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A non denseness result

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Abstract

We prove that the set of orbits of analytic automorphisms on the complex unit disk is a non dense subspace of the Helly product space $D^{\mathbb{N}_0}$.

1 Introduction

Certain Banach spaces $\mathcal{B}(X)$ are realized by norming a collection of real or complex valued functions on a given non empty set X. If φ maps X into itself, it is natural to consider the *linear composition operator* C_φ defined by

$$(C_\varphi f)(x) = f(\varphi(x)), \ f \in \mathcal{B}(X), \ \ x \in X.$$

The classical Banach spaces of analytic functions are derived from various L^p spaces. Hardy and Bergman spaces, as well as Dirichlet ones with an additional Hilbert space structure, were performed on the unit ball, the unit polydisk, the torus and the unit sphere of $\mathbb{C}^n$, while a broad research goes on more general domains (cf. [3], [5], [6]). If $z \in D$ and $\varphi \in \text{Aut}(D)$ we shall call the set $\mathcal{O}(z, \varphi) = (z, \varphi(z), \varphi^2(z), ...)$ the *orbit of z under φ*. The study of orbits, cyclicity and iteration (see [1]) is strongly motivated because $C_\varphi^n = C_{\varphi^n}$, where $\varphi^n = \varphi \circ \varphi \circ ... \circ \varphi$ (n times). Of course, fixed poir.s play a major role and they allow a classification of the elements of Aut(D). In fact, accordingly to the location of its fixed points an analytic automorphism is: *elliptic,* if one fixed point is in the disk and the other is in the complement of the closed disk; *parabolic,* if there is one fixed point on the unit circle of double multiplicity; *hyperbolic,* if both fixed points are different on the unit circle. Moreover, every map of D into itself has an attractive fixed point (the *Denjoy - Wolff point*).

Theorem 1 *(Cf. [2], [8]) If φ, not the identity and not an elliptic automorphism of D, is an analytic self - map of D, there is a unique point $a \in \overline{D}$ so that the iterates φ^n of φ converge to a uniformly on compact subsets of D.*

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In considering analytic functions, it is known that the usual euclidean me-tric on the disk is inappropiate. So, we shall consider the Poincaré metric in which the length of a curve Γ in D is $\int_\Gamma |dz| / (1-|z|^2)$, or the equivalent pseudohyperbolic metric in which the distance between two points $z_1,\ z_2 \in D$ is $|(z_1 - z_2)/(1-\overline{z_1}\ z_2)|$. If $d_P(z_1, z_2)$ denotes the Poincaré distance between z_1 and z_2 then

$$d_P(z_1,z_2) = \frac{1}{2}\ \ln \frac{1+\frac{|z_1-z_2|}{|1-\overline{z_1}\ z_2|}}{1-\frac{|z_1-z_2|}{|1-\overline{z_1}\ z_2|}} = \arg\tanh \frac{|z_1-z_2|}{|1-\overline{z_1}\ z_2|}.$$

Moreover, analytic automorphims on D become Poincaré isometries, i.e. a res-triction of how much they can be accomplished as change of variables.[1] For instance, if $|(z_1 - z_2)/(1-\overline{z_1}\ z_2)| = |(w_1 - w_2)/(1-\overline{w_1}\ w_2)| \neq 0$ in D there is a unique $\varphi \in \mathrm{Aut}(D)$ such that $\varphi(z_1) = w_1$ and $\varphi(z_2) = w_2$.

2 On orbits

If $z \in D$ and $S_{\mathrm{Aut}(D)}[z]$ is the stabilizer of z in $\mathrm{Aut}(D)$ then

$$S_{\mathrm{Aut}(D)}[z] = \{Id_D\} \cup \left\{ f_\theta(w) = \alpha_\theta \frac{w-a_\theta}{1-\overline{a_\theta}\ w},\quad w\in D,\ 0\le\theta<2\pi \right\},$$

where

$$\begin{aligned} a_\theta &= \frac{z+|z|\ \exp(i\theta)}{1+|z|^2}, \\ & \textit{and} \\ \alpha_\theta &= \frac{|z|^2 - \exp[i(\arg(z)-\theta)]}{1-|z|^2\ \exp[i(\theta-\arg(z))]},\quad 0\le\theta<2\pi. \end{aligned}$$

For instance,

$$S_{\mathrm{Aut}(D)}\left[\frac{1}{2}\right] = \left\{ f_\theta(w) = e^{-i\theta}\ \frac{2\,(1+e^{i\theta})-5w}{5-2(1+e^{-i\theta})\ w},\ w\in D,\ 0\le\theta<2\pi \right\}.$$

More generally, if $\varphi \in \mathrm{Aut}(D)$ then $z \in D$ is a fixed point of φ if there is a positive integer n such that $\varphi^n \in S_{\mathrm{Aut}(D)}[z]$. We´ll write

$$\text{length}\ _\varphi[z] = \min\{n\in\mathbb{N}:\ \varphi^n(z) = z\}.$$

Some authors ([4]) also define the *enter length* $\mathrm{ent}_\varphi(z)$ and the cycle length $\mathrm{cyc}_\varphi(z)$ of z at φ by

$$\begin{aligned} \text{ent}\ _\varphi[z] &= \min\left\{k\in\mathbb{N}\cup\{0\} : \exists\,(m\in\mathbb{N})\,/\ \varphi^k(z) = \varphi^{k+m}(z)\right\}, \\ \text{cyc}\ _\varphi[z] &= \min\left\{m\in\mathbb{N} : \varphi^{\text{ent}\ _\varphi[z]+m}(z) = \varphi^{\text{ent}\ _\varphi[z]}(z)\right\}. \end{aligned}$$

[1] Cf. [7], Ch. 2, Section 7.

Of course, if length $_{\varphi}[z] = 1$ then $\mathcal{O}(z,\ \varphi) = (z,\ z,\ z,\ \ldots)$. If length $_{\varphi}[z] = 2$ then $\mathcal{O}(z,\ \varphi) = (z,\ \varphi(z),\ z,\ \varphi(z),\ \ldots.)$. Moreover, if $z,\ w$ are two points of D three possible arranges of $\mathcal{O}(z,\ \varphi)$ arise:

$$\begin{array}{ll} I: & (z,\ w,\ z,\ w,\ \ldots), \\ II: & (z,\ w,\ z,\ z,\ \ldots), \\ III: & (z,\ w,\ w,\ w,\ \ldots). \end{array}$$

Only in the first case there is $\varphi \in \mathrm{Aut}(D)$ such that $\mathcal{O}(z,\ \varphi) = (z,\ w,\ z,\ w,\ \ldots)$, where ent $_{\varphi}[z] = 0$ and cyc $_{\varphi}[z] = 2$. The second case fails by the require injectiviness. For the thirth case, we consider the equations

$$\lambda\,\frac{z-a}{1-\overline{a}\,z} = \lambda\,\frac{w-a}{1-\overline{a}\,w} = w. \tag{1}$$

Since $z \neq w$ it must be $w \neq 0$. Therefore (1) is equivalent to

$$(z-w)\left(1-|a|^2\right) = 0,$$

which is no possible if $|a| < 1$. Analogously, if t is a third point in D three new arranges arise:

$$\begin{array}{ll} IV: & (z,\ w,\ t,\ z,\ w,\ t,\ \ldots), \\ V: & (z,\ w,\ t,\ w,\ t,\ w,\ \ldots), \\ VI: & (z,\ w,\ t,\ t,\ t,\ t,\ \ldots). \end{array}$$

The cases V and VI fail by no inyectiviness. The fourth case is realized as an orbit automorphism under the following restriction

$$\left|\frac{z-w}{1-z\,\overline{w}}\right| = \left|\frac{w-t}{1-w\,\overline{t}}\right| = \left|\frac{t-z}{1-t\,\overline{z}}\right|,$$

and then ent $_{\varphi}[z] = 0$ and cyc $_{\varphi}[z] = 3$. The relationship between points that are in the orbit of rotations is geometrically easy to describe. However, finite orbits of more general automorphisms involve more severe restrictions. On the other hand, every non elliptic automorphism as well as their iteractions has infinite orbits at all points within D.

Proposition 2 *Let the unit circle ∂D to be considered as a subspace of the complex plane and the unit disk D endowed with the Poincaré metric. For $\alpha,\ \beta \in \partial D$ and $a,\ b \in D$ we shall write*

$$(\alpha, a)\cdot(\beta, b) = \left(\overline{\alpha}\,\beta\,\frac{\alpha+\overline{a}\,b}{\overline{\alpha+\overline{a}\,b}}, \frac{\alpha\,a+b}{\alpha+\overline{a}\,b}\right) \tag{2}$$

and $(\partial D \times D,\ \cdot)$ becomes a topological group. Moreover, $\partial D \times D$ becomes isometrically isomorphic to Aut (D) *if* Aut(D) *is endowed with the quotient topo-logy by the map*

$$\Phi \ :\ \partial D \times D \to \mathrm{Aut}\,(D),$$

$$\Phi\,(\alpha, a)\,(z) \ \doteq\ \alpha\,\frac{z-a}{1-\overline{a}\,z},\ z \in D.$$

Proof. We observe that the product $\cdot$ in (2) is well defined. By the general form of analytic disk automorphisms Φ is bijective. Since

$$\Phi\left[(\alpha,a)\cdot(\beta,b)\right]=\Phi(\beta,b)\circ\Phi(\alpha,a)\quad if\quad (\alpha,a),\ (\beta,b)\in\partial D\times D$$

the product $\cdot$ is associative. We have $(1,0)\in\partial D\times D$ and

$$(\alpha,a)\cdot(1,0)=(1,0)\cdot(\alpha,a)=(\alpha,a)\quad if\quad (\alpha,a)\in\partial D\times D.$$

Moreover $(\alpha,a)\cdot(\overline{\alpha},-\alpha\ a)=(\overline{\alpha},-\alpha\ a)\cdot(\alpha,a)=(1,0)$ and we can write $(\alpha,a)^{-1}=(\overline{\alpha},-\alpha\ a)$, i.e. $(\partial D\times D,\ \cdot)$ is a group. let $\{(\alpha_i,\ a_i)\}_{i\in I}$, $\{(\beta_i,\ b_i)\}_{i\in I}$ be nets in $\partial D\times D$ which converge to (α,a) and (β,b) respectively. For $i\in I$ we write

$$(3)\qquad \begin{aligned}(\alpha_i,\ a_i)\cdot(\beta_i,\ b_i)^{-1} &= (\alpha_i,\ a_i)\cdot(\overline{\beta_i},\ -\beta_i\ b_i)\\ &= \left(\frac{\alpha_i}{\beta_i}\frac{1-a_i\ \overline{b_i}}{\overline{1-a_i\ \overline{b_i}}},\ \beta_i\ \frac{a_i-b_i}{1-a_i\ \overline{b_i}}\right).\end{aligned}$$

Since $\lim_{i\in I} d_P(a_i,a)=0$ then

$$0=\lim_{i\in I}\tanh d_P(a_i,a)=\lim_{i\in I}\left|\frac{a_i-a}{1-\overline{a}\ a_i}\right|.$$

Further, $|a_i-a|\,/\,|1-\overline{a}\ a_i|\geq|a_i-a|\,/2$ and $\lim_{i\in I}|a_i-a|=0$. Analogously $\lim_{i\in I}|b_i-b|=0$ and

$$\lim_{i\in I}\frac{\alpha_i}{\beta_i}\frac{1-a_i\ \overline{b_i}}{\overline{1-a_i\ \overline{b_i}}}=\frac{\alpha}{\beta}\frac{1-a\ \overline{b}}{\overline{1-a\ \overline{b}}}.$$

Since

$$d_P\left(\beta_i\ \frac{a_i-b_i}{1-a_i\ \overline{b_i}},\ \beta\ \frac{a-b}{1-a\ \overline{b}}\right)=\arg\tanh\left|\frac{\beta_i\ \frac{a_i-b_i}{1-a_i\ \overline{b_i}}-\beta\ \frac{a-b}{1-a\ \overline{b}}}{1-\beta\ \frac{a-b}{1-a\ \overline{b}}\overline{\left(\beta_i\frac{a_i-b_i}{1-a_i\ \overline{b_i}}\right)}}\right|$$

we obtain

$$\lim_{i\in I}d_P\left(\beta_i\ \frac{a_i-b_i}{1-a_i\ \overline{b_i}},\ \beta\ \frac{a-b}{1-a\ \overline{b}}\right)=0.$$

By (3)

$$\lim_{i\in I}\left[(\alpha_i,\ a_i)\cdot(\beta_i,\ b_i)^{-1}\right]=(\alpha,\ a)\cdot(\beta,\ b)^{-1}$$

and the claim follows. ■

Theorem 3 *The map $\Theta: D\times\operatorname{Aut}(D)\to D^{\mathbb{N}_0}$, $\Theta(z,\ \varphi)=\mathcal{O}(z,\ \varphi)$ is continuous onto a non dense subspace $\mathcal{O}$ of $D^{\mathbb{N}_0}$.*

Proof. Let $\{(z_l,\ \Phi(\alpha_l,a_l))\}_{l\in L}$ be a convergent net to $(z,\Phi(\alpha,a))$ and $n\in\mathbb{N}_0$. Then

$$d_P\left[\Phi^n(\alpha_l,a_l)(z_l),\Phi^n(\alpha,a)(z)\right]\le$$

$$\begin{aligned}&\le\ d_P\left[\Phi^n(\alpha_l,a_l)(z_l),\Phi^n(\alpha_l,a_l)(z)\right]+d_P\left[\Phi^n(\alpha_l,a_l)(z),\Phi^n(\alpha,a)(z)\right]\\&=\ d_P(z_l,z)+d_P\left[\Phi^n(\alpha_l,a_l)(z),\Phi^n(\alpha,a)(z)\right].\end{aligned}$$

It will suffice to show that

$$\lim_{l\in L}d_P\left[\Phi^n(\alpha_l,a_l)(z),\Phi^n(\alpha,a)(z)\right]=0 \tag{4}$$

for each n. The case $n=0$ is trivial and

$$d_P\left[\Phi(\alpha_l,a_l)(z),\Phi(\alpha,a)(z)\right]=d_P\left[\left(\Phi(\alpha,a)^{-1}\circ\Phi(\alpha_l,a_l)\right)(z),z\right]$$

$$\begin{aligned}&=\ \arg\tanh\left|\frac{\Phi(\alpha,a)^{-1}\left(\Phi(\alpha_l,a_l)(z)\right)-z}{1-\overline{z}\,\Phi(\alpha,a)^{-1}\left(\Phi(\alpha_l,a_l)(z)\right)}\right|\\&=\ \arg\tanh\left|\frac{\Phi\left(\overline{\alpha\,\alpha_l}\,\frac{\alpha_l-\alpha\,a\,\overline{a_l}}{\overline{\alpha_l-\alpha\,a\,\overline{a_l}}},\ \frac{\alpha_l\,a_l-\alpha\,a}{\alpha_l-\alpha\,a\,\overline{a_l}}\right)(z)-z}{1-\overline{z}\,\Phi\left(\overline{\alpha\,\alpha_l}\,\frac{\alpha_l-\alpha\,a\,\overline{a_l}}{\overline{\alpha_l-\alpha\,a\,\overline{a_l}}},\ \frac{\alpha_l\,a_l-\alpha\,a}{\alpha_l-\alpha\,a\,\overline{a_l}}\right)(z)}\right|\\&=\ \arg\tanh\left|\frac{\overline{\alpha\,\alpha_l}\,\frac{\alpha_l-\alpha\,a\,\overline{a_l}}{\overline{\alpha_l-\alpha\,a\,\overline{a_l}}}\,\frac{z-\frac{\alpha_l\,a_l-\alpha\,a}{\alpha_l-\alpha\,a\,\overline{a_l}}}{1-\left(\frac{\alpha_l\,a_l-\alpha\,a}{\alpha_l-\alpha\,a\,\overline{a_l}}\right)^{-}z}-z}{1-\overline{\alpha\,\alpha_l}\,\frac{\alpha_l-\alpha\,a\,\overline{a_l}}{\overline{\alpha_l-\alpha\,a\,\overline{a_l}}}\,\frac{z-\frac{\alpha_l\,a_l-\alpha\,a}{\alpha_l-\alpha\,a\,\overline{a_l}}}{1-\left(\frac{\alpha_l\,a_l-\alpha\,a}{\alpha_l-\alpha\,a\,\overline{a_l}}\right)^{-}z}\,\overline{z}}\right|.\end{aligned}$$

Since $\lim_{l\in L}\ \alpha_l=\alpha$ in ∂D and $\lim_{l\in L}\ |a_l-a|=0$ in D we obtain

$$\lim_{l\in L}d_P\left[\Phi(\alpha_l,a_l)(z),\Phi(\alpha,a)(z)\right]=0.$$

If (4) is assumed to be true for $\le n$ for $n\in\mathbb{N}$, let $w_l=\Phi^n(\alpha_l,a_l)(z)$ if $l\in L$ and $w=\Phi^n(\alpha,a)(z)$. Thus

$$d_P\left[\Phi^{n+1}(\alpha_l,a_l)(z),\Phi^{n+1}(\alpha,a)(z)\right]=d_P\left[\Phi(\alpha_l,a_l)(w_l),\Phi(\alpha,a)(w)\right]. \tag{5}$$

By the inductive hypothesis $\lim_{l\in L}d_P(w_l,w)=0$ and by the case $n=1$ and (5) Θ becomes continuous. On the other hand, let $z_0,\ z_1\in D,\ r_0>0,\ r_1>0$. We shall show that there exist $z_2\in D,\ s>0$ such that for all $\varphi\in\mathrm{Aut}(D)$ there isn´t $z\in D$ so that

$$d_P(z,\ z_0)<r_0,\ d_P(\varphi(z),\ z_1)<r_1\ \textit{and}\ d_P(\varphi^2(z),\ z_2)<s.$$

The set $\mathcal{P}=\{\varphi\in\mathrm{Aut}(D):\ D_P(\varphi(z_0),\ r_0)\cap D_P(z_1,\ r_1)\neq\emptyset\}$ is clearly non empty. If $\varphi\in\mathcal{P}$ we shall write

$$J_\varphi=D_P(\varphi(z_0),\ r_0)\cap D_P(z_1,\ r_1). \tag{6}$$

If $z \in D$ and $t > 0$ then

$$\varphi(J_\varphi) \cap D_P(z,\ t) \neq \emptyset \Leftrightarrow dist_P(z,\ \varphi(J_\varphi)) < t \tag{7}$$

If z and t satisfies (7) and $w \in \varphi(J_\varphi) \cap D_P(z,\ t)$ then

$$\begin{aligned} d_P(\varphi^{-2}(w),\ z_0) &= d_P(w,\ \varphi^2(z_0)) < r_0, \\ d_P(\varphi(\varphi^{-2}(w)),\ z_1) &= d_P(w,\ \varphi(z_1)) < r_1, \\ d_P(\varphi^2(\varphi^{-2}(w)),\ z) &= d_P(w,\ z) < t, \end{aligned}$$

i.e.

$$\mathcal{O} \cap \left[D_P(z_0,\ r_0) \times D_P(z_1,\ r_1) \times D_P(z,\ t) \times D^{\mathbb{N}_0 - \{0,1,2\}}\right] \neq \emptyset. \tag{8}$$

Moreover, it is easy to see that (8) $\Rightarrow$ (7). It will be enough to show that (8) does not hold for some z and t, i.e. we can choose $z_2 = z$ and $s = t$. Let us observe that

$$\cup_{\varphi \in \mathcal{P}} D_P(\varphi(z_0),\ r_0) \subseteq D_P(z_1,\ 2r_0 + r_1) \tag{9}$$

and if $t = 3r_0 + r_1 + d_P(z_0,\ z_1)$ then

$$\operatorname{diam}\left[D_P(\varphi(z_0),\ r_0) \cup D_P(z_1,\ 2r_0 + r_1)\right] = t. \tag{10}$$

Given $\varphi \in \mathcal{P}$, $w_1,\ w_2 \in J_\varphi$ by (6) there is $w_3 \in D_P(z_0,\ r_0)$ so that $w_1 = \varphi(w_3)$. Moreover, by (6) y (9) we have $J_\varphi \subseteq D_P(\varphi(z_0),\ r_0) \subseteq D_P(z_1,\ 2r_0 + r_1)$. By (10) we deduce

$$d_P(w_1,\ \varphi(w_2)) = d_P(\varphi(w_3),\ \varphi(w_2)) = d_P(w_2,\ w_3) \leq t.$$

Therefore

$$\sup\{d_P(w_1,\ \varphi(w_2)) : \quad \varphi \in \mathcal{P}, \quad w_1,\ w_2 \in J_\varphi\} \leq t. \tag{11}$$

Finally, let $z \in D$ so that $dist_P(z,\ D_P(z_1,\ 2r_0 + r_1)) \geq 2t$. Since (11) for $\varphi \in \mathcal{P}$, $w \in J_\varphi$ we can write

$$\begin{aligned} d_P(z,\ \varphi(w)) &\geq |d_P(z,\ w) - d_P(w,\ \varphi(w))| \\ &\geq d_P(z,\ w) - d_P(w,\ \varphi(w)) \\ &\geq dist_P(z,\ D_P(z_1,\ 2r_0 + r_1)) - t \\ &\geq t, \end{aligned}$$

i.e. $dist_P(z,\ \varphi(J_\varphi)) \geq t$ for each $\varphi \in \mathcal{P}$. ∎

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