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REMARKS ON $\sum_{n\leqslant N} \frac{(-1)^{n-1}}{n^{\frac{1}{2}+it}}$ AND $\zeta(\frac{1}{2}+it)$.

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ABSTRACT. Using a combinatorial identity we give a different expression for the sum $\sum_{n\leqslant N} \frac{(-1)^{n-1}}{n^{\frac{1}{2}+it}}$. We also show an approximate functional equation for $\zeta(\frac{1}{2}+it)$ given **only** in terms of sums of above type with $N = O(\sqrt{t})$.

0. Introduction.

It is well known that $\zeta(\frac{1}{2}+it) = O(\sum_{n\leqslant\sqrt{\frac{t}{2\pi}}} \frac{1}{n^{\frac{1}{2}+it}})$. See [2].

In this note we derive some results concerning the sum $\sum_{n\leqslant N} \frac{(-1)^{n-1}}{n^s}$ where $1 \leqslant N \leqslant ct$, c is some fixed positive constant and, as usual, $s = \sigma + it$. We describe briefly this note.

The main result of section 1 is Corollary 1 below which gives an equivalent sum, if $s = \frac{1}{2}+it$, for the above expression. This is done using a combinatorial identity (Lemma 1) which allows to write the above sum in a different manner (Theorem 1). Lemma 2 is an auxiliary lemma that we use to simplify the expression that finally appears.

The main result of section 2 is Theorem 2 which gives an approximate functional equation for $\zeta(\frac{1}{2}+it)$ depending **only** on $\sum_{n\leqslant c\sqrt{t}} \frac{(-1)^{n-1}}{n^{1/2+it}}$, with $c = \sqrt{\frac{1}{2\pi}}, \sqrt{\frac{2}{\pi}}$. This is done using the Hardy-Littlewood functional equation and a variant of it (Lemma 4).

Finally in section 3 we present an identity suitable for treating other sums.

Sections 1 and 2 can be read independently and are almost self-contained. Section 3 depends only on Lemma 1.

1. On the sum $\sum_{1\leqslant n\leqslant N} \frac{(-1)^{n-1}}{n^s}$.

We begin with a combinatorial result.

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Lemma 1. *Let $N = 1, 2, \ldots$. The following identity holds:*

$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{(n+x)} = \sum_{n=1}^{N} \frac{(2n-2)!}{4^{n-1}(2n+x)\ldots(1+x)} \left\{ \frac{3}{2}n + \frac{1}{2}x - \frac{1}{4} \right\} +$$

$$+ \sum_{n=1}^{N} (-1)^{n+N} \frac{(2n-2)!}{4^{n-1}(N+n+x)\ldots(N-n+x+1)} \left\{ \frac{N+x}{2} + \frac{1}{4} \right\}.. \qquad \text{(L1.1)}$$

Proof. We have

$$\sum_{k=1}^{K} \frac{b_1 \ldots b_{k-1}}{x(x+a_1)\ldots(x+a_k)} (x + a_k - b_k) = \frac{1}{x} - \frac{b_1 \ldots b_K}{x(x+a_1)\ldots(x+a_K)}, \qquad \text{(L1.2)}$$

which follows from writing the right hand side as $A_0 - A_K$ and noticing that each term on the left is $A_{k-1} - A_k$. Replace x by $(n+x)^2$, a_k by $-k^2$ and b_k by $k(\frac{1}{2} - k)$. Multiply everything by $(n+x)(-1)^{n-1}$ and add from $n = 1$ to N. Then $b_1 \ldots b_{k-1} = (-1)^{k-1} \frac{(2k-2)!}{4^{k-1}}$ and

$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{(n+x)} - \sum_{n=1}^{N} \frac{(-1)^{n-1} b_1 \ldots b_{n-1}}{(2n-1+x)\ldots(1+x)} =$$

$$= \sum_{n=1}^{N} \sum_{k=1}^{n-1} \underbrace{\frac{(-1)^{n-1}}{(n+k+x)\ldots(n-k+x)} b_1 \ldots b_{k-1} \left((n+x)^2 - \frac{k}{2} \right)}. \qquad \text{(L1.3)}$$

If we define

$$\epsilon_{n,k}(x) := (-1)^{n+k} \frac{(2k-2)!(\frac{n+x}{2} + \frac{1}{4})}{4^{k-1}(n+k+x)\ldots(n-k+1+x)},$$

then $\epsilon_{n,k}(x) - \epsilon_{n-1,k}(x) = \underbrace{\quad}$. Therefore the right-hand side of (L1.3) is equal to

$$\sum_{n=1}^{N} \sum_{k=1}^{n-1} \epsilon_{n,k}(x) - \epsilon_{n-1,k}(x) = \sum_{k=1}^{N} \epsilon_{N,k}(x) - \sum_{k=1}^{N} \epsilon_{k,k}(x),$$

and this yields formula (L1.1). □

The above lemma allows to rearrange the sum of the title of this section in a different way.

Theorem 1. *For any $s = \sigma + it$ and $N = 1, 2, \ldots$ we have*

$$\sum_{1 \leqslant n \leqslant N} \frac{(-1)^{n-1}}{n^s} =$$

$$\sum_{1\leqslant n\leqslant N}\frac{3n}{2(2n-1)4^{n-1}}\sum_{0\leqslant j\leqslant 2n-1}\binom{2n-1}{j}(-1)^j(1+j)^{-s}-$$

$$-\sum_{1\leqslant n\leqslant N}\frac{1}{2(2n-1)4^{n-1}}\sum_{0\leqslant j\leqslant 2n-1}\binom{2n-1}{j}(-1)^j(1+j)^{-s+1}-$$

$$-\sum_{1\leqslant n\leqslant N}\frac{1}{4(2n-1)4^{n-1}}\sum_{0\leqslant j\leqslant 2n-1}\binom{2n-1}{j}(-1)^j(1+j)^{-s}+$$

$$+\sum_{1\leqslant n\leqslant N}\frac{(-1)^{n+N}}{2(2n-1)4^{n-1}}\sum_{0\leqslant j\leqslant 2n-1}\binom{2n-1}{j}(-1)^j(N-n+1+j)^{-s}(n-1-j)+$$

$$+\sum_{1\leqslant n\leqslant N}\frac{(-1)^{n+N}}{4(2n-1)4^{n-1}}\sum_{0\leqslant j\leqslant 2n-1}\binom{2n-1}{j}(-1)^j(N-n+1+j)^{-s}.$$

Proof. Let $(-z)^{-s}=e^{-s\,log(-z)}$ where $-\pi\leqslant Arg(-z)<\pi$. Then $\sum_{1\leqslant n\leqslant N}\frac{(-1)^{n-1}}{n^s}=\frac{1}{2\pi i}\int_\gamma(-z)^{-s}\sum_1^N\frac{(-1)^{n-1}}{(n+z)}dz$ where γ is a positively oriented curve enclosing $-1,-2,\dots,$ $-2N$ and not touching the positive real axis. Now use Lemma 1 and the theorem of residues again. The first three double sums of Theorem 1 correspond to the three terms of the bracket $\{\frac{3n}{2}+\frac{x}{2}-\frac14\}$ and the other two double sums to the terms of $\{\frac{N+x}{2}+\frac14\}$. □

The following lemma is a tool that will be used to simplify the number of double sums in Theorem 1. From now on c will denote a fixed positive constant.

Lemma 2.

(i) *Let* $\sigma=\frac12$. *We have that*

$$A_{n,k}(s):=\frac{1}{4^{n-1}}\sum_{j=0}^{2n-1}\binom{2n-1}{j}(-1)^j(k+j+1)^{-s}(1+j-n),$$

is $O(t^\epsilon)$ *uniformly if* $0<\epsilon<1/4$, $1<t$ *and* n,k *are such that* $(\log t)^{\frac{1}{\epsilon}}\leqslant n\leqslant ct$, $k=0,1,2,\dots$.

(ii) *Let* $\sigma=\frac12$. *Then* $A_{n,k}(s)=O(\sqrt{n})$ *uniformly for all* $0<t$, $n=1,2,\dots,$ $k=0,1,2,\dots$.

(iii) *If* $\frac34<\sigma\leqslant1$, *then*

$$A_{n,k}(s)=O(n^{\frac34-\sigma}),$$

uniformly for all $0<t$, $n=1,2,\dots,$ $k=0,1,2,\dots$.

(iv) *Let* $\sigma=\frac12$. *Then there exist a real number* β, $1<\beta<e$ *such that the sum*

$$\frac{1}{4^{n-1}}\sum_{\substack{n-2n^{\frac34}-\frac12+\frac{1}{n^{1/4}}<j\leqslant n-n^{\frac12+\epsilon}\\ n+n^{\frac12+\epsilon}\leqslant j<n+2n^{\frac34}-\frac12-\frac{1}{n^{1/4}}}}\binom{2n-1}{j}(-1)^j(1+j)^{-s},$$

is $O(t^{\frac{7}{4}-\log\beta.\log t})$ *uniformly on* $0<\epsilon<1/4$, $1<t$ *and* n *such that* $(\log t)^{\frac{1}{\epsilon}}\leqslant n\leqslant ct$, $12\leqslant n$.

Proof. Recall the inequality

$$\sum_{\delta\leqslant|\frac{j}{2n-1}-\frac{1}{2}|}\binom{2n-1}{j}\leqslant\frac{4^n}{8(2n-1)\delta^2},\qquad\text{(L2.1)}$$

which is (7) of [1] page 6, with $x=1/2$; $2n-1$ instead of n.

We first prove i). Let $0<\epsilon<1/4$. Divide $0\leqslant j\leqslant 2n-1$, $j\in Z$, into three pairwise disjoint sets

$$\mathcal{P}_1=\{j/0\leqslant j\leqslant n-2n^{\frac{3}{4}}-\frac{1}{2}+\frac{1}{n^{1/4}};n+2n^{\frac{3}{4}}-\frac{1}{2}-\frac{1}{n^{1/4}}\leqslant j\leqslant 2n-1\},$$

$$\mathcal{P}_2=\{j/n-2n^{\frac{3}{4}}-\frac{1}{2}+\frac{1}{n^{1/4}}<j\leqslant n-n^{\frac{1}{2}+\epsilon};n+n^{\frac{1}{2}+\epsilon}\leqslant j<n+2n^{\frac{3}{4}}-\frac{1}{2}-\frac{1}{n^{1/4}}\},$$

$$\mathcal{P}_3=\{j/n-n^{\frac{1}{2}+\epsilon}<j<n+n^{\frac{1}{2}+\epsilon}\}.$$

We rewrite the sum $A_{n,k}(s)$ as $\sum_{j\in\mathcal{P}_1}+\sum_{j\in\mathcal{P}_2}+\sum_{j\in\mathcal{P}_3}$. We have

$$|\sum_{j\in\mathcal{P}_3}|\leqslant\frac{2}{(n-n^{1/2+\epsilon}+1)^\sigma}(1+n^{1/2+\epsilon})\leqslant n^\epsilon O(1)\leqslant O(t^\epsilon).\qquad\text{(L2.2)}$$

For $j\in\mathcal{P}_1$ we have ($[x]$ means as usual the integer part of x)

$$|\sum_{j\in\mathcal{P}_1}|\leqslant|\sum_{j\in\mathcal{P}_1;[n/2]\leqslant j}|+|\sum_{j\in\mathcal{P}_1;j<[n/2]}|\leqslant$$

$$\frac{1}{4^{n-1}}\frac{n}{(1+[n/2])^\sigma}\sum_{j\in\mathcal{P}_1;[n/2]\leqslant j}\binom{2n-1}{j}+\frac{1}{4^{n-1}}(n+1)\sum_{j\in\mathcal{P}_1;j<[n/2]}\binom{2n-1}{j}.$$

The first summand of this last expression is readily seen to be $O(1)$ using (L2.1) and the fact that $\mathcal{P}_1$ is the set $\{j/|\frac{j}{2n-1}-\frac{1}{2}|\geqslant n^{-1/4}\}$. Also

$$\sum_{j<[n/2]}\binom{2n-1}{j}\leqslant\binom{2n-1}{[n/2]}[n/2]\leqslant O(\alpha^n),\qquad\text{(L2.3)}$$

for some α, $0<\alpha<4$. This follows using Stirling formula. Therefore

$$|\sum_{j\in\mathcal{P}_1}|\leqslant O(1).\qquad\text{(L2.4)}$$

If $(-z)^{-s}$ is defined as $e^{-s\log(-z)}$ and $-\pi<Arg(-z)\leqslant\pi$ then $\sum_{j\in\mathcal{P}_2}$ is equal to

$$\frac{1}{2\pi i}\frac{(2n-1)!}{4^{n-1}}\sum_{i=1}^{2}\int_{\gamma_i}\frac{(-z+k)^{-s}(-z-n)}{(2n+z)(2n-1+z)\dots(1+z)}dz,$$

where γ_1 is a positively oriented curve enclosing $z=-j$ where $j\in Z$ satisfies $n-2n^{\frac{3}{4}}-\frac{1}{2}+\frac{1}{n^{1/4}}+1<j\leqslant n-n^{\frac{1}{2}+\epsilon}+1$. Denote j_0, j_1 the smallest and largest integer numbers satisfying the preceding inequality. We assume $12\leqslant n$, this gives that $0<j_0$ and $2\leqslant j_1$ (if $n<12$ then $A_{n,k}(s)=O(1)$ for all $t\in R$, $k=0,1,2,\dots$). The same for γ_2 with $z=-j$ and j such that $n+n^{\frac{1}{2}+\epsilon}+1\leqslant j<n+2n^{\frac{3}{4}}-\frac{1}{2}-\frac{1}{n^{1/4}}+1$. We consider first γ_1. The same argument applies to γ_2. Now take γ_1 as the rectangle $-j_0+\frac{1}{2}+ih$, $-j_1-\frac{1}{2}+ih$, $-j_1-\frac{1}{2}-ih$, $-j_0+\frac{1}{2}-ih$, $h>0$. Thus γ_1 is a rectangle of side j_1-j_0+1 and height $2h$, where h will be defined below. We have

$$|\frac{(2n-1)!}{4^{n-1}}\int_{\gamma_1}\frac{(-z+k)^{-s}(-z-n)}{(2n+z)(2n-1+z)\dots(1+z)}dz|\leqslant$$

$$\frac{(2n-1)!}{4^{n-1}}.\max_{z\in\gamma_1}\frac{|z-k|^{-\sigma}|z+n|}{|(2n+z)(2n-1+z)\dots(1+z)|}.\max_{z\in\gamma_1}e^{t.Arg(-z+k)}.\text{length }\gamma_1. \quad \text{(L2.5)}$$

Notice that the maximum $\max_{z\in\gamma_1}e^{t.Arg(-z+k)}$ is taken on $z_0=-j_0+1/2-ih$. Also $\max_{z\in\gamma_1}e^{t.Arg(-z+k)}\leqslant\max_{z\in\gamma_1}e^{t.Arg(-z)}$ if $k=0,1,2,\dots$. If one wished $\max_{z\in\gamma_1}e^{t.Arg(-z)}=e$ then one must have $Arctan\frac{h}{j_0-1/2}=\frac{1}{t}$ and therefore $h=(j_0-1/2)tan(\frac{1}{t})$. Thus we set $h:=min\{1,(j_0-1/2)tan(\frac{1}{t})\}$. From the above discussion we then have

$$\max_{z\in\gamma_1}e^{t.Arg(-z+k)}\leqslant e,\ \text{if } k=0,1,2,\dots \quad \text{(L2.6)}$$

Also using the definition of j_0 one sees that for some positive constant a

$$\frac{an}{t}\leqslant h, \quad \text{(L2.7)}$$

and

$$\text{length }\gamma_1=O(n^{3/4})\ ;\ max_{z\in\gamma_1}|z-k|^{-\sigma}|z+n|=O(n^{1/4}). \quad \text{(L2.8)}$$

It remains to estimate

$$\min_{z\in\gamma_1}|(2n+z)\dots(1+z)|:=\min_{z\in\gamma_1}w(z).$$

$w(z)$ is the product of distances from z to $-1,-2,\dots,-2n$. If z is on the vertical segment $[-j_1-1/2+ih,-j_1-1/2-ih]$ one sees that the minimun is taken on $z_1=-j_1-1/2$ since moving z upwards or downwards increases all these distances. In the same way for the other vertical segment one sees that the minimum is taken on $z_2=-j_0+1/2$. It is easy to see that the minimum is on z_1 if both segments are taken into account (for $w(-j_1-1/2)\leqslant w(-j_1+1/2)\leqslant w(-j_1+3/2)\leqslant\dots$) and is greater than $(2n-j_1-1)!(j_1-1)!\frac{1}{4}$. If z and $z+1$ belong to the horizontal segment $[-j_1-1/2+ih,-j_0+1/2+ih]$ then one sees by elementary considerations that $w(z)\leqslant w(z+1)$ and therefore the minimum on this segment is equal to $\min_{z\in[-j_1-1/2+ih,-j_1+1/2+ih]}w(z)$. This last expression is seen to be greater than $(2n-j_1-1)!(j_1-2)!\frac{h}{4}$. By symmetry the bound on the other horizontal segment is the same. The above discussion yields (recall $2 \leqslant j_1$ and by definition $h \leqslant 1$)

$$(2n-j_1-1)!(j_1-2)!\frac{h}{4} \leqslant \min_{z\in\gamma_1} w(z).$$

Using this last estimate, (L2.6) and (L2.8) one has that (L2.5) is bounded by

$$\frac{1}{4^{n-1}}\binom{2n-1}{j_1} j_1(j_1-1)\frac{1}{h}O(n). \tag{L2.9}$$

But $|n-n^{1/2+\epsilon}-j_1| \leqslant 1$. By Stirling formula $\binom{2n-1}{j_1} = O(\frac{4^n}{\sqrt{n}\beta^{n^{2\epsilon}}})$ for any n with β some fixed number such that $1<\beta<e$ (this is proved in the Appendix). Using this and (L2.7) in (L2.9) one obtains $|\sum_{j\in\mathcal{P}_2}| \leqslant O(\frac{tn^{3/2}}{\beta^{n^{2\epsilon}}})$. As $(\log t)^{\frac{1}{\epsilon}} \leqslant n \leqslant ct$ then $t^{(\log \beta)(\log t)} \leqslant \beta^{n^{2\epsilon}}$. Therefore

$$|\sum_{j\in\mathcal{P}_2}| \leqslant O(t^{5/2-\log \beta..\log t}). \tag{L2.10}$$

From (L2.10), (L2.2) and (L2.4), i) follows. We observe again that the O-term is uniform in the stated range of the variables.

Proof of ii). This is easy upon using (L2.3).

Proof of iii). Write $A_{n,k}(s)$ again as $\sum_{j\in\mathcal{P}_3}+\sum_{j\in\mathcal{P}_4}$ where $\mathcal{P}_3$ is defined as in the proof of i) and take $\mathcal{P}_4$ as its complement. Take ϵ in $\mathcal{P}_3$ to be $\frac{1}{8}$. Thus

$$|\sum_{j\in\mathcal{P}_3}| \leqslant \frac{1}{(n-n^{5/8}+1)^{\sigma}}(1+n^{5/8}) \leqslant O(n^{5/8-\sigma}),$$

and $|\sum_{j\in\mathcal{P}_4}| \leqslant \frac{1}{4^{n-1}}\sum_{j\in\mathcal{P}_4, j<[n/2]}\binom{2n-1}{j}(1+j)^{-\sigma}(1+j-n)+$

$$+\frac{1}{4^{n-1}}\sum_{j\in\mathcal{P}_4,[n/2]\leqslant j}\binom{2n-1}{j}(1+j)^{-\sigma}(1+j-n).$$

The first term is $O(n(\frac{\alpha}{4})^n)$ (use (L2.3)) and the second is $O(\frac{n^{1-\sigma}}{4^{n-1}}\sum_{j\in\mathcal{P}_4}\binom{2n-1}{j})$. From (L2.1) with $\delta=\frac{n^{-3/8}}{3}$ one obtains that $\sum_{j\in\mathcal{P}_4}\binom{2n-1}{j}=O(\frac{4^{n-1}}{n^{1/4}})$. Therefore

$$|\sum_{j\in\mathcal{P}_4}| \leqslant O(n^{\frac{3}{4}-\sigma}).$$

From this iii) follows.

Proof of iv). The stated sum is

$$\frac{1}{2\pi i}\frac{(2n-1)!}{4^{n-1}}\sum_{i=1}^{2}\int_{\gamma_i}\frac{(-z)^{-s}}{(2n+z)(2n-1+z)\dots(1+z)}dz, \tag{L2.11}$$

where the curves γ_1 and γ_2 are those defined in the proof of $\sum_{j\in\mathcal{P}_2}$ in i) above. The proof goes unchanged, except for the right-hand side of (L2.8) which must be substitued by $\max_{z\in\gamma_1}|z|^{-\sigma} = O(n^{-1/2})$. Then, if $12 = n_0 \leqslant n$, (L2.11) is $O(t^{\frac{7}{4}-\log\beta.\log t})$, and the lemma follows. □

Now we simplify the number of double sums appearing in Theorem 1. We do this in two steps, the first being

Lemma 3. *Let $t > e^2$ and $1 \leqslant N \leqslant ct$. Then*

$$\sum_{1\leqslant n\leqslant N}\frac{(-1)^{n-1}}{n^s} = \frac{1}{2}\sum_{n=1}^{N}\frac{1}{4^{n-1}}\sum_{j=0}^{2n-1}\binom{2n-1}{j}(-1)^j(1+j)^{-s} + \textit{error term},$$

where the error term is

i) $O(\log t.\exp(\sqrt{\frac{\log t.\log(\log t)}{2}}))$ if $\sigma = 1/2$, (here $\exp(z) = e^z$).

ii) $O(1)$ uniformly in σ if $\frac{3}{4} < \sigma_0 \leqslant \sigma \leqslant 1$.

Proof. Set

$$\sum_{n=1}^{N}\frac{1}{4^{n-1}}\sum_{j=0}^{2n-1}\binom{2n-1}{j}(-1)^j(1+j)^{-s} = \sum_{n=1}^{N}B_n(s).$$

Now, the third and the fifth double sums of Theorem 1 are easily seen to be $O(1)$. In fact, if $1/2 \leqslant \sigma \leqslant 1$ we have, using (L2.1), that

$$\sum_{j=0}^{2n-1}\binom{2n-1}{j}(-1)^j(1+j)^{-s} = O(\frac{4^n}{\sqrt{n}}). \tag{L3.1}$$

This gives that the third double sum is $O(1)$. The same argument applies to the fifth double sum.

The first, second and fourth double sum of Theorem 1 are written by rearrangement respectively as

$$\sum_{n=1}^{N}\frac{3}{4}B_n(s) + \sum_{n=1}^{N}\frac{3}{4(2n-1)}B_n(s), \tag{L3.2}$$

$$\sum_{n=1}^{N}\frac{1}{2(2n-1)}A_{n,0}(s) + \sum_{n=1}^{N}\frac{1}{4}B_n(s) + \sum_{n=1}^{N}\frac{1}{4(2n-1)}B_n(s), \tag{L3.3}$$

$$-\sum_{n=1}^{N}\frac{(-1)^{n+N}}{2(2n-1)}A_{n,N-n}(s), \tag{L3.4}$$

where $A_{n,k}(s)$ is defined in Lemma 2. Again the second sum of (L3.2) and the third of (L3.3) are $O(1)$. The main term of Lemma 3 is equal to the first term of (L3.2) minus the second of (L3.3). The first term of (L3.3) and (L3.4) are estimated using Lemma 2. They yield the 'error term' of Lemma 3. From this we readily get ii) of Lemma 3.

To prove i) consider, for example, (L3.4). It is bounded in absolute value, because of Lemma 2 i) and ii), by

$$\sum_{1\leqslant n<(log\ t)^{\frac{1}{\epsilon}}} \frac{1}{2(2n-1)}|A_{n,N-n}(s)| + \sum_{(log\ t)^{\frac{1}{\epsilon}}\leqslant n\leqslant N} \frac{1}{2(2n-1)}|A_{n,N-n}(s)| \leqslant$$

$$O(\sum_{1\leqslant n<(log\ t)^{\frac{1}{\epsilon}}} \frac{1}{\sqrt{n}}) + O(t^{\epsilon}log\ t) = O((log\ t)^{\frac{1}{2\epsilon}}) + O(t^{\epsilon}log\ t). \quad \text{(L3.5)}$$

Now if we take $\epsilon = \epsilon(t) = \sqrt{\frac{log(log\,t)}{2log\,t}}$ then $t^{\epsilon} = (log\,t)^{\frac{1}{2\epsilon}} = exp(\sqrt{\frac{log\,t.log(log\,t)}{2}})$. From this and (L3.5), i) follows. □

Finally we prove the main result of this section:

Corollary 1. *Let* $t > e^2$, $\epsilon := \epsilon(t) = \sqrt{\frac{log(log\,t)}{2log\,t}}$ *and* $exp(\sqrt{2.log\,t.log(log\,t)}) \leqslant N \leqslant ct$. *Then*

$$\sum_{1\leqslant n\leqslant N} \frac{(-1)^{n-1}}{n^{1/2+it}} =$$

$$= \frac{1}{2} \sum_{exp(\sqrt{2.log\,t.log(log\,t)})\leqslant n\leqslant N} \frac{1}{4^{n-1}} \sum_{n-n^{\frac{1}{2}+\epsilon}<j<n+n^{\frac{1}{2}+\epsilon}} \binom{2n-1}{j}(-1)^j(1+j)^{-\frac{1}{2}-it}+$$

$$+O\left(log\ t.exp(\sqrt{\frac{log\,t.log(log\,t)}{2}})\right)$$

Proof. From the definition of ϵ one has $t^{2\epsilon} = (log\,t)^{\frac{1}{\epsilon}} = exp(\sqrt{2.log\,t.log(log\,t)})$. The difference between the double sum of Corollary 1 and the double sum of Lemma 3 (with $\sigma = 1/2$) is

$$\frac{1}{2}\sum_{1\leqslant n<t^{2\epsilon}} \frac{1}{4^{n-1}} \sum_{j=0}^{2n-1} \binom{2n-1}{j}(-1)^j(1+j)^{-s}+$$

$$+\frac{1}{2}\sum_{t^{2\epsilon}\leqslant n\leqslant N} \frac{1}{4^{n-1}} \sum_{\substack{j\leqslant n-2n^{\frac{3}{4}}-\frac{1}{2}+\frac{1}{n^{1/4}} \\ n+2n^{\frac{3}{4}}-\frac{1}{2}-\frac{1}{n^{1/4}}\leqslant j}} \binom{2n-1}{j}(-1)^j(1+j)^{-s}+$$

$$+\frac{1}{2}\sum_{t^{2\epsilon}\leqslant n\leqslant N} \frac{1}{4^{n-1}} \sum_{\substack{n-2n^{\frac{3}{4}}-\frac{1}{2}+\frac{1}{n^{1/4}}<j\leqslant n-n^{1/2+\epsilon} \\ n+n^{1/2+\epsilon}\leqslant j<n+2n^{\frac{3}{4}}-\frac{1}{2}-\frac{1}{n^{1/4}}}} \binom{2n-1}{j}(-1)^j(1+j)^{-s}.$$

Using (L3.1) one obtains that the first sum is $O(t^{\epsilon})$. The second one is treated analogously to (L2.4) and it is easily seen to be $O(log\ t)$ (this is left to the reader). To get a bound for the third sum use Lemma 2 iv), obtaining that this sum is $O(1)$. Collecting the $O-$terms and the $O-$term of Lemma 3, the corollary follows. □

2. Relation between the sum $\sum_{1\leqslant n\leqslant c\sqrt{t}}\frac{(-1)^{n-1}}{n^{1/2+it}}$ and $\zeta(1/2+it)$.

The main result of this section is the following theorem which is proved using the Hardy-Littlewood approximate functional equation and a variant of it. To simplify the notation we write $L(x,s) := \sum_{n\leqslant x}\frac{(-1)^{n-1}}{n^s}$. Also $\bar{\ }$ means conjugation and $[x]$ denotes the integer part of a real number x.

Theorem 2. *If $t>2$ and $s=1/2+it$ then*

$$\zeta(s)\overline{\chi(s)}(3-2\sqrt{2}cos(t\,log2)) = -\overline{\frac{\chi(s)}{2^{\bar{s}-1}}}L(2[\sqrt{\frac{t}{2\pi}}]+1,s)+\overline{\chi(s)}L(\sqrt{\frac{t}{2\pi}},s)-$$

$$-\overline{\frac{1}{2^{s-1}}L(2[\sqrt{\frac{t}{2\pi}}]+1,s)}+\overline{L(\sqrt{\frac{t}{2\pi}},s)}+O(t^{-1/4}log\,t),$$

where, as usual, $\chi(s)=2^{s-1}\pi^s sec(\frac{s\pi}{2})/\Gamma(s)$.

Using Theorem 2 and (T2.1) below, it is easily seen that the growth of $\zeta(\frac{1}{2}+it)$ depends on $\sum_{n\leqslant c\sqrt{t}}\frac{(-1)^{n-1}}{n^{1/2+it}}$ with $c=\sqrt{\frac{1}{2\pi}},\sqrt{\frac{2}{\pi}}$. Thus, estimations of these sums yields an estimation of $\zeta(\frac{1}{2}+it)$.

To prove the above theorem we need to recall

The Hardy-Littlewood approximate functional equation ([2] pg. 79). *If h is a positive constant and $0<\sigma<1$, $2\pi xy=t$, $0<h<x$, $0<h<y$, then*

$$\zeta(s)=\sum_{n\leqslant x}\frac{1}{n^s}+\chi(s)\sum_{n\leqslant y}\frac{1}{n^{1-s}}+O(x^{-\sigma}log\,|t|)+O(|t|^{1/2-\sigma}y^{\sigma-1}) \qquad \text{(H-L)}$$

Our proof requires also a variant of this equation.

Lemma 4. *If $0<t$, $0<\sigma<1$, $2\pi xy=t$, $1/2<x\leqslant y$ then*

$$(1-2^{1-s})\zeta(s)=\sum_{n\leqslant x}\frac{(-1)^{n-1}}{n^s}-\frac{\chi(s)}{2^{s-1}}\sum_{0\leqslant n\leqslant[y+\frac{1}{2}]-1}\frac{1}{(2n+1)^{1-s}}+O(x^{-\sigma}). \qquad \text{(L4.1)}$$

Proof. We have for $0<\sigma<1$

$$(1-2^{1-s})\zeta(s)=\sum_1^\infty\frac{(-1)^{n-1}}{n^s}=\sum_1^m\frac{(-1)^{n-1}}{n^s}+\frac{(-1)^m}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}e^{-mx}}{e^x+1}dx.$$

This last integral can be transformed in the way of 2.4 of [2] and give

$$(1-2^{1-s})\zeta(s)=\sum_1^m\frac{(-1)^{n-1}}{n^s}+\frac{(-1)^m e^{-i\pi s}\Gamma(1-s)}{2\pi i}\int_C\frac{w^{s-1}e^{-mw}}{e^w+1}dw,$$

where C is a curve coming from $+\infty$, going around 0 and going back to $+\infty$ (it excludes all zeros of $e^w + 1$). Suppose $0 < t$, $1/2 < x \leqslant y$. Set $m = [x]$, $y = \frac{t}{2\pi x}$, $q = [y + \frac{1}{2}]$ (thus $1 \leqslant q$), $\eta = 2\pi y$. Deform the contour into straight lines $C_1, \ldots, C_4$ joining $+\infty, c_0\eta + i\eta(1 + c_0)$, $-c_0\eta + i\eta(1 - c_0)$, $-c_0\eta - i2\pi q$, $+\infty$ where c_0 is an absolute constant $0 < c_0 \leqslant 1/2$. We have then

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^s} - \frac{\chi(s)}{2^{s-1}} \sum_{n=0}^{q-1} \frac{1}{(2n+1)^{1-s}}$$
$$+ \frac{(-1)^m e^{-\pi i s}\Gamma(1-s)}{2\pi i}\{\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\}.$$

The last summand of the above expression is $O(x^{-\sigma})$. The proof of this fact is exactly the same as that given in pages 82 through 84 of [2]. □

Proof of Theorem 2. Take $x = y = \sqrt{\frac{t}{2\pi}}$, $s = \frac{1}{2} + it$, $t > 2$ in (H-L) and (L4.1). Recall ([2] pg.78) that

$$\chi(\frac{1}{2} + it) = (\frac{2\pi}{t})^{it} e^{i(t+\frac{\pi}{4})}(1 + O(\frac{1}{t})). \tag{T2.1}$$

Perhaps with the addition of one term to the second sum of (L4.1) one arrives to the formula

$$(1 - 2^{1-s})\zeta(s) = L(\sqrt{\frac{t}{2\pi}}, s) - \frac{\chi(s)}{2^{s-1}} \sum_{0 \leqslant n \leqslant \sqrt{\frac{t}{2\pi}}} \frac{1}{(2n+1)^{1-s}} + O(t^{-1/4}) \tag{T2.2}$$

Now add (H-L) and (T2.2) to yield

$$\zeta(s)2(1 - 2^{-s}) = \sum_{n \leqslant \sqrt{\frac{t}{2\pi}}} \frac{1}{n^s} + L(\sqrt{\frac{t}{2\pi}}, s) - \frac{\chi(s)}{2^{s-1}} \overline{L(2[\sqrt{\frac{t}{2\pi}}] + 1, s)} + O(t^{-1/4} \log t). \tag{T2.3}$$

Multiply (T2.3) by $\frac{\overline{\chi(s)}}{2^{\bar{s}+s-1}} = \overline{\chi(s)}$ and add to the complex conjugate of (T2.2) to obtain

$$\zeta(s)2(1 - 2^{-s})\overline{\chi(s)} + \overline{\zeta(s)}(1 - 2^{1-\bar{s}}) =$$
$$= -\frac{\overline{\chi(s)}}{2^{\bar{s}-1}} L(2[\sqrt{\frac{t}{2\pi}}] + 1, s) + \overline{\chi(s)} L(\sqrt{\frac{t}{2\pi}}, s) -$$
$$-\frac{|\chi(s)|^2}{2^{s-1}} \overline{L(2[\sqrt{\frac{t}{2\pi}}] + 1, s)} + \overline{L(\sqrt{\frac{t}{2\pi}}, s)} + O(t^{-1/4} \log t). \tag{T2.4}$$

The left-hand side of (T2.4) can be simplified a bit using the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ or $\overline{\zeta(s)} = \overline{\chi(s)}\zeta(s)$ if $s = 1/2 + it$. Also $|\chi(1/2 + it)|^2 = 1$. This gives the theorem. □

3. Final remarks.

Finally we prove the following identity :

Generalization of L1.1. *If $N = 1, 2, \ldots,$ and $x, \theta \in C$, then*

$$\sum_{n=1}^{N} \frac{e^{i\theta n}}{(n+x)} =$$

$$\sum_{n=1}^{N} \frac{(2n-2)!e^{i\theta(2n-1)}}{(e^{i\theta}-1)^{2(n-1)}(2n+x)\ldots(1+x)} \left\{2n+x - \frac{e^{i\theta}(n+x)}{e^{i\theta}-1} - \frac{e^{i\theta}(e^{i\theta}n+n-1)}{(e^{i\theta}-1)^2}\right\} +$$

$$\sum_{n=1}^{N} \frac{e^{i\theta(n+N-1)}(2n-2)!}{(e^{i\theta}-1)^{2(n-1)}(N+n+x)\ldots(N-n+x+1)} \left\{\frac{e^{i\theta}(N+x)}{e^{i\theta}-1} + \frac{e^{i\theta}(e^{i\theta}n+n-1)}{(e^{i\theta}-1)^2}\right\}$$

.....

Proof. If in the passage from (L1.2) to (L1.3) we multiply by $e^{i\theta n}(n+x)$, instead of multiplying by $(-1)^{n-1}(n+x)$, and leave b_k undefined for the moment, we get

$$\sum_{n=1}^{N} \frac{e^{i\theta n}}{(n+x)} - \sum_{n=1}^{N} \frac{e^{i\theta n} b_1 \ldots b_{n-1}}{(2n-1+x)\ldots(1+x)} =$$

$$= \underbrace{\sum_{n=1}^{N}\sum_{k=1}^{n-1} \frac{e^{i\theta n}}{(n+k+x)\ldots(n-k+x)} b_1 \ldots b_{k-1}\left((n+x)^2 - \frac{k}{2}\right)}.$$

Let $b_k = \frac{2e^{i\theta}k(2k-1)}{(e^{i\theta}-1)^2}$ (thus $b_1 \ldots b_{k-1} = \frac{e^{i\theta(k-1)}}{(e^{i\theta}-1)^{2(k-1)}}(2k-2)!$) and define

$$\epsilon_{n,k}(x) := \frac{e^{i\theta(k-1)}}{(e^{i\theta}-1)^{2(k-1)}}(2k-2)!e^{i\theta n} \frac{\left(\frac{e^{i\theta}}{e^{i\theta}-1}(n+x) + \frac{e^{i\theta}(ke^{i\theta}+k-1)}{(e^{i\theta}-1)^2}\right)}{(n+k+x)\ldots(n-k+1+x)}.$$

Check that $\epsilon_{n,k}(x) - \epsilon_{n-1,k}(x) = \underbrace{}$. Now the proof follows as in the Lemma 1 and is left to the reader. □

Appendix.

Proposition. *There exist positive constants a, β, $1 < \beta < e$ such that for all $0 < \epsilon < 1/4$, $n, j_1 \in N$, $|n - n^{1/2+\epsilon} - j_1| \leqslant 1$, one has*

$$\binom{2n-1}{j_1} \leqslant a \frac{4^n}{\sqrt{n}\beta^{(n^{2\epsilon})}}. \tag{P.1}$$

Proof. We shall denote by a_1, a_2, a_3 absolute positive constants. Now writing $n - j_1 = k$ one sees that $k = O(n^{3/4})$. Stirling formula yields

$$\binom{2n-1}{j_1} = \binom{2n-1}{n-k} = \frac{(2n-1)!}{(n+k-1)!(n-k)!} \leqslant a_1 \frac{(2n-1)^{2n-1/2}}{(n+k-1)^{n+k-1/2}(n-k)^{n-k+1/2}}$$

$$\leqslant a_2 \frac{(2n)^{2n-1/2}}{(n+k)^{n+k}(n-k)^{n-k}} = a_2 \frac{4^n}{\sqrt{2n}(1+\frac{k}{n})^{n+k}(1-\frac{k}{n})^{n-k}}$$

We have for δ some small positive number independent of ϵ and j_1, that

$$(1+\frac{k}{n})^{n+k}(1-\frac{k}{n})^{n-k} = \{(1-\frac{k^2}{n^2})^{\frac{n^2}{k^2}}\}^{\frac{k^2}{n}}\{(1+\frac{2k}{n-k})^{\frac{n-k}{2k}}\}^{\frac{2k^2}{n-k}} \geqslant$$

$$\frac{1}{(e+\delta)^{\frac{k^2}{n}}}(e-\delta)^{\frac{2k^2}{n-k}} \geqslant \{\frac{(e-\delta)^2}{e+\delta}\}^{\frac{k^2}{n-k}} = \beta^{\frac{k^2}{n-k}} \geqslant \beta^{\frac{k^2}{n}} \geqslant a_3\beta^{(n^{2\epsilon})},$$

this last inequality because of $\frac{k^2}{n} - n^{2\epsilon} = o(1)$. □

We point out that Lemma 1 (an other identities derived by the author) can be used to obtain an approximate functional equation for $\zeta(s)$ in the critical strip. This will appear elsewhere.

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