbb{C}[X,Y]/\text{Ideal} = \bigoplus_{n\geq 0} L(np)$ can be embedded into a commutative ring R of differential operators in $\mathbb{C}[[t\!t]][D]$, having elements of the form D^n + (lower order) for any $n \geq 1$. Such an algebra of differential operators is isomorphic (as $\mathbb{C}$-algebra) with a $\mathbb{C}$-subalgebra $A = W_0^{-1}RW_0$, $W_0 \in$ **G**, of $\mathbb{C}((D^{-1}))$ having the property $A \cap \mathbb{C}[[D^{-1}]]D^{-1} = \{0\}$ (cf. Lemma 1.1 in [Mul]).

Now, consider the mapping $\varphi : \mathbb{C} \to \mathbb{C}^\infty$ defined by $\varphi(z) = (0,0,z,0,0,\dots)$. Then, the tangent vector $\frac{\partial}{\partial z}$ goes into the **KdV** tangent vector $\frac{\partial}{\partial t_3}$. We want to define the pseudodifferential operator $W(t\!t) \in \mathbf{G}$ in such a way that the map

(24)

$$\begin{array}{ccccccc} \mathbb{C} & \xrightarrow{\varphi} & \mathbb{C}^\infty & \xrightarrow{u} & \mathbf{G} & \longrightarrow & \mathbf{UGM} \\ & & t\!t & \longrightarrow & W(t\!t) & \longrightarrow & \text{frame}\{\dots, W^{-1}D^n, \dots, W^{-1}D, W^{-1}\}, \end{array}$$

coincides with the usual mapping embedding the elliptic curve in $\mathbb{P}^2$ and then $\mathbb{P}^2$ into **UGM**.

One can view the **UGM** as the set of all $\mathbb{N}^c$-frames modulo the group $GL(\mathbb{N}^c)$ acting by right action. An $\mathbb{N}^c$-frame is a matrix whose rows are indexed by the integers $\mathbb{Z}$, whose columns are indexed by the set $\mathbb{N}^c = \{-1,-2,\dots\}$, and has the form

(25)

$$\left[\begin{array}{cccc|ccc} \ddots & & & & \vdots & \vdots & \vdots \\ & 1 & & & 0 & \dots & 0 \\ & & 1 & & 0 & \dots & 0 \\ & & & 1 & 0 & \dots & 0 \\ \hline \dots & * & * & * & & * & \\ \dots & * & * & * & & (n+m)\times m & \\ \dots & * & * & * & & \text{frame} & \\ \hline & & * & & & * & \end{array}\right] \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ -m-1 \\ -m \\ \vdots \\ n-1 \\ \cdot \end{array}$$

$$\dots\dots\dots\dots \mid -m \quad \dots \quad -1$$

An element of $GL(\mathbb{N}^c)$ is a $\mathbb{N}^c \times \mathbb{N}^c$-matrix with the lower-right corner an element of $GL(m)$ for some m, and the upper-left corner an infinite identity matrix $\mathbb{I}$. So they are as follows

$$\begin{bmatrix} \mathbb{I} & \mathbf{0} \\ * & GL(m) \end{bmatrix} \tag{26}$$

The Grassmann manifold $GM(m, m+n)$ of m subspaces in $(m+n)$-space can be viewed as the set of $(m+n)\times m$-frames modulo $GL(m)$ acting by right action. A $(m+n)\times m$-frame can be put as a $\mathbb{N}^c$-frame like (24) in an obvious way which induces an embedding of $GM(m, n+m)$ into **UGM**. Thus, the homogeneous coordinates $[x_0 : \cdots : x_N]$ of the projective space $\mathbb{P}^N = GM(1, N+1)$ will appear as a $\mathbb{N}^c$-frame as follows

$$\begin{array}{c} \left[\begin{array}{cccc|c} \ddots & & & & \vdots \\ & 1 & & & 0 \\ & & 1 & & 0 \\ & & & 1 & 0 \\ \hline \cdots & * & * & * & x_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & * & * & * & x_N \\ \hline & & * & & * \end{array}\right] \begin{array}{c} \vdots \\ \vdots \\ \cdot \\ -2 \\ -1 \\ \vdots \\ N-1 \\ \cdot \end{array} \\ \cdots\ \cdots\ .\ -2 \mid -1 \end{array} \tag{27}$$

Now, the space $V = \mathcal{P}/\mathcal{P}x$ of pseudodifferential operators with constant coefficients has a basis of the form $e_i = (\frac{\partial}{\partial x})^{-i-1} \bmod \mathcal{P}x$, $i \in \mathbb{Z}$. Thus, a typical point of the elliptic curve in $\mathbb{P}^2$ via the covering map $\pi : \mathbb{C} \to \mathbb{P}^2$, $z \mapsto [1 : \wp(z) : \wp'(z)]$, has the asoociated frame (27) $U = \{\ldots, e_{-n} + \sum_{j\geq -1} \lambda_{-nj} e_j, \ldots, e_{-1} + \sum_{j\geq -1} \lambda_{-1j} e_j, e_{-1} + \wp(z)e_0 + \wp'(z)e_1\}$. This frame must differ from the frame in (24) by an element of $GL(\mathbb{N}^c)$ which has the form

$$\left[\begin{array}{cccc|ccc} \ddots & & & & \vdots & \vdots & \vdots \\ & 1 & & & 0 & \cdots & 0 \\ & & 1 & & 0 & \cdots & 0 \\ & & & 1 & 0 & \cdots & 0 \\ \hline \cdots & * & * & * & 1 & & 0 \\ \cdots & \vdots & \vdots & \vdots & & \ddots & \\ \cdots & * & * & * & * & & 1 \\ \hline & & * & & & * & \end{array}\right] \tag{28}$$

and therefore we must have $W^{-1}(\varphi(z)) = 1 + \wp(z)D^{-1} + \wp'(z)D^{-2}$.

Finally, starting with a given Baker-Akhiezer function, one can obtain the above W by multiplying by a convenient element of $\mathbf{G}$.

5. Multicomponent KP hierarchy.

Let us introduce some notation to consider the multicomponent KP equations. See [Ad-B]. We will consider wave functions of the form

$$w(\mathbb{t}) = \left(I + \sum_{i>0} w_i z^i\right)\phi(\mathbb{t})$$

where w_i are $k \times k$ matrices depending on $\mathbb{t}$ and $\phi(\mathbb{t})$ is the exponential diagonal matrix

$$\phi(\mathbb{t}) = \exp\left(\sum_{i>0}\begin{pmatrix} t_i^1 & & & \\ & t_i^2 & & \\ & & \ddots & \\ & & & t_i^k \end{pmatrix} z^{-i}\right)$$

and $\mathbb{t} = (t_1^1, t_1^2, \ldots, t_1^k, \ldots)$ is the vector of time variables t_i^j.

We have

$$\partial_{t_i^j}\phi(\mathbb{t}) = \frac{1}{z^i}\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}\phi(\mathbb{t}),$$

and if $\partial = \sum_{j=1}^k \partial_{t_1^j}$, $\partial\phi(\mathbb{t}) = \frac{1}{z}\phi(\mathbb{t})$.

Given the matrix pseudodifferential operator $W = I + \sum_{i=1}^{\infty} w_i \partial^{-i}$ we have

$$W\phi(\mathbb{t}) = \left(I + \sum_{i>0} w_i z^i\right)\phi(\mathbb{t}) = w(\mathbb{t}).$$

The multicomponent KP equations can be written as the set of Lax equations

(29) $$\partial_{t_i^j} Q = [Q, [R_j^i]_+]$$

where $Q = W^{-1}(A\partial)W$, $A =$ constant diagonal matrix with nonzero entries and $R_j^i = W^{-1}E_{jj}\partial^i W$, $E_{jj} = \operatorname{diag}(0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)$, and $[\ \ ]_+$ indicates the differential operator part of R_j^i.

The set of equations (1) is also equivalent to the equations in the wave operator W

$$\partial_{t_i^j} W = -W[R_j^i]_- \tag{30}$$

where $[\ ]_-$ is the formal pseudodifferential operator part.

PROPOSITION 4. *Given the wave function $w^*(t) = W^{-1}\phi(t)$, there is a matrix differential operator $P_i^j = \sum_{k=0}^n \partial^i a_i$, a_i = diag constant such that*

$$\frac{\partial}{\partial t_i^j} w(t) = P_i^j w(t).$$

Proof:

$\frac{\partial}{\partial t_i^j} w^*(t) = O(z)\phi(t) + (I + \sum_{i>0} w_i^* z^i) E_{jj} z^{-i}\phi(t)$. On the other hand $\partial^i w^*(t) = O(z)\phi(t) + (I + \sum_{i>0} w_i z^i)\frac{1}{z^i}\phi(t)$. Therefore $\frac{\partial}{\partial t_i^j} - \partial^i E_{jj}$ is a differential operator that acting on $w(t)$ has order $O(\frac{1}{z^{i-1}})$ and we continue by induction.

Suppose there is a ψ do differential operator K such that

$$P_n^j = K^{-1}(E_{jj}\partial)^n K = K^{-1}.$$

Then the expression (1) can be written as

$$\begin{aligned}\frac{\partial}{\partial t_i^j} W^{-1}\phi(t) &= \frac{\partial W^{-1}}{\partial t_i^j}\phi(t) + W^{-1}\frac{\partial}{\partial t_i^j}\phi(t) = \frac{\partial W^{-1}}{\partial t_i^j}\phi(t) + W^{-1}E_{jj}\frac{1}{z^i}\phi(t)\\ &= \frac{\partial W^{-1}}{\partial t_i^j}\phi(t) + W^{-1}E_{jj}\partial^i\phi(t) = P_i^j W^{-1}\phi(t).\end{aligned}$$

Thus, (1) is equivalent with

$$\frac{\partial W^{-1}}{\partial t_i^j} + W^{-1}(E_{jj}\partial^i) = P_i^j W^{-1} \tag{31}$$

and in particular $P_i^j = [W^{-1}(E_{jj}\partial^i)W]_+ = [R_j^i]_+$.

Now, if $Q = W^{-1}A\partial W$, we have

$$\begin{aligned}\frac{\partial Q}{\partial t_i^j} &= \frac{\partial W^{-1}}{\partial t_i^j} W.Q - Q\frac{\partial W^{-1}}{\partial t_i^j} W = (P_i^j - R_j^i)Q - Q(P_i^j - R_j^i)\\ &= [Q, [R_j^i]_-] = -[Q, [R_j^i]_+].\end{aligned}$$

PROPOSITION 5. *The operator Q satisfies the multicomponent K.P. hierarchy.*

Our purpose is to identify the Euler top flow with a multicomponent K.P. flow under a suitable embedding.

6. Baker functions defined on an elliptic curve.

In this section we present several examples. They are different attempts of defining a Baker-Akhiezer function for the divisor $\mathcal{D} = p_1 + p_2 + p_3 + p_4 =$ sum of points. The relevant examples 2 and 3 allow us to identify the Euler Top flow with a Multicomponent **KP** flow.

Let $z_i = O(t)$ be the local parameter at the reduced piece p_i of the divisor $\mathcal{D} = p_1 + p_2 + p_3 + p_4$. Ω_i^n the normalized differential of 2^{nd} kind with a single pole of order n at p_i and holomorphic everywhere else.

Consider the map $\varphi : E \to \operatorname{Pic}^0(E)$ defined by $\varphi(x) = [\tau_x \mathcal{D} - \mathcal{D}]$ (the canonical map). This has a finite kernel (the translation group $H(\mathcal{D})$). Let $\mathcal{E}$ be a divisor in $\operatorname{Pic}^0(E)$ such that $\mathcal{D} = \varphi^{-1}\mathcal{E}$. Then $\theta(\varphi(p))$ is a theta function for the divisor $\mathcal{D}$.

A Baker function can be obtained as

$$\psi_{i,\nu}^n(x) = \exp\left(\nu\, t_n \int_{p_0}^x \Omega_i^n\right) \frac{\theta_1\left(\int_{p_0}^x \omega + t_n U_i^n + \xi\right)}{\theta_2\left(\int_{p_0}^x \omega + \xi\right)}, \tag{32}$$

where ω is a basis of holomorphic differentials, and θ_i theta functions associated to translates of $\mathcal{D}$. As we go around a b-cycle of E we pick a b-period of Ω_i^n. So the exponential gets increased by the factor $\exp(t_n \underbrace{\int_{b_n} \Omega_i^n}_{U_i^n})$, which will cancel out with factors of θ_1 and θ_2.

LEMMA 6. *The expression (32) is a Baker function at p_i associated to the divisor $\mathcal{D}$. It has the expression*

$$\psi_{i,\nu}^n(x) = \begin{cases} e^{\nu\, t_n / z_i^n}\,(1 + O(z_i)) & \text{around } p_i \\ e^{\nu\, t_n \alpha_{ij}}\,(1 + O(z_j)) & \text{around } p_j,\ j \neq i. \end{cases} \tag{33}$$

Proof. Assume θ_1, θ_2 are θ functions of order ν with characteristics $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$, i.e. satisfy a relation of the type

$$\theta_n \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z+2\pi iN+BM) = \exp\left\{-\frac{\nu}{2}\langle BM,M\rangle - \nu\langle M,z\rangle + 2\pi i(\langle\alpha,N\rangle - \langle\beta,M\rangle)\right\}\theta_n\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z).$$

If θ_1 and θ_2 are of the same type and order then all the factors cancel except the factor $\exp\{-\nu\langle M, t_n U_i^n\rangle\} = \exp\left\{-\nu\, t_n \int_{Mb_i} \Omega_i^n\right\}$.

So if we add the factor ν in the exponential of the Baker function we obtain the desired cancelling, i.e. (32) is a well defined meromorphic function outside p_i, with zeroes at $\theta_1\left(\int_{p_0}^x \omega + t_n U_i^n + \xi\right) = 0$ and poles at $\theta_2\left(\int_{p_0}^x \omega + \xi\right) = 0$

As candidates for θ_1 and θ_2 one can pick the functions $\theta\left[\begin{smallmatrix}(\alpha+\gamma)/n\\ \beta\end{smallmatrix}\right](nz|nB)[D]$.

Around p_i we have

$$\Omega_i^n = -\frac{dz_i}{z_i^{n+1}} + O(1), \tag{34}$$

so

$$\int_{p_0}^x \Omega_i^n = \frac{1}{z_i^n(x)} + O(1) \qquad \text{and} \qquad e^{\nu\, t_n \int_{p_0}^x \Omega_i^n} = e^{\nu\, t_n / z_i^n}(1 + O(t_n, z_i)). \tag{35}$$

One can pick as z_i the time parameter t of a holomorphic vector field in E. One also has the expressions (33) around p_i.

Let τ_{ij} be the translation that sends $p_i \to p_j$, i.e., addition by $p_j - p_i$, and let $\Omega_j = \tau_{ij}^* \Omega_i$ be the pull back of Ω_i. Then Ω_j blows up at p_j.

Now we have the formula $\int_a \Omega_j = \int_{\tau_{ij}a} \Omega_i = \int_a \Omega_i$ (since $a + \tau_{ij}$ is homologous to a) for a period a of Ω_i (i.e., the periods are the same). (Notice that one can choose a so that all the translates $\tau_{ij}a$ of a do not meet the poles of Ω_i or its translate Ω_j.)

If we let $c_{ij} = \int_{p_0}^{p_0+p_i-p_j} \Omega_j$ then we have $\int_{p_0}^x -\Omega_i - \int_{p_0}^{x+p_i-p_j} \Omega_j = -c_{ij}$. In other words, one can interpret the cycle c_{ij} as the difference between the infinite integrals $\int_{p_0}^{x+p_i-p_j} \Omega_j - \int_{p_0}^x \Omega_i$ as $x \to p_i$. One has $c_{ij} + c_{jk} = c_{ik}$.

The correlation function f_{ij} defined by $df_{ij} = \Omega_i - \Omega_j$ is defined on the universal cover of E. Up to a constant we can pick $f_{ij} = \int^x \Omega_i - \int^x \Omega_j$ which is a function that blows up at p_i and p_j.

Now, we can write $\int^x \Omega_i = \int^x \Omega_j + f_{ij}$, and let $\alpha_{ij} = \int^{p_j} \Omega_i$, $i \neq j$. Thus, we have

$$\begin{aligned} \psi_i^n &= e^{\nu\, t_n \alpha_{ij}}(1 + O(z_j)) \qquad \text{about } p_j \\ &= O(1) \quad \square \end{aligned} \tag{36}$$

LEMMA 7. *We have the estimates*

$$\frac{d}{dt_n}\psi_{i,\nu}^n(x) = \begin{cases} \left(\dfrac{\nu}{z_i^n(x)} + O(z_i)\right)\psi_{i,\nu}^n(x) & \text{if } x \text{ is around } p_i, \\ O(1) & \text{if } x \text{ is around } p_j,\ j \neq i. \end{cases} \tag{37}$$

Proof.

$$\begin{aligned} \frac{d}{dt_n} &= e^{\nu\, t_n/z_i^n}(1 + O(z_i)) \\ &= \frac{\nu}{z_i^n} e^{\nu\, t_n/z_i^n}(1 + O(z_i)) + e^{\nu\, t_n/z_i^n} O_1(z_i) \\ &= \left(\frac{\nu}{z_i^n} + O_2(z_i)\right)\psi_{i,\nu}^n. \end{aligned}$$

$$\frac{d}{dt_n} e^{\nu\, t_n \alpha_{ij}}(1 + O(z_j)) = e^{\nu\, t_n \alpha_{ij}}\left(\nu\alpha_{ij} + O_1(z_j)\right). \quad \square$$

PROPOSITION 8. *There is a unique function, up to an element in $H(\mathcal{E}_0)$, having essential singularity at the point p_i, zeroes at $\mathcal{E}_0$ and blowing up at $\mathcal{E}_\infty$.*

Proof. If ψ and $\tilde{\psi}$ are two Baker functions then $\tilde{\psi}/\psi$ is meromorphic on the elliptic curve because the essential singularities cancel. The poles at $\mathcal{E}_\infty$ also cancel. Thus, the divisor of $\tilde{\psi}/\psi$ comes from the zeroes of $\tilde{\psi}$ and ψ, namely $\tilde{\mathcal{E}}_0$ and $\mathcal{E}_0$. So, we have $\tilde{\mathcal{E}}_0$ linearly equivalent to $\mathcal{E}_0$ for all $|t_n| \ll 1$. Since the group of divisors linearly equivalent to $\mathcal{E}_0$ is finite (the translation group $H(\mathcal{E}_0)$) we have that such a Baker function is unique up to an element in the Translation group of $\mathcal{E}_0$. $\square$

Note 1. It follows from PROPOSITION 8 the following lemma:

LEMMA 9. *On an abelian variety, a Baker function with expansion $\psi = O(z)e^{t_1/z}$ and no other zero or pole has to be zero.*

Note 2. Let $g_{\alpha\beta} = h_\alpha/h_\beta$, h_α defining the divisor $\mathcal{E}_0$ on A = abelian variety, and $[\mathcal{E}_0]$ the corresponding bundle. For any z fixed, define the bundle of transition functions $e^{\Sigma t_i/z_i}$ and denote by $L_{(z,t)}$ such bundle. Then define $L_{(z,t)} \otimes [\mathcal{D}] = \mathcal{L}_z$. A section of $\mathcal{L}_z$ is given by holomorphic s_α such that $s_\alpha = e^{\Sigma t_i/z_i}(h_\alpha/h_\beta)s_\beta$. Clearly, the quotient $\left\{\dfrac{s_\alpha}{h_\alpha}\right\}$ gives a meromorphic section of $L_{(z,t)}$.

Note 3. We can show the following for general abelian varieties

PROPOSITION 10. *There exists a monomorphism embedding $\mathcal{R} = \Gamma(A - \mathcal{D}, \mathcal{O}_A)$ into a ring of commutative differential operators with matrix coefficients.*

For elliptic curves this was shown in PROPOSITION 3.

Example 1. In order to illustrate Note 3 we draw Table I with the expansions of $\psi_1, \ldots, \psi_4$ and $D\psi_1, \ldots, D\psi_4$ around the points $p_1, \ldots, p_4$ and let $\{z_1', z_2', z_3'\}$ be the generators of the affine ring associated to the Euler top system. This system has divisor $\mathcal{D} = p(1,1) + p(1,-1) + p(-1,1) + p(-1,-1) = p_1 + p_2 + p_3 + p_4$ and the expansion of the functions $\{z_1', z_2', z_3'\}$ about $\mathcal{D}$ in terms of the time evolution parameter t are

$$\begin{cases} z_1' = \sqrt{\alpha\beta}\, z_1 = \delta_1 \left(\dfrac{1}{t} - (u+v)t + \cdots\right) \\ z_2' = \sqrt{\alpha\gamma}\, z_2 = \delta_2 \left(\dfrac{1}{t} + ut + \cdots\right) \\ z_3' = \sqrt{\gamma\beta}\, z_3 = \delta_1\delta_2 \left(\dfrac{1}{t} + vt + \cdots\right) \end{cases} \tag{38}$$

Table I

	$p(1,1)$	$p(1,-1)$	$p(-1,1)$	$p(-1,-1)$
ψ_1	$e^{\nu t_1/z_1}(1+O(z_1))$	$e^{\nu t_1\alpha_{12}}(1+O(z_2))$	$e^{\nu t_1\alpha_{13}}(1+O(z_3))$	$e^{\nu t_1\alpha_{14}}(1+O(z_4))$
ψ_2	$e^{\nu t_1\alpha_{21}}(1+O(z_1))$	$e^{\nu t_1/z_2}(1+O(z_2))$	$e^{\nu t_1\alpha_{23}}(1+O(z_3))$	$e^{\nu t_1\alpha_{24}}(1+O(z_4))$
ψ_3	$e^{\nu t_1\alpha_{31}}(1+O(z_1))$	$e^{\nu t_1\alpha_{32}}(1+O(z_2))$	$e^{\nu t_1/z_3}(1+O(z_3))$	$e^{\nu t_1\alpha_{34}}(1+O(z_4))$
ψ_4	$e^{\nu t_1\alpha_{41}}(1+O(z_1))$	$e^{\nu t_1\alpha_{42}}(1+O(z_2))$	$e^{\nu t_1\alpha_{43}}(1+O(z_3))$	$e^{\nu t_1/z_4}(1+O(z_4))$
$D\psi_1$	$\left(\frac{\nu}{z_1}+O(z_1)\right)\psi_1$	$O(1)$	$O(1)$	$O(1)$
$D\psi_2$	$O(1)$	$\left(\frac{\nu}{z_2}+O(z_2)\right)\psi_2$		
$D\psi_3$			$\left(\frac{\nu}{z_3}+O(z_3)\right)\psi_3$	
$D\psi_4$				$\left(\frac{\nu}{z_4}+O(z_4)\right)\psi_4$
$z_1'\psi_1$	$\frac{1}{t}\psi_1+\cdots$	$\frac{1}{t}\psi_1$	$-\frac{1}{t}\psi_1$	$-\frac{1}{t}\psi_1$

Notice that $\psi_j(x+p_i-p_j)=\exp(\nu\, t_n c_{ij})\,\psi_i(x)$ (with $\int_{p_0}^{p_j}\Omega_i=\int_{p_o}^{p_i}\Omega_j-c_{ij}$), once one chooses convenient θ functions to construct the remaining Baker functions from a given one. This is because we have

$$\int_{p_o}^{x}\Omega_i=\int_{p_0}^{x+p_i-p_j}\Omega_j-c_{ij} \tag{39}$$

and

$$\int_{p_0}^{x}\omega=\int_{p_0}^{x+p_i-p_j}\omega+\int_{x+p_i-p_j}^{x}\omega=\int_{p_0}^{x+p_i-p_j}\omega+\int_{p_i}^{p_j}\omega, \tag{40}$$

since ω are translation invariant 1-forms on an elliptic curve.

Example 2. Consider now a 2^{nd} kind normalized differential form Ω that blows up at the $\frac{1}{2}$-periods p_1, p_2, p_3 and p_4 of order $-\dfrac{dz_i}{z_i^2}$, where z_i is the local parameter at the points p_i. Let τ_{ij} be the translation by the vectors $p_j - p_i$. Assume that these translations are all $\frac{1}{2}$-periods.

We assume that the differential Ω is invariant under the group of translations $\tau_{ij}(x) = x + p_j - p_i$. This is a subgroup of the group of translations associated to the divisor $\mathcal{D} = p_1 + p_2 + p_3 + p_4$. We have the following relation:

$$(41) \quad \int_{p_0}^{x_i} \Omega = \int_{p_0}^{x_i} \tau_{ij}^* \Omega = \int_{p_0+p_j-p_i}^{x_i+p_j-p_i} \Omega = \int_{p_0}^{x_j} \Omega - \int_{p_0}^{p_0+p_j-p_i} \Omega = \int_{p_0}^{x_j} \Omega - c_{ij},$$

where $x_j = x_i + p_j - p_i$ and x_i is close to p_i so that x_j is close to p_j.

One can pick p_0 so that $\int_{p_0}^{x_1} \Omega = \dfrac{1}{z_1(x_1)} + O(z_1(x_1)) = \dfrac{1}{z_j(x_j)} - c'_{ij} + O(z_j(x_j))$ with $x_j = x_1 + p_j - p_1$ and for certain coefficients c'_{ij} satisfying the cocycle condition $c'_{ij} + c'_{jk} = c'_{ik}$, $c'_{ij} = -c'_{ji}$.

Now on the long range curve γ_i we have

$$\int_{\gamma_i(x)} \Omega = \int_{p_0}^{x_i} \Omega + \int_{x_i}^{x} \Omega = \int_{\gamma_j(x)} \Omega + \text{periods} = \int_{p_0}^{x_0} \Omega + \int_{x_j}^{x} \Omega + \text{periods of } \Omega.$$

Namely

$$(42) \qquad c_{ij} = \int_{p_0}^{x_j} - \int_{p_0}^{x_i} \Omega = \int_{x_i}^{x} \Omega - \int_{x_j}^{x} \Omega + \text{ periods of } \Omega \qquad (x_i \text{ close to } p_i).$$

On the other hand, for a holomorphic normalized *translation invariant* differential ω we have

$$(43) \qquad \int_{p_0}^{x_j} \omega = \int_{p_0}^{x_i+p_j-p_i} \omega = \int_{p_0}^{x_i} \omega + \int_{x_i}^{x_i+p_j-p_i} \omega = \int_{p_0}^{x_i} \omega + \int_{p_i}^{p_j} \omega,$$

where we assume $\int_{p_i}^{p_j} \omega$ is a $\frac{1}{2}$-period. Also, modulo a period

$$(44) \qquad \int_{x_i}^{x} \omega \equiv \int_{x_j}^{x} \omega + \int_{p_i}^{p_j} \omega.$$

Given the ϑ-functions ϑ_1, ϑ_2 related to $\mathcal{D}$, and of the same order, we define the following Baker functions

$$(45) \qquad \psi_i(x) = \exp\left(\nu t_n \int_{x_i}^{x} \Omega\right) \frac{\vartheta_1\left(\int_{x_i}^{x} \omega + t_n U + \xi\right)}{\vartheta_2\left(\int_{x_i}^{x} \omega + \xi\right)},$$

where the points x_i are very close to p_i.

One can relate the behaviour of ψ_i as x approaches p_j. We have

$$\begin{aligned}
\psi_i(x) &= \exp(\nu\, t_n c_{ij}) \exp\left(\nu\, t_n \int_{x_j}^{x} \Omega\right) \frac{\vartheta_1\left(\int_{x_j}^{x} \omega + \int_{p_i}^{p_j} \omega + t_n U + \xi\right)}{\vartheta_2\left(\int_{x_j}^{x} \omega + \int_{p_i}^{p_j} \omega + \xi\right)} \cdot \\
&\qquad \frac{\vartheta_1(\ldots)}{\vartheta_2(\ldots)} \cdot \frac{\vartheta_2(\ldots)}{\vartheta_1(\ldots)} \\
&= \exp(\nu\, t_n c_{ij})\, \psi_j(x) \left\{ \frac{\tau^*\vartheta_1\left(\int_{x_j}^{x} \omega + t_n U + \xi\right)}{\vartheta_1\left(\int_{x_j}^{x} \omega + t_n U + \xi\right)} \cdot \frac{\vartheta_2\left(\int_{x_j}^{x} \omega + \xi\right)}{\tau^*\vartheta_2\left(\int_{x_j}^{x} \omega + \xi\right)} \right\} \\
&= \exp(\nu\, t_n c_{ij})\ \psi_j(x)\ \psi_{ij}(x)
\end{aligned}$$

where τ^* represents translation by the $\frac{1}{2}$-period $\int_{p_i}^{p_j} \omega$.

Now, we would like to estimate the term within braces as $x \to p_j$ and $t_n \to 0$. We assume $\vartheta_1 = \vartheta_{00}$ and $\vartheta_2 = \vartheta_{11}$, the elliptic θ-functions with $\frac{1}{2}$-integer characteristics. If ϑ represents the Riemann θ-function associated to the elliptic curve of lattice $\mathbb{Z}\{1, \tau\}$, then we have the usual relations:

$$\begin{aligned}
&\vartheta_{00}(z,\tau) = \vartheta(z,\tau), \quad \vartheta_{01}(z,\tau) = \vartheta(z+\tfrac{1}{2},\tau), \\
&\vartheta_{10}(z,\tau) = \exp(\pi i \tau/4 + \pi i z)\, \vartheta(z + \tfrac{1}{2}\tau, \tau), \\
&\vartheta_{11}(z,\tau) = \exp(\pi i \tau/4 + \pi i (z + \tfrac{1}{2}))\, \vartheta(z + \tfrac{1}{2}(1+\tau), \tau) \\
&\vartheta(z + \alpha\tau + \beta, \tau) = \exp(-\pi i \alpha^2 \tau - 2\pi i \alpha z)\, \vartheta(z,\tau).
\end{aligned}$$

and the relations on page 19 [Mu 2].

Now, let $p_{ij} = \int_{p_i}^{p_j} \omega$, so that $p_{12} = \frac{1}{2}$, $p_{13} = \frac{1}{2}\tau$, $p_{14} = \frac{1}{2}(1+\tau)$. By our choice and use of tables we obtain

$$\begin{aligned}
\psi_{12} &= -\frac{\vartheta_{01}(U)}{\vartheta_{00}(U)} \cdot \frac{\vartheta_{11}(V)}{\vartheta_{10}(V)} = \frac{\vartheta(U + \frac{1}{2})}{\vartheta(U)} \cdot \frac{\vartheta(V + \frac{1}{2}(1+\tau))}{\vartheta(V + 1 + \frac{1}{2}\tau)} \cdot \frac{\exp(\pi i (V + \frac{1}{2}))}{\exp(\pi i (V))}, \\
\psi_{13} &= -i \frac{\exp(-\pi i \tau/4 - \pi i U)}{\exp(-\pi i \tau/4 - \pi i V)} \cdot \frac{\vartheta_{10}(U)}{\vartheta_{00}(U)} \cdot \frac{\vartheta_{11}(V)}{\vartheta_{01}(V)}, \\
\psi_{14} &= i \frac{\exp(-\pi i \tau/4 - \pi i U)}{\exp(-\pi i \tau/4 - \pi i V)} \cdot \frac{\vartheta_{11}(U)}{\vartheta_{00}(U)} \cdot \frac{\vartheta_{11}(V)}{\vartheta_{00}(V)}.
\end{aligned}$$

One uses the period relations

$$\vartheta_{01}(z+\alpha\tau+\beta) = \exp(-\pi i\alpha - \pi i\alpha^2\tau - 2\pi i\alpha z)\,\vartheta_{01}(z),$$
$$\vartheta_{00}(z+\alpha\tau+\beta) = \exp(\pi i\beta - \pi i\alpha^2\tau - 2\pi i\alpha z)\,\vartheta_{10}(z),$$
$$\vartheta_{11}(z+\alpha\tau+\beta) = \exp(\pi i(\beta-\alpha) - \pi i\alpha^2\tau - 2\pi i\alpha z)\,\vartheta_{11}(z),$$

to find

$$\psi_{21} = \frac{\vartheta_{01}(U)}{\vartheta_{00}(U)}\cdot\frac{(+\vartheta_{11}(V))}{\{-(-\vartheta_{10}(V))\}} = -\psi_{12} = \psi_{43},$$
$$\psi_{23} = i\exp(-\pi i(U-V))\,\frac{\{-\vartheta_{11}(U)\}\,\vartheta_{11}(V)}{\vartheta_{00}(U)\,\{\vartheta_{00}(V)\}} = -\psi_{14},$$
$$\psi_{31} = -i\exp(-\pi i(U-V))\,\frac{\{\exp(-\pi i\tau+2\pi iU)\vartheta_{10}(U)\}\,\vartheta_{11}(V)}{\vartheta_{00}(U)\,\{\exp(+\pi i-\pi i\tau+2\pi iV)\,\vartheta_{01}(V)\}}$$
$$= -\exp(2\pi i(U-V))\,\psi_{13},$$
$$\psi_{32} = i\exp(-\pi i(U-V))\,\frac{\{\exp(\pi i-\pi i\tau+2\pi iU)\,\vartheta_{11}(U)\}\,\vartheta_{11}(V)}{\vartheta_{00}(U)\,\{\exp(-\pi i\tau+2\pi iV)\,\vartheta_{00}(V)\}}$$
$$= -\exp(2\pi i(U-V))\,\psi_{14},$$
$$\psi_{41} = +i\exp(-\pi i(U-V))\,\frac{\{\exp(-\pi i\tau+2\pi iU)\,\vartheta_{11}(U)\}\,\vartheta_{11}(V)}{\vartheta_{00}(U)\,\{\exp(-\pi i\tau+2\pi iV)\,\vartheta_{00}(U)\}}$$
$$= \exp(2\pi i(U-V))\,\psi_{14},$$
$$\psi_{24} = \psi_{13},$$
$$\psi_{34} = \psi_{12},$$
$$\psi_{42} = \psi_{31},$$
$$\psi_{ii} = 1.$$

Thus a suitable change of basis matrix (or of the coefficients ψ_{ij}) is

$$M = \begin{pmatrix} 1 & \psi_{12} & \psi_{13} & \psi_{14} \\ -\psi_{12} & 1 & -\psi_{14} & \psi_{13} \\ -e^{2\pi i(U-V)}\psi_{13} & -e^{2\pi i(U-V)}\psi_{14} & 1 & \psi_{12} \\ e^{2\pi i(U-V)}\psi_{14} & -e^{2\pi i(U-V)}\psi_{13} & -\psi_{12} & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ -e^{2\pi i(U-V)}B & A \end{pmatrix}.$$

We can obtain other expressions for ψ_{12}, ψ_{13} and ψ_{14}:

$$\psi_{12} = -i\frac{\vartheta(U+\frac{1}{2})}{\vartheta(U)} \cdot \frac{\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(V+\frac{1}{2}\tau)} = -\exp(\pi i \tfrac{1}{2}) \cdots$$

$$\psi_{13} = -i\exp(-\pi i(U-V)) \cdot$$
$$\cdot \frac{\exp(\pi i\tau/4 + \pi i U)\,\vartheta(U+\frac{1}{2}\tau)\,\exp(\pi i\tau/4 + \pi i(V+\frac{1}{2}))\,\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(U)\,\vartheta(V+\frac{1}{2})}$$
$$= -\exp(\pi i\tfrac{1}{2}(1+\tau))\ \exp(2\pi i V) \cdot \frac{\vartheta(U+\frac{1}{2}\tau)\,\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(U)\,\vartheta(V+\frac{1}{2})}$$

$$\psi_{14} = i\exp(-\pi i(U-V)) \cdot$$
$$\cdot \frac{\exp(\pi i\tau/4 + \pi i(U+\frac{1}{2}) + \pi i\tau/4 + \pi i(V+\frac{1}{2}))}{\vartheta(U)}$$
$$\cdot \frac{\vartheta(U+\frac{1}{2}(1+\tau))\,\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(V)}$$
$$= -\exp(\pi i\tau/2)\ \exp(2\pi i V) \cdot \frac{\vartheta(U+\frac{1}{2}(1+\tau))\,\vartheta(V+\frac{1}{2}(1+\tau))}{\vartheta(U)\,\vartheta(V)}$$

LEMMA 11. $\det M \neq 0$.

Proof. If $t_n = 0$ then $U = V$ and $\psi_{12} = -\frac{a_1 a_3}{a_0 a_2}$, $\psi_{13} = -i\frac{a_2 a_3}{a_0 a_1}$, $\psi_{14} = i\frac{a_3^2}{a_0^2}$ and

$$\det M = \left(1 + \left(\frac{a_1 a_3}{a_0 a_2}\right)^2\right)^2 \left(1 - \left(\frac{a_3 a_2}{a_0 a_1}\right)^2\right)^2 \neq 0$$

for appropriate values of a_1, a_2, a_3, a_0. □

In a similar fashion as we did in the previous example we can construct a table of the expansions for the functions ψ_i around the points p_i.

We have $\psi_{ij}(x) = \alpha_{ij}(\xi) + O(z_j)$ and the expansions in Table II:

Table II

	ψ_1	permutations	$D\psi_1$	permutations	$z_1'\psi_1$
$p(1,1)$	$e^{t_1/z_1}(1+O(z_1))$	...	$\left(\frac{1}{z_1}+O(z_1)\right)\psi_1$	...	$\frac{1}{t}\psi_1$
$p(1,-1)$	$e^{t_1 c_{12}}\psi_2(x)(\alpha_{12}+\cdots)$	...	...	...	$\frac{1}{t}\psi_1$
$p(-1,1)$	$e^{t_1 c_{13}}\psi_3(x)(\alpha_{13}+\cdots)$	...	...	...	$-\frac{1}{t}\psi_1$
$p(-1,-1)$	$e^{t_1 c_{14}}\psi_4(x)(\alpha_{14}+\cdots)$	...	...	...	$-\frac{1}{t}\psi_1$

With the expansions we have for $D\psi_1$, $D\psi_2$, $D\psi_3$, $D\psi_4$ we get an expression $z_1'.\psi_1 = \Sigma\lambda_{ij}D\psi_j + O(1)$, since the matrix (α_{ij}) is nonsingular by the lemma. Therefore obtaining a very complicated matrix differential operator in $M_4[[t]][D]$. Also, we obtain a commutative ring of differential operators in $M_4[\mathbb{C}[[t]]][D]$, as follows from the representation to be obtained for the z_i's.

We want to study in more detail the relations arisen from the action of the translation group $G \cong \mathbb{Z}/2\times\mathbb{Z}/2$, whose elements we indicate by $\tau_{ij} = p_j - p_i$. The action on functions is defined by $\tau_{ij}^* f(x) = f(x+\tau_{ij})$ and one can show the formula

$$\psi_{p_k}(x+\tau_{ij}) = e^{tc_{ij}}\psi_{\tau_{ij}(p_k)}(x+\tau_{ij})\,\psi_{ij}^{\tau_{ij}(p_k)}(x+\tau_{ij}), \tag{46}$$

where $\psi_{ij}^{\tau_{ij}(p_k)}$ is defined as follows:

$$\psi_{ij}^{\tau_{ij}(p_k)}(x+\tau_{ij}) = \frac{\vartheta_1\left(\int_{\tau_{ij}(p_k)}^{x+\tau_{ij}}\omega + tU + \xi + \int_{p_i}^{p_j}\omega\right)}{\vartheta_2\left(\int_{\tau_{ij}(p_k)}^{x+\tau_{ij}}\omega + \xi + \int_{p_i}^{p_j}\omega\right)}\cdot\frac{\vartheta_2\left(\int_{\tau_{ij}(p_k)}^{x+\tau_{ij}}\omega + \xi\right)}{\vartheta_1\left(\int_{\tau_{ij}(p_k)}^{x+\tau_{ij}}\omega + tU + \xi\right)}.$$

The above formula translates into the multiplicative cocycle formula

$$\psi_{\tau_{ij}(p_k)}(y) = \phi_{ij}^{\tau_{ij}(p_k)}(y)\,\psi_{p_k}(y). \tag{$*$}$$

Indeed, identifying the elements of G with the translation points $\{p_i\}$ and with the translations $\tau_{ij} = p_j - p_i$ for a chosen base point, we have the elements $\{\psi_\sigma\}$, $\psi_\sigma \in \Gamma(E\times\{|t|<\epsilon\}, \mathcal{F}^*(*\mathcal{D})) = \hat{\mathcal{S}}$, which is a ring that contains $\mathcal{S} = \Gamma(E, \mathcal{O}(*\mathcal{D}))$ and $\phi_{ij}^{\tau(\sigma)} = \tau\cdot\psi_\sigma = 1 + O(t, z)$ which are also elements in $\hat{\mathcal{S}}$. Thus, equation $(*)$ is the cocycle relation $\tau\cdot\psi_\sigma = \psi_{\tau\sigma}/\psi_\sigma$.

Now, if we differentiate with respect to t we obtain

$$D\psi_{\tau_{ij}(p_k)}(y) = \left\{D\phi_{ij}^{\tau_{ij}(p_k)}(y) + \phi_{ij}^{\tau_{ij}(p_k)}(y)\,D\log\psi_{p_k}(y)\right\}\psi_{p_k}(y). \tag{47}$$

Let the cocycle relation $(*)$ be written $\phi_{\sigma,\tau} = \tau\cdot\psi_\sigma = \psi_{\tau\sigma}/\psi_\sigma$.

Now, one has the expansions around the points $\nu \in \{p_i\}$

$$\psi_\sigma(\text{about }\nu) = e^{t/z}(\alpha_{\sigma,\nu}(t) + \beta_{\sigma,\nu}(t)z + \cdots).$$

One obviously has $D\psi_\sigma(\text{about } \nu) = \left[\frac{1}{z} + O(1)\right] \psi_\sigma(\text{about } \nu)$ (assuming $\alpha_{\sigma,\nu}(0) \neq 0$). Around the points ν the expansions of the coordinate $z'_i\,(\text{about } \nu) = \frac{\alpha_\nu}{z} + O(1) = \frac{\alpha_\nu}{z} + \beta_\nu + O(z)$, where α_ν is a constant. Thus

$$z'_i\ (\text{about } \nu)\ \cdot \psi_\sigma\ (\text{about } \nu)\ = \Sigma\lambda_{\sigma,\rho} D\psi_\rho\ (\text{about } \nu)\ + O(1)e^{\frac{t}{z}}.$$

Since the poles in z have to be peeled off, this leads to the equation

$$\alpha_\nu\ \alpha_{\sigma,\nu} = \sum_\rho \lambda_{\sigma,\rho}\,\alpha_{\rho,\nu}(t). \tag{48}$$

This means that $(\lambda_{\sigma\rho})(\alpha_{\rho,\nu}(t)) = (\alpha_{\sigma,\nu}(t))\,\text{diag}(\alpha_\nu)$; namely

LEMMA 12. *$(\lambda_{\sigma\rho})$ is diagonalizable and nonsingular if* $\det \text{diag}(\alpha_\nu) \neq 0$.

In an analogous way we obtain a relation for the coefficients $\lambda_{\sigma\rho}$ of the 0^{th} order part: we have the equations

$$\alpha_\nu\beta_{\sigma,\nu} + \beta_\nu\alpha_{\sigma,\nu} = \sum_\rho \lambda_{\sigma,\rho}(\beta_{\rho,\nu} + \alpha'_{\rho,\nu}) + \sum_\rho \mu_{\sigma,\rho}\alpha_{\rho,\nu}. \tag{49}$$

Namely

$$(\mu_{\sigma,\rho})(\alpha_{\rho,\nu}(t)) = (\beta_{\sigma,\nu}(t))\,\text{diag}(\alpha_\nu) + (\alpha_{\sigma,\nu}(t)\,\text{diag}(\beta_\nu) - (\lambda_{\sigma\rho})[(\beta_{\rho,\nu}(t)) + (\alpha'_{\rho\nu})].$$

Let $\mu = (\mu_{\sigma,\rho})$, $\alpha = (\alpha_{\rho,\nu})$, $\beta = (\beta_{\sigma,\nu})$, $r = \text{diag}(\alpha_\nu)$, $s = \text{diag}(\beta_\nu)$, $\lambda = (\lambda_{\sigma,\rho})$; then we can write the operator as follows: $\lambda D + \mu$, but $\alpha^{-1}(\lambda D + \mu)\alpha = \alpha^{-1}\lambda\alpha D + \alpha^{-1}\lambda\alpha' + \alpha^{-1}\mu\alpha = rD + s + [\alpha^{-1}\beta, r]$. Thus, by an appropriate conjugation the operator is almost with constant coefficients.

Actually, by looking at the expansions of z'_i we obtain $s = 0$ so that the representation of z'_i as differential operator is $r_iD + [a, r_i]$, $a = \alpha^{-1}\beta$.

PROPOSITION 13. *There is a unique pseudodifferential operator ψ_i associated with $\mathbb{D}_i = r_i\partial + [a, r_i]$, and a unique $W = 1 + \sum_{i=1}^{\infty} s_{-i}\partial^{-i}$ pseudodifferential operator such that $\mathbb{D}_i = W^{-1}\psi_i W$ for any i. Any such W differ by a diagonal matrix.*

If $\psi_i = r_i\partial + \sum_{k=1}^{\infty} a_{-k}\partial^{-k}$, then the equality $W\mathbb{D}_i = \psi_i W$ yields the following equations:

(50)
$$\begin{gathered}
[s_{-1}+a, r_i] = 0 \\
[s_{-2}, r_i] = a_{-1} + r_i s'_{-1} - s_{-1}[a, r_i] \\
[s_{-(n+1)}, r_i] - a_{-n} = r_i s'_{-n} + \sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1}(-1)^k\binom{n-j-1}{k} a_{-(n-k-j)} s_{-j}^{(k)} \\
- \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k} s_{-(n-k)}[a, r_i]^{(k)}.
\end{gathered}$$

One can choose $s_{-1} = -a$, and the remaining s_{-k} such that $[s_{-k}, r_i] = 0$ for any $i = 1, 2, 3$. Since $r_1 = \mathrm{diag}(1,1,-1,-1)$, $r_2 = \mathrm{diag}(1,-1,1,-1)$, $r_3 = \mathrm{diag}(1,-1,-1,1)$, being commutative with the group of matrices generated by $\langle r_1, r_2\rangle$ means that s_{-k} is diagonal, $k > 1$. Then, the values of the a_{-k} are uniquely determined. If we perturbe the coefficients of W by diagonal matrices we obtain another solution to this representation.

PROPOSITION 14. *Given the operator $r\partial + [a, r]$, with r constant diagonal matrix, $r^2 = 1$, then, there exists a pseudo-differential operator $K = 1 + \Sigma w_{-i}\partial^{-i}$ such that $r\partial + [a, r] = K(r\partial)K^{-1}$. Such a solution differs by a constant operator K commuting with r.*

Proof. Let $L(x) = [x, r]$; this is a linear derivation and satisfies $rL(x) + L(x)r = 0$. We want to find a solution K to the equation

(51)
$$(r\partial + L(a))(1 + \Sigma w_{-i}\partial^{-i}) = (1 + \Sigma w_{-i}\partial^{-i})(r\partial).$$

This gives a system that implies the differential equations in w_{-i}

(**)
$$\begin{cases} L(a) = L(w_{-1}) \\ L(w_{-(i+1)} = rw'_{-i} + L(a)w_{-i} = P(w_{-i}). \end{cases}$$

Notice that we have the following identities:

(52)
$$LP(x) = PL(x) - 2rL(a)x \quad \text{and} \quad P(rx) = rP(x) - 2rL(a)x.$$

Also

$$L^2(x) = [L(x), r] = 2(x - rxr) = -2rL(x) = L(x)(2r). \tag{53}$$

Now any 4×4 matrix x can be written as $x = -\frac{1}{2}rL(x) + d$, with $L(d) = 0$. Indeed, this follows from the above properties of the operator L. Let us decompose $w_{-i} = -\frac{1}{2}rL(w_{-i}) + d_{-i}$. On one hand we have

$$-2rL(w_{-(i+1)}) = L^2(w_{-(i+1)}) = LP(w_{-i}) = PL(w_{-i}) - 2rL(a)w_{-i}.$$

Namely,

$$L(w_{-(i+1)}) = -\tfrac{1}{2}rPL(w_{-i}) + L(a)w_{-i}. \tag{54}$$

Replacing, we obtain

$$\begin{aligned} L(w_{-(i+1)}) &= -\frac{1}{2}L(w'_{-i}) - \tfrac{1}{2}rL(a)L(w_{-i}) - \tfrac{1}{2}L(a)rL(w_{-i}) + L(a)d_{-i} \\ &= -\tfrac{1}{2}L(w_{-i}) + L(a)d_{-i} = L(-\tfrac{1}{2}w'_{-i} + ad_{-i}). \end{aligned}$$

This implies that $w_{-(i+1)} = -\frac{1}{2}w'_{-i} + ad_{-i} + d_{-(i+1)}$, where $d_{-(i+1)}$ belongs to the kernel of L.

In order to solve $(**)$, we will represent the solution $w_{-(i+1)}$ as the sum of a term in Image of L + a term in $\operatorname{Ker} L$. Thus, we can write the following recursion formula for w_{-i}:

$$w_{-(i+1)} = \tfrac{1}{4}rL(w_{-i})' - \tfrac{1}{2}rL(a)d_{-i} + d_{-(i+1)}, \tag{55}$$

where the d_{-i} are to be determined so as to satisfy the system $(**)$ since we have

$$\begin{aligned} L(w_{-(i+1)}) &= L(\tfrac{1}{4}rL(w'_{-i}) - \tfrac{1}{2}rL(a)d_{-i} + d_{-(i+1)}) = -\tfrac{1}{2}L(w_{-i})' + L(a)d_{-i} \\ &= rw'_{-i} - rd'_{-i} + L(a)\left(w_{-i} + \tfrac{1}{2}rL(w_{-i})\right) \\ &= P(w_{-i}) - rd'_{-i} + \tfrac{1}{2}L(a)rL(w_{-i}). \end{aligned}$$

Assuming that $L(w_{-i})$ is known, it follows that $d'_{-i} = -\frac{1}{2}L(a)L(w_{-i})$. This element belongs to $\operatorname{Ker} L$ since $L(L(x)L(y)) = -2(rL(x) + L(x)r)L(y) = 0$, and gives, up to a constant matrix commuting with r, the solution we want. The first terms are

$$w_{-1} = -\tfrac{1}{2}rL(a) + d_{-1} \text{ where } d_{-1} = -\tfrac{1}{2}\int L(a)^2$$
$$w_{-2} = \tfrac{1}{4}rL(a)' - \tfrac{1}{2}rL(a)d_{-1} + d_{-2} \text{ where } d'_{-2} = -\tfrac{1}{2}L(a)\left(\tfrac{1}{4}rL(a)' - \tfrac{1}{2}rL(a)d_{-1}\right)$$

We now determine the differential operator part of the pseudo-differential operator $K(r\partial^2)K^{-1}$. If $K = 1 + \Sigma w_{-i}\partial^{-i}$, $K^{-1} = 1 - w_{-1}\partial^{-1} + (w_{-1}^2 - w_{-2})\partial^{-2} + \cdots$.

$$\begin{aligned} K(r\partial^2)K^{-1} &= (1 + w_{-1}\partial^{-1} + w_{-2}\partial^{-2} + \cdots) \\ &\quad (r\partial^2 - rw_{-1}\partial - 2rw'_{-1} + r(w_{-1}^2 - w_{-2}) + \cdots) \\ &= r\partial^2 + L(w_{-1})\partial + L(w_{-2}) - 2rw'_{-1} + rw_{-1}^2 - w_{-1}rw_{-1} + \cdots \end{aligned}$$

The independent term can be written as:

$$\begin{aligned} rw'_{-1} + L(a)w_{-1} - 2r\left(-\tfrac{1}{2}rL(a)' - \tfrac{1}{2}L(a)^2\right) + (rw_{-1} - w_{-1}r)w_{-1} = \\ = -rw'_{-1} = \tfrac{1}{2}L(a)' + \tfrac{1}{2}rL(a)^2. \end{aligned}$$

Thus

$$(*_i) \qquad (K(r_i\partial^2)K^{-1})_+ = r_i\partial^2 + L_i(a)\partial + \tfrac{1}{2}L_i(a)' + \tfrac{1}{2}r_iL_i(a)^2. \quad \square$$

Example 3. Assume now that the coordinates $z_1 = (z_1')$, $z_2 = (z_2')$, $z_3 = (z_3')$ (having the expansions shown in Example 1) satisfy the Euler top equations

$$(56) \quad \begin{cases} \dfrac{dz_1}{dt} = -z_2z_3 \\ \dfrac{dz_2}{dt} = -z_1z_3 \\ \dfrac{dz_3}{dt} = -z_1z_2 \end{cases} \quad \text{with relations} \quad \begin{aligned} \frac{z_1^2}{\alpha_2\alpha_3} + \frac{z_2^2}{\alpha_3\alpha_1} + \frac{z_3^2}{\alpha_1\alpha_2} = 1 \\ \frac{\lambda_1 z_1^2}{\alpha_2\alpha_3} + \frac{\lambda_2 z_2^2}{\alpha_3\alpha_1} + \frac{\lambda_3 z_3^2}{\alpha_1\alpha_2} = h. \end{aligned}$$

Here $z_1 = \frac{\epsilon_1}{t} - \epsilon_1(u+v)t + \cdots$, $z_2 = \frac{\epsilon_2}{t} + \epsilon_2 ut + \cdots$, $z_3 = \frac{\epsilon_1\epsilon_2}{t} + \epsilon_1\epsilon_2 vt$, $\epsilon_1^2 = \epsilon_2^2 = 1$, $u = \frac{1}{6}((\lambda_3 - h)\alpha_3 + (h - \lambda_1)\alpha_1)$, $v = \frac{1}{6}((h - \lambda_2)\alpha_2 + (\lambda_1 - h)\alpha_1)$, $w = -(u+v) = \frac{1}{6}((h - \lambda_3) + (\lambda_2 - h)\alpha_2)$.

We have seen that the differential operator associated with z_i is $D_i = r_i\partial + L_i(a)$, $L_i(a) = [a, r_i]$. Thus, $D_i^2 = \partial^2 + r_i L_i(a') + L_i(a)^2$ and $D_i D_j = D_j D_i = r_k \partial^2 + L_k(a)\partial + r_i L_j(a') + L_i(a) L_j(a)$ (cycle $i \to j \to k$). We wish to compare the operator $(*_k)$ with the operator associated to the function $-\frac{dz_k}{dt}$, i.e., $D_i D_j$.

Since the operators D_i satisfy the equations

$$(57) \qquad \sum_{i=1}^{3} \alpha_i D_i^2 = \alpha_1\alpha_2\alpha_3, \qquad \sum_{i=1}^{3} \lambda_i \alpha_i D_i^2 = \alpha_1\alpha_2\alpha_3 h,$$

we obtain the relation $D_i^2 - D_j^2 = \alpha_k(h - \lambda_k)$, $i \to j \to k \to i$.

If $s_i = r_i L_i(a') + L_i(a)^2$, then we can also write $s_i - s_j = \alpha_k(h - \lambda_k)$. Let $T = r_i L_j(a') + L_i(a)L_j(a) = r_j L_i(a') + L_j(a) L_i(a)$, then it follows

$$(58) \qquad [L_i(a), L_j(a)] = r_j L_i(a') - r_i L_j(a') = r_j L_k(a') r_j \qquad (i \to j \to k \to i).$$

Also

$$(58') \quad r_k T = s_j - L_j(a)^2 + r_k L_i(a) L_j(a) = s_i - L_i(a)^2 + r_k L_j(a) L_i(a) \quad (r_k = r_i r_j),$$

which yields, using the relations

$$(58'') \qquad L_k(a) = r_i L_j(a) + L_i(a) r_j = r_j L_i(a) + L_j(a) r_i \quad (i \to j \to k)$$

$$(59) \qquad \begin{aligned} L_k(a)(r_j L_i(a) - r_i L_j(a)) &\overset{\text{by (1)}}{=} L_k(a) r_j L_k(a) r_j \\ &= s_i - s_j = \alpha_k(h - \lambda_k) \quad (i \to j \to k \to i) \end{aligned}$$

Let us compute the differences between the independent terms of the operators $(K_k(r_k\partial^2)K_k^{-1})_+$ and $D_i D_j$. This is $2S = 2T - r_k s_k$

$$\begin{aligned} 2r_k S &= r_k T + r_k T - s_k \\ &\overset{\text{by (58')}}{=} s_i - L_i(a)^2 + r_k L_j(a) L_i(a) + r_j L_j(a') + r_k L_i(a) L_j(a) - s_k \\ &\overset{\text{by (58)}}{=} s_i - s_k - L_i(a)^2 + r_k L_j(a) L_i(a) + \\ &\qquad + r_k (L_k(a) L_i(a) - L_i(a) L_k(a)) r_i + r_k L_i(a) L_j(a) \\ &\overset{\text{by (58'')}}{=} s_i - s_k - L_k(a) r_j L_i(a) + r_k L_k(a) L_i(a) r_i + r_k L_i(a) r_k L_i(a) \\ &= s_i - s_k + r_k (L_i(a) r_k L_i(a) r_k) r_k \\ &= s_i - s_k + r_k(\alpha_i(h - \lambda_i) r_k \\ &= \alpha_i(h - \lambda_i) - \alpha_j(h - \lambda_j). \end{aligned}$$

Thus

$$\left(K_k(r_k\partial^2)K_k^{-1}\right)_+ = D_1D_2 + \tfrac{1}{2}\left\{\alpha_j(h-\lambda_j) - \alpha_i(h-\lambda_i)\right\} r_k = D_1D_2 + c_k r_k,$$

and we can write

$$\left(K_k(r_k\partial^2 - c_k r_k)K_k^{-1}\right)_+ = D_1D_2,$$

with $r_k\partial^2 - c_k r_k = G_k(r_k\partial^2)G_k^{-1}$, G_k being a scalar differential operator and therefore commuting with r_k.

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