The Convolution Product of k-th derivative of Dirac delta in $P \pm i0 - m^2$.

Manuel A. Aguirre
Núcleo Consolidado de
Matemática Pura y Aplicada(NUCOMPA),
Facultad de Ciencias Exactas,
UNCentro,Pinto 399,7000 Tandil,
ARGENTINA.
E-mail:maguirre@exa.unicen.edu.ar

Abstract

In this paper we give a sense to the convolution product of certain kinds of distribution $\delta^{(k)}(P \pm i0 - m^2)$ which has been introduced by ([1],page339-348).

1 Introduction.

Let $x = (x_1, ...x_n)$ a point of the n- dimensional Euclidean R^n.
Consider a quadratic form in n variables defined by

$$P = P(x) = x_1^2 + ...x_p^2 - x_{p+1}^2 ... - x_{p+q}^2 \tag{1}$$

where $p + q = n$ is the dimension of the space.

The distribution $(m^2 + P \pm i0)^\lambda$ are defined by

$$(m^2 + P \pm i0)^\lambda = \lim_{\epsilon \to 0}(m^2 + P \pm i\epsilon \mid x \mid^2)^\lambda, \tag{2}$$

where $\epsilon > 0, m$ is a positive real number,λ is a complex number and

$$\mid x \mid^2 = x_1^2 + ...x_n^2. \tag{3}$$

It is useful to state an equivalent definition of distribution $(m^2 + P \pm i0)^\lambda$. In this definition the following distribution appear:

$$(m^2 + P)_+^\lambda = \begin{cases} (m^2 + P)^\lambda & if \quad m^2 + P \geq 0, \\ 0 & if \quad m^2 + P < 0. \end{cases} \tag{4}$$

and

$$(m^2+P)^\lambda_- = \begin{cases} (-(m^2+P))^\lambda & if \quad m^2+P \le 0, \\ 0 & if \quad m^2+P > 0; \end{cases} \tag{5}$$

where m a is positive real number and λ is a complex number.

From([5],page566), we have

$$(m^2+P\pm i0)^\lambda = (m^2+P)^\lambda_+ + e^{\pm\lambda\pi i}(m^2+P)^\lambda_- \tag{6}$$

On the other hand the distribution $(P \pm i0)^\lambda$ is defined by

$$(P\pm i0)^\lambda = \lim_{\varepsilon\to 0}(P \pm i\epsilon \mid x \mid^2)^\lambda, \tag{7}$$

where $\epsilon > 0, m$ is a positive real number,λ is a complex number and $\mid x \mid^2$is defined by (3).

The distribution $(P \pm i0)^\lambda$ have poles at the points $\lambda = -\frac{n}{2} - k, k = 0, 1, 2, ...$

¿From ([2],page 276,formulas (2) and (2')),we have

$$(P\pm i0)^\lambda = P^\lambda_+ + e^{\pm\lambda\pi i}P^\lambda_-, \tag{8}$$

where

$$P^\lambda_+ = \begin{cases} P^\lambda & if \quad P \ge 0, \\ 0 & if \quad P < 0, \end{cases} \tag{9}$$

and

$$P^\lambda_- = \begin{cases} (-P)^\lambda & if \quad P \le 0, \\ 0 & if \quad P > 0. \end{cases} \tag{10}$$

We note that (c.f.[5],page 566), the following result is valid

$$\mathrm{Re}\, s_{\lambda=-k}(m^2+P)^\lambda_+ = \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(m^2+P), \tag{11}$$

and

$$\mathrm{Re}\, s_{\lambda=-k}(m^2+P)^\lambda_- = \frac{1}{(k-1)!}\delta^{(k-1)}(m^2+P). \tag{12}$$

2 The Convolution Product of $\delta^{(k)}(P \pm i0 - m^2) * \delta^{(r)}(P \pm i0 - m^2)$

In this paragraph we need the following formulae:

$$\delta^{(k)}(P \pm i0 - m^2) = (-1)^k \pi^{\frac{n}{2}} e^{\frac{\mp q\pi i}{2}} \sum_{j\geq 0} \{\frac{(m^2)^{j+\frac{n}{2}-k-1} L^j \delta}{2^{2j} j! \Gamma(\frac{n}{2}+j-k)}\} \quad \text{for n odd} \quad (13)$$

([1],page 344,formula 3.13),.

$$\delta^{(k)}(P \pm i0 - m^2) = (-1)^k \pi^{\frac{n}{2}} e^{\frac{\mp q\pi i}{2}} \sum_{j\geq 0} \frac{(m^2)^{j+\frac{n}{2}-k-1} L^j \delta}{2^{2j} j! \Gamma(\frac{n}{2}+j-k)} \quad \text{for n even if } k < \frac{n}{2} \quad (14)$$

([1],page 344,formula 3.13),

$$\delta^{(k)}(P \pm i0 - m^2) = \frac{(-1)^k}{\pi^{-\frac{n}{2}}} e^{\frac{\mp q\pi i}{2}} . \sum_{j\geq k-\frac{n}{2}+1} \frac{(m^2)^{j+\frac{n}{2}-k-1} L^j \delta}{2^{2j} j! \Gamma(j-k+\frac{n}{2})} \quad \text{for n even if } k \geq \frac{n}{2} \quad (15)$$

([1],page 344,formula 3.14) and

$$L^j \{\delta(x)\} * L^r \{\delta(x)\} = L^{j+r} \{\delta(x)\} \quad (16)$$

([1],page 347,formula(5.10)). Where j and r are integers non negative and L^s is the ultrahyperbolic operator iterated j times defined by

$$L^s = \left\{ \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - ... - \frac{\partial^2}{\partial x_{p+q}^2} \right\}^s . \quad (17)$$

In this paragraph the simbol $*$ will mean convolution.
Now,we prove the following theorem:

Theorem 1 *Let n be the odd dimension of the space,k,r non-negative integers such $k + r \geq n - 2$ then the following formula is valid:*

$$\{\delta^{(k)}(P \pm i0 - m^2)\} * \{\delta^{(r)}(P \pm i0 - m^2)\} = (-1)^{k+r} \pi^{\frac{n}{2}} \pi^{\frac{n}{2}} . e^{\frac{\mp q\pi i}{2}} . e^{\frac{\mp q\pi i}{2}} . \sum_{s\geq k+r-n+2} \binom{n-k-r+2s-2}{s} \frac{(m^2)^{n-k-r+s-2}}{4^s \Gamma(\frac{n}{2}-r+s)\Gamma(\frac{n}{2}-k+s)} L^s \delta\{x\}, \quad (18)$$

Where

$$\left(\begin{smallmatrix} t \\ s \end{smallmatrix} \right) = \frac{t!}{j!(t-j)!}, \tag{19}$$

$$t! = \Gamma(t+1) \tag{20}$$

and L^s is the ultrahyperbolic operator iterated j times defined the formula (17).

Proof. From (13) and considering (16) we have,

$$\delta^{(k)}(P \pm i0 - m^2) * \delta^{(r)}(P \pm i0 - m^2) = (-1)^{k+r} \pi^{\frac{n}{2}} \pi^{\frac{n}{2}} . e^{\frac{\mp q\pi i}{2}} . e^{\frac{\mp q\pi i}{2}} .$$

$$\cdot \sum_{j \geq 0} \frac{(m^2)^{\frac{n}{2}-k-1 \dagger j}}{j! 4^j \Gamma(\frac{n}{2}-k+j)} \sum_{\nu \geq 0} \frac{(m^2)^{\frac{n}{2}-r-1 \dagger \nu}}{\nu! 4^\nu \Gamma(\frac{n}{2}-r+\nu)} \left[L^j \delta\{x\} * L^\nu \delta\{x\} \right] = (-1)^{k+r} . e^{\frac{\mp q\pi i}{2}} .$$

$$e^{\frac{\mp q\pi i}{2}} . \pi^n . \sum_{j \geq 0} \frac{(m^2)^{\frac{n}{2}-k-1 \dagger}}{j! 4^s \Gamma(\frac{n}{2}-k+j)} \sum_{\nu \geq 0} \frac{(m^2)^{\frac{n}{2}-r-1 \dagger \nu}}{\nu! 4^\nu \Gamma(\frac{n}{2}-r+\nu)} L^{j+\nu} \{\delta(x)\} = (-1)^{k+r} e^{\frac{\mp q\pi i}{2}} .$$

$$e^{\frac{\mp q\pi i}{2}} . \pi^n \sum_{s \geq 0} \frac{(m^2)^{\frac{n}{2}-k-1 \dagger \frac{n}{2}-r-1 \dagger s}}{4^s} \left(\sum_{j=0}^{s} \frac{1}{j!(s-j)! \Gamma(\frac{n}{2}-k+j) \Gamma(\frac{n}{2}-r+s-j)} \right) L^s \{\delta(x)\} \quad . \tag{21}$$

On the other hand the following relation are valid,

$$\frac{1}{\Gamma(\frac{n}{2}+j-k)} = \frac{(s-j)!}{\Gamma(\frac{n}{2}+s-k)} \binom{\frac{n}{2}-k+s-1}{s-j} \tag{22}$$

and

$$\frac{1}{\Gamma(\frac{n}{2}+s-j-r)} = \frac{(j)!}{\Gamma(\frac{n}{2}+s-r)} \binom{\frac{n}{2}-r+s-1}{j} \tag{23}$$

Using (22) and (23) we have,

$$\sum_{j=0}^{s} \frac{1}{j!(s-j)! \Gamma(\frac{n}{2}-k+j) \Gamma(\frac{n}{2}-r+s-j)} =$$

$$\sum_{j=0}^{s} \binom{\frac{n}{2}-k+s-1}{s-j} \binom{\frac{n}{2}-r+s-1}{s-j} \cdot \frac{1}{\Gamma(\frac{n}{2}-k+s) \Gamma(\frac{n}{2}-r+s)} = \tag{24}$$

$$\binom{\frac{n}{2}-k+s-1+\frac{n}{2}-r+s-1}{s} \cdot \frac{1}{\Gamma(\frac{n}{2}-k+s) \Gamma(\frac{n}{2}-r+s)}$$

¿From (21) and using (24) we have,

$$\{\delta^{(k)}(P \pm i0 - m^2)\} * \{\delta^{(r)}(P \pm i0 - m^2)\} = (-1)^{k+r} \pi^n e^{\frac{\mp q\pi i}{2}} . e^{\frac{\mp q\pi i}{2}} .$$

$$\sum_{s \geq 0} \binom{n-k-r+2s-2}{s} \frac{(m^2)^{n-k-r+s-2}}{4^s \Gamma(\frac{n}{2}-r+s) \Gamma(\frac{n}{2}-k+s)} L^s \delta\{x\} . \tag{25}$$

On the other hand using (19) and (20) and the formula

$$\Gamma(z).\Gamma(1-z) = \frac{\pi}{sen(z\pi)}, \tag{26}$$

we have

$$\binom{n-k-r+2s-2}{s} = \frac{(n-k-r+2s-2)!}{s!(n-k-r+s-2)!} = \frac{(n-k-r+2s-2)!}{s!\Gamma(n-k-r+s-2+1)} =$$

$$(n-k-r+2s-2)!.\frac{\Gamma((2-s+k+r-n)).sen(2-s+k+r-n)\pi}{s!.\pi} = 0 \tag{27}$$

$$\text{if } \; s < k+r-n+2,$$

here k+r-n+2$\geq$ 0.
¿From (25) and (27), we obtain

$$\{\delta^{(k)}(P \pm i0 - m^2)\} * \{\delta^{(r)}(P \pm i0 - m^2)\} = (-1)^{k+r}\pi^n e^{\frac{\mp q\pi i}{2}}.e^{\frac{\mp q\pi i}{2}}.$$

$$\textstyle\sum_{s \geq k+r-n+2} \binom{n-k-r+2s-2}{s}. \tag{28}$$

$$\frac{(m^2)^{n-k-r+s-2}}{4^s\Gamma(\frac{n}{2}-r+s)\Gamma(\frac{n}{2}-k+s)} L^s \delta\{x\} \text{ if k+r-n+2} \geq 0.$$

¿From (28) we conclude the proof of theorem 1. ■

Theorem 2 *Let n be the even dimension of the space, k, r non-negative integers such* $k<\frac{n}{2}$ *and* $r<\frac{n}{2}$ *then the following formula is valid*

$$\{\delta^{(k)}(P \pm i0 - m^2)\} * \{\delta^{(r)}(P \pm i0 - m^2)\} = \quad (-1)^{k+r}\pi^{\frac{n}{2}}\pi^{\frac{n}{2}}.e^{\frac{\mp q\pi i}{2}}.e^{\frac{\mp q\pi i}{2}}.$$

$$\textstyle\sum_{s \geq k+r-n+2} \binom{n-k-r+2s-2}{s} \frac{(m^2)^{n-k-r+s-2}}{4^s\Gamma(\frac{n}{2}-r+s)\Gamma(\frac{n}{2}-k+s)} \,\Delta^s\, \delta\,\{x\}$$

$$\textit{under condition} \;\; n-2 \leq k+r < n. \tag{29}$$

The proof of Theorem 2 is the same form of Theorem 1 except that in this case from(14) using the conditions $k<\frac{n}{2}$ *and* $r<\frac{n}{2}$ *we have k+r <n, therefore*

from the condition k+r-n+2$\geq$ 0 *and k+r<n, we obtain the condition n-2<k+r <n under which the Theorem 2 is valid.*

Theorem 3 *Let n be the even dimension of the space, k, r non-negative integers such $k \geq \frac{n}{2}$ and $r \geq \frac{n}{2}$ then the following formula is valid*

$$\delta^{(k)}(P \pm i0 - m^2) * \delta^{(r)}(P \pm i0 - m^2) = (-1)^{k+r} \pi^{\frac{n}{2}} \pi^{\frac{n}{2}} . e^{\frac{\mp q\pi i}{2}} . e^{\frac{\mp q\pi i}{2}} .$$
$$\sum_{s \geq 0} \binom{k+r+2s-n+2}{s} \frac{(m^2)^s}{4^{s+k+r-n+2}\Gamma(r-\frac{n}{2}+s+2)\Gamma(k-\frac{n}{2}+s+2)} L^{s+k+r-n+2} \delta\{x\} \quad (30)$$

Proof. From (15) and considering (16), we have

$$\delta^{(k)}(P \pm i0 - m^2) * \delta^{(r)}(P \pm i0 - m^2) = \frac{(-1)^{k+r} \pi^{\frac{n}{2}} \pi^{\frac{n}{2}} . e^{\frac{\mp q\pi i}{2}} . e^{\frac{\mp q\pi i}{2}}}{4^{k+r-n+2}} . \quad (31)$$
$$\sum_{s \geq 0} \left(\frac{m^2}{4}\right)^s . \sum_{j=0}^{s} \frac{1}{j!(s-j)!(k-\frac{n}{2}+j+1)!(r-\frac{n}{2}+s-j+1)!} L^{s+k+r-n+2}\{\delta(x)\}.$$

Using the formulae(20), (21) and (22) from (31), we have

$$\delta^{(k)}(P \pm i0 - m^2) * \delta^{(r)}(P \pm i0 - m^2) = (-1)^{k+r} \pi^n . e^{\frac{\mp q\pi i}{2}} . e^{\frac{\mp q\pi i}{2}} .$$

$$\sum_{s \geq 0} \binom{k+r+2s-n+2}{s} \frac{(\frac{m^2}{4})^s}{4^{k+r-n+2}\Gamma(k-\frac{n}{2}+s+2)\Gamma(r-\frac{n}{2}+s+2)!} L^{s+k+r-n+2}\{\delta(x)\} \quad (32)$$

$$if \quad k + r \geq n$$

¿From (32) we conclude the proof of therem 3. ∎

In particular putting $m^2 = 0$ in (18), (29) and (30) we have the following formulae:

$$\delta^{(k)}(P \pm i0) * \delta^{(r)}(P \pm i0) = \frac{(-1)^{k+r} \pi^n}{4^{k+r-n+2}\Gamma(k-\frac{n}{2}+2)} e^{\frac{\mp q\pi i}{2}} . e^{\frac{\mp q\pi i}{2}} .$$

$$\frac{1}{\Gamma(r-\frac{n}{2}+2)} . L^{k+r-n+2}\{\delta(x)\} \qquad \text{is if } n \text{ is odd,} \quad (33)$$

$$\delta^{(k)}(P \pm i0) * \delta^{(r)}(P \pm i0) = \frac{(-1)^{k+r} \pi^n}{4^{k+r-n+2}\Gamma(k-\frac{n}{2}+2)} .$$
$$\frac{1}{\Gamma(r-\frac{n}{2}+2)} . L^{k+r-n+2}\{\delta(x)\} \quad (34)$$

if n is even for $k < \frac{n}{2}$ and $r < \frac{n}{2}$

and

$$\delta^{(k)}(P \pm i0) * \delta^{(r)}(P \pm i0) = \frac{(-1)^{k+r} \pi^n}{4^{k+r-n+2}\Gamma(k-\frac{n}{2}+2)} e^{\frac{\mp q\pi i}{2}} . e^{\frac{\mp q\pi i}{2}} .$$
$$\frac{1}{\Gamma(r-\frac{n}{2}+2)} . L^{k+r-n+2}\{\delta(x)\} \quad (35)$$

if n is even for $k \geq \frac{n}{2}$ and $r \geq \frac{n}{2}$.

Using the formula (26), under conditions k< $\frac{n}{2}$ and k< $\frac{n}{2}$ for n even, we have,

$$\frac{1}{\Gamma(k-\frac{n}{2}+2)}=0 \quad \text{if } k<\frac{n}{2}-1 \tag{36}$$

and

$$\frac{1}{\Gamma(r-\frac{n}{2}+2)}=0 \ \text{ if } \ r<\frac{n}{2}-1 \tag{37}$$

¿From (34) and using (36) and (37) we have

$$\delta^{(k)}(P\pm i0)*\delta^{(r)}(P\pm i0)=0 \text{ if } k<\frac{n}{2}-1 \text{ and } r<\frac{n}{2}-1 \tag{38}$$

Therefore from (34) and (35), we have

$$\delta^{(k)}(P\pm i0)*\delta^{(r)}(P\pm i0)=\frac{(-1)^{k+r}\pi^n}{4^{k+r-n+2}\Gamma(k-\frac{n}{2}+2)}.e^{\frac{\mp q\pi i}{2}}.e^{\frac{\mp q\pi i}{2}}.$$
$$\frac{1}{\Gamma(r-\frac{n}{2}+2)}.L^{k+r-n+2} \text{ if } n \text{ is even for } k\geq\frac{n}{2}-1 \text{ and } r\geq\frac{n}{2}-1 \ . \tag{39}$$

¿From (35), (38) and (39), we obtain the following formulae:

$$\delta^{(k)}(P\pm i0)*\delta^{(r)}(P\pm i0)=e^{\frac{\mp q\pi i}{2}}.e^{\frac{\mp q\pi i}{2}}.a_{k,r,n}L^{k+r-n+2}\delta \text{ if n is odd,} \tag{40}$$

$$\delta^{(k)}(P\pm i0)*\{\delta^{(r)}(P\pm i0)=0 \ \text{ if } n \text{ is even for } k<\frac{n}{2}-1\text{and } r<\frac{n}{2}-1 \tag{41}$$

and

$$\delta^{(k)}(P\pm i0)*\delta^{(r)}(P\pm i0)=e^{\frac{\mp q\pi i}{2}}.e^{\frac{\mp q\pi i}{2}}.a_{k,r,n}L^{k+r-n+2}\delta$$
$$\text{if n is even } for \ \ k\geq\frac{n}{2}-1 \text{ and } r\geq\frac{n}{2}-1. \tag{42}$$

here

$$a_{k,r,n}=\frac{(-1)^{k+r}\pi^n}{4^{k+r-n+2}\Gamma(k-\frac{n}{2}+2)\Gamma(r-\frac{n}{2}+2)}. \tag{43}$$

On the other hand using the formula([1],page 344,formula (4.2))

$$\delta^{(k)}(P \pm i0) = \frac{e^{\frac{\mp q\pi i}{2}}(-1)^k \pi^{\frac{n}{2}}.}{4^{k-\frac{n}{2}+1}\Gamma(k-\frac{n}{2}+2)} \text{if } n \text{ is even and } k \geq \frac{n}{2} \tag{44}$$

from(42),we have the following formula

$$\delta^{(k)}(P \pm i0) * \delta^{(r)}(P \pm i0) = e^{\frac{\mp q\pi i}{2}}.\frac{(k+r-n+2)!\pi^{\frac{n}{2}}}{(k-\frac{n}{2}+1)!(r-\frac{n}{2}+1)!(-1)^{1-\frac{n}{2}}}\delta^{(r+k+1-\frac{n}{2})}(P \pm i0)$$

$$\text{if n is even } for \;\; k \geq \tfrac{n}{2} - 1 \text{ and } r \geq \tfrac{n}{2} - 1. \tag{45}$$

The formula (45) appears in ([1],formula (5.1) page 346.

References

[1] M.A.Aguirre T. The distribution $\delta^{(k)}(P \pm i0 - m^2)$,Journal of Computational and Applied Math.88(1997), Elservier,N.H, 309-348.

[2] I.M.Gelfand and G.EShilov.Generalized Functions,Vol.I,Academic Press,New York,1964.

[3] L.S.Schwartz.Theorie des Distributions,Herman,Paris,1966.

[4] A.Erdelyi(Editor).Higher Trascendental Functions,Vol.I and II,McGraw-Hill,New York,1953.

[5] D.N.Brester.On the distribution connected with quadratic forms,SIAM J. Appl. Math. 16 (1968) 563-581.