

# COMPUTER SIMULATIONS AND THEORETICAL PROBES FOR NEURODYNAMIC CONVERGENCE PROPERTIES

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**ABSTRACT :** The work deals simultaneously with two different neurodynamic contexts: The magnetic spin systems and the neurophysiological systems. Both are analyzed within a unifying set of modular tools which enable us to handle all initial conditions at once. Our generic neurodynamic model mimicks the behaviour of a single layer of neurons that are activated or deactivated according to a step function and its algorithmic implementation. The latter allows us to verify the convergence properties of the system, thus, reaching conclusions about the stability and/or the attractive character of certain configurations, focusing on the interrelationship of such properties and the choices of synaptic tensors and updating strategies. Specifically, we investigate the dynamics of recurrent periodic and deterministic networks updated by parallel and serial operation modes. The work involves computer experiments which provide support for our theoretical tenets. The work is organized as follows: First we discuss our inspirational source, rooted in a neurobiological model, then we provide an algorithmic formulation of the model, supplemented by novel theoretical developments and finally, we cast our results making use of new representational tools.

## INTRODUCTION AND OUTLINE OF THE WORK

This work presents a comparative study unifying by means of a single set of representational modular tools the behavior of two different neurodynamic systems representing differentiated contexts: The magnetic spin systems, typically lacking self-coupling of their units, and the neurophysiological systems endowed with self-coupling of their units.

Two different operational regimes are handled for the sake of comparison: synchronic and asynchronic updating.

The advantage of our approach is a representational one, enabling us to visualize in matrix form, all conformations of the systems.

Thus, mapping the conformation into a two dimensional object, facilitates the assessment of the dynamical visually traits determined by specified initial conditions. This representational tool considerably simplifies the comparative analysis.

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This paper is organized as follows, in Section 1 we outline our biological motivation for our neurodynamic model which will be subsequently compared. Section 2 deals with the theoretical tenets of the algorithm built upon the models. Section 3 is devoted to the comparative characterization of the dynamic properties such as stability and attractivity of persistent states *vis a vis* different updating and synaptic schemes. Section 4 is concerned with numerical simulations of the most revealing dynamical features, and finally the conclusions are presented.

## 1. ELEMENTS OF THE MODEL

### 1.1 Basic building blocks of the model

The basic physiological structure of the brain has been intensely studied<sup>1-6</sup>. The fundamental constructive blocks are neurons. Each neuron can be activated by the flow of a chemical activator through synaptic bindings between different neurons. The neuron can be inhibited or excited, since this chemical activators change the ionic concentration resulting in a change on the electrochemical potential. These electric effects are known as activator or inhibitor postsynaptic potentials. If the whole potential reaching a specific neuron surpasses certain threshold, the neuron is activated; if this is null, the neuron remains in its previous state and, otherwise, the neuron is deactivated. The potential of a neuron is determined by the integrated effect of all inhibitor and excitator potentials that reach it.

### 1.2 The canonic neural scheme

The model is based on assuming a simplified form of reality. In the first place, the neural network is considered as isolated from any external stimulus. Secondly, neurons are activated or not according to the sign function and the net input that excites or inhibits each neuron. In particular the potential is determined by the sum of all the postsynaptic inputs resulting from the previous stage. Its value will determine the state of the neuron being considered. Finally, it is supposed that connections between neurons and synaptic binding properties remain stable, i.e. they do not change with time. These simplifications are justified in order to solve the problem mathematically.

The algorithmic implementation of the model has four stages: *learning, initiation, iteration up to convergence and output* according to the following standard scheme: The only two possible neuronal states are active  $s_i = +1$  and deactive  $s_i = -1$ . The learning stage consists in specifying the  $W$  synaptic matrix through Hebb's rule, once a set of persistent states  $L_p$  is selected. The family  $L_p$  belongs to the  $2^N$  space of binary vectors of  $N$  components with each entry adopting the values  $+1$  or  $-1$ , from Hebbian rule it follows that  $W$  is symmetric. The initiation stage consists of presenting a vector from configuration space.

- **Learning**

$L_p = \{\xi^1, \xi^2, \dots, \xi^p\}$  family of  $p$  vectors of  $N$  components, representing the  $p$  *a-priori* patterns to be learned by the system.

Hebbian synaptic connectivities without self-feedbacks:

$$\omega_{ji} = \frac{1}{N} \sum_{\mu=1}^p \xi_j^\mu \xi_i^\mu \quad j \neq i \quad (1)$$

$$\omega_{ii} = 0$$

- **Initiation**

$s_j(t=0)$  initial state of the  $j$  neuron

- **Iteration up to convergence**

Update of state vector  $\vec{s}(t)$

$$s_j(t+1) = \text{sign}(v_j(t)) \quad (2)$$

with

$$v_j = \sum_{i=1}^N w_{ji} s_i(t)$$

- **Output**

Record the vector obtained in the previous stage.

## 2. ALGORITHMIC CONVERGENCE PROPERTIES

All the results presented in this and the next section provide the theoretical background for our comparative study. Their validity is confirmed by our computations, as indication in Section 4.

*The algorithmic convergence properties<sup>7</sup> depend on the operation mode and the structure of synaptic matrix, these issues are investigated in the next forthcoming two subsections.*

## 2.1 Warranting convergence in serial operation mode

The asynchronous updating process is carried out following Eq. (2), and the output is the vector to which it converges after iteration.

We introduce a quadratic function to the model

$$Q = -\frac{1}{2} \sum_{i,j} w_{ij} s_i s_j \quad \forall i \quad \forall j \quad (3)$$

similar to that defined by Hopfield<sup>8</sup>. This quadratic function is called “energy” of these recurrent dynamics. Then, it is known that the asynchronous operation of the algorithm warrants convergence.

Let  $s'_j$  the new value of the unit  $j$ , and  $s_j$  its previous value. Then the following relations hold :

$$\begin{aligned} s'_j &= \text{sign} \left( \sum_i w_{ji} s_i \right) \\ \Delta Q &= \Delta Q_j = \sum_i w_{ji} s_i s'_j \\ \text{sign} \left( \sum_i w_{ji} s_i \right) &\neq \text{sign}(s_j) \\ \Delta Q &< 0 \end{aligned}$$

*The asynchronous algorithmic operation in the  $2^N$  state discrete binary space with constant Hebbian matrix (1), dynamic rule (2) evolves with an “energy” decrease (3). (Otherwise,  $s'_j = s_j$  and  $\Delta Q = 0$ , there is not change of the unit  $j$ ). From the relations indicated above, it follows also that the system undergoes the same convergence properties once self-feedback is allowed (i.e.  $w_{ii} \neq 0$ ).*

## 2.2 Warranting convergence in parallel operation mode

### 2.2.1 Mutually orthogonal patterns

*This recurrent model updates in synchronic operation mode with Hebb's rule (1) and self-feedbacks ( $w_{ii} \neq 0$ ) and evolves with an ever-decreasing “energy” function for mutually orthogonal patterns. No cycles can occur in this particular symmetric case in which the synaptic matrix is directly  $W = \frac{1}{N} \Sigma \Sigma^T$ . It statement is valid in the more general analysis at the next subsection. Otherwise, the self-feedbacks lack ( $w_{ii} = 0$ ) becomes the previous avowal not longer true and cycles of length 2 could appear so the algorithm may be trapped in a 2-cycle between two states with the same  $Q$  quadratic function value<sup>7,11</sup>.*

### 2.2.2 Linearly independent patterns

*The Hebb's rule (1) with self-coupling does not warrant an ever decreasing energy function or algorithmic convergence in parallel operation mode for general linearly independent patterns and updating by Eq. (2). In this situation, Eq. (1) must be changed for the projection rule in which the synaptic tensor is constructed as*

$$W = \Sigma \Sigma^+ \quad (4)$$

where  $\Sigma^+$  is the Moore-Penrose pseudoinverse.

Thus, during the evolution of the system performing parallel iterations, the energy is an ever decreasing function. Therefore, no cycles can occur. A similar result holds valid for the evolution of a network in which one neuron only reevaluates its state at each time step ( sequential operation ) as indicated below.

We consider the evolution of a network, designed after the projection rule, from a state  $\bar{\kappa} = \bar{\kappa}(t)$  to a state  $\bar{\kappa}(t + \tau)$  in one parallel iteration: first, the network computes the potential vector  $\bar{v}$  by the relation  $\bar{v} = W\bar{\kappa}(t)$ ; then, the update rule (2), sets the network into the state  $\bar{\kappa}(t + \tau)$  such that  $(\bar{\kappa}(t + \tau))^T \bar{v} = \sum_i |v_i|$ .

Consider a vector  $\bar{\kappa}^* \neq \bar{\kappa}(t + \tau)$ , the difference between them arising from at least one component  $j$  with  $(v_j \neq 0)$ , then  $(\bar{\kappa}^*)^T \bar{v} = \sum_i |v_i| \kappa_i^* \kappa(t + \tau)_i$ . At least the  $j$ th term of the sum is equal to  $-|v_j|$ , so  $\sum_i |v_i| \kappa_i^* \kappa(t + \tau)_i < \sum_i |v_i|$  it leads to Eq. (5) :

$$\|\bar{\kappa}(t + \tau) - \bar{v}\| < \|\bar{\kappa}^* - \bar{v}\| \quad (5)$$

Eq. 5 sets that if the state  $\bar{\kappa}(t)$  has all components of  $\bar{v} = W\bar{\kappa}(t)$  different from zero, the next state  $\bar{\kappa}(t + \tau)$  will be the vector of  $\{-1, +1\}^n$  which is closest to  $\bar{v}$  ( with respect to the Euclidean distance).

*The quadratic function associated to these recurrent networks is proportional to the square of the synaptic potential vector<sup>11</sup>. If the network evolves from a state to another different state, it is performed with a decreasing the associated "energy" function  $Q$ .*

After the learning stage is completed, one parallel iteration takes place, so, the  $\bar{\kappa}(t)$  state becomes the  $\bar{\kappa}(t + \tau)$  state, where the delay time is selected equal to one, i. e.,  $\tau = 1$ . It consists in the evaluation of potencial vector  $\bar{v} = W\bar{\kappa}$  and applies the updating rule (2). Let  $\bar{\kappa}^*$  a vector that verifies  $\bar{\kappa}^* \neq \bar{\kappa}(t + \tau)$  and it assumes that the  $\bar{v}$  entries are nonzero. In this particular case  $\bar{\kappa}(t + \tau)$  will be the  $2^N$  vector which is closest to  $\bar{v}$  with respect to the Euclidean distance, therefore, by Eq. 5

$$\|\vec{k}^* - \vec{v}\| > \|\vec{k}(t+1) - \vec{v}\| \quad \forall \vec{k}(t+1) \neq \vec{k}(t) \quad (6)$$

Since  $W\vec{k}(t+1)$  is the vector belonging to the subspace spanned by the patterns closest to  $\vec{k}(t+1)$ ,

$$\|\vec{k}(t+1) - W\vec{k}(t+1)\| < \|\vec{k}(t+1) - W\vec{k}(t)\| \text{ if } \vec{k}(t+1) \neq \vec{k}(t) \quad (7)$$

thus the previous two inequalities gives  $\|\vec{k}(t+1) - W\vec{k}(t+1)\| < \|\vec{k}(t) - W\vec{k}(t)\|$  and it is equivalent to

$$Q(\vec{k}(t+1)) < Q(\vec{k}(t)) \quad (8)$$

For practical purposes, however, parallel operation is more efficient as far as computation time are concerned. Otherwise when operating in asynchronic mode, will always lead to a stable state which corresponds to a local minimum in the quadratic function  $Q$ . This property suggests the use of the model as a device for performing a local search algorithm in order to find a local minimum value of the quadratic function.

### 3. STABILITY AND ATTRACTIVITY PROPERTIES

#### 3.1 Parallel dynamics with orthogonal patterns

Let  $\{\xi^1, \xi^2, \dots, \xi^p\}$  be a family of  $p$  orthogonal patterns  $p \leq N$  and  $\Sigma$  the  $p$ -range matrix formed by them. Matrix  $W$  is calculated by the projection rule which, in this case, it is reduced to Hebb's rule (1) with  $w_{ii} \neq 0$ . Let  $\vec{\eta}$  be a state different from the patterns. We investigate the evolution of the system when started in state  $\vec{\eta}$ . Thus  $W\vec{\eta}$  is a linear combination of the  $p$  orthogonal patterns, the coefficients of which are the inner products of  $\vec{\xi}^\mu$  and  $\vec{\eta}$ . Since the components of these vectors are -1's and +1's, we get:

$$(\vec{\xi}^\mu)^T \vec{\eta} = n - 2d_H(\vec{\xi}^\mu, \vec{\eta}) \quad (9)$$

where  $d_H(\vec{\xi}^\mu, \vec{\eta})$  is the Hamming distance between the states  $\vec{\xi}^\mu$  and  $\vec{\eta}$ . We denote  $d_H(\vec{\xi}^\mu, \vec{\eta})$  by  $H_\mu$ , then  $W\vec{\eta} = \left(\frac{1}{n}\right) \sum_{\mu=1}^p (n - 2H_\mu) \vec{\xi}^\mu$ . A sufficient condition for the attractivity of the pattern  $\vec{\xi}^\nu$ , while the system evolves from a state  $\vec{\eta}$  to pattern  $\vec{\xi}^\nu$  in one iteration is  $(W\vec{\eta})_i \xi_i^\nu > 0$ , for all  $i$ , it is satisfied if

$$\left| \sum_{\substack{\mu=1 \\ \mu \neq \nu}}^p (n - 2H_\mu) \xi_i^\mu \xi_i^\nu \right| < n - 2H_\nu, \quad \forall i \quad (10)$$

The triangular inequalities of Hamming distance and the orthogonality of the patterns  $\bar{\xi}^\mu$  and  $\bar{\xi}^\nu$  direct to Eq. 11

$$\left| \sum_{\substack{\mu=1 \\ \mu \neq \nu}}^p (n - 2H_\mu) \xi_i^\mu \xi_i^\nu \right| \leq 2H_\nu(p-1) \quad (11)$$

Consequently, if  $2H_\nu(p-1) < n - 2H_\nu$  is satisfied, relation (10) will be verified, thus the network will certainly evolve from state  $\bar{\eta}$  to state  $\bar{\xi}^\nu$ . The last relation implies that :

$$H_\nu < n/2p \quad (12).$$

*In summary, if a state  $\bar{\eta}$  lies within a distance of  $n/2p$  of a pattern  $\bar{\xi}^\nu$  : First,  $\bar{\xi}^\nu$  is the nearest pattern, and finally the network will evolve from state  $\bar{\eta}$  to state  $\bar{\xi}^\nu$  in one iteration<sup>7</sup>. It should be noticed that  $n/2p$  is a lower limit of the size of basin of attraction of a pattern; starting states lying at larger distance may lead the network to that pattern. It can be shown similarly that any state lying within a distance of  $n/2p$  of a state  $\bar{\xi}^\nu$  will converge to that state in one iteration<sup>11,12</sup>.*

### 3.2 Parallel dynamics and linearly independent patterns

If the patterns are not orthogonal, an order of magnitude of the average minimum attractivity is given by  $n/2r$ , where  $r$  is the rank of the pattern family<sup>12</sup>.

### 3.3 Serial dynamics

The stability for multi patterns  $1 < \mu \leq p$  in serial operation mode, Hebbian transition matrix (1) with  $w_{ii} \neq 0$ .

$$\text{sign} \left( \sum_j w_{ij} \xi_j^\nu \right) = \xi_i^\nu \quad (\forall i) \quad (13)$$

$$\begin{aligned} &= \xi_i^\nu + \frac{1}{N} \sum_{j=1}^N \sum_{\substack{\mu=1 \\ \mu \neq \nu}}^p \left( \xi_i^\mu \xi_j^\mu \xi_j^\nu \right) = \\ &0 < \left( -\xi_i^\nu \right) \left( \frac{1}{N} \sum_{j=1}^N \sum_{\substack{\mu=1 \\ \mu \neq \nu}}^p \left( \xi_i^\mu \xi_j^\mu \xi_j^\nu \right) \right) \leq 1 \end{aligned} \quad (14)$$



The stability condition (13) for several persistent states of the Hebbian matrix with  $w_{ii} \neq 0$  imposes the validity of (14), that shows that if  $\xi$  is stable,  $-\xi$  is also stable.

*The asynchronous updating with Hebbian synaptic matrix (1) with  $w_{ii} \neq 0$  and  $p$  - vectors  $\{\xi^1, \xi^2, \dots, \xi^p\}$  stability of the  $2^N$  state discrete binary configuration space and dynamic rule (2) has stability in its respective negatives.*

#### 4. RELEVANT DYNAMICAL FEATURES AS REVEALED BY SIMULATION

Prior to a detailed analysis, it is worth mentioning that the stability of the learned patterns in the systems we focus upon is not sufficient to warrant the emergence of associative memory. Clearly, the most extreme case of uselessness as memory associative function is furnished by the orthogonal patterns of complete range, i. e., when  $w = n^{-1} \text{hadamard}(N) * \text{hadamard}(N)^T$ .

An important requirement for a system to feature an associative memory is that the stable patterns must have a sizable basin of attraction. This ensures the retrieval of the full memory by an input which contains only partial information on it. Let be  $\alpha$  the ratio between the numbers of linearly - independent patterns over the numbers of units, so,  $\alpha = p / N$ . On the other hand, whenever we have mutually orthogonal patterns with  $\alpha = 1/2$  we get poor attractive properties. This represents another limit situation which is almost useless in yielding an associative memory function. For instance, let  $\Sigma = \text{hadamard}(1:N, 1:N/2)$  and  $w = n^{-1} \Sigma (\Sigma)^T$ , almost all states of configuration space are stable but they are weak attractors.

In general terms, if we have linearly-independent patterns with  $\alpha$  approaching  $1/2$ , the systems are losing its attractive character for the patterns, so that, they cease to provide associative memory if  $\alpha > 1/2$ , although they have pattern stability for  $\alpha < 1/2$ . Thus, the attractivity of the patterns falls sharply if  $p$  becomes of the order of  $N/2$ , and concurrently, the number of stable nonprototype states increases<sup>9,11,12</sup>.

Let's consider the set of ten target vectors represented below in Figure 1. They have been sketched by our graphic tool. These patterns are the ten digits in the decimal digits and we assume that they are the *a-priori* memories of the system.

First at all, we have checked the validity of our algorithms using the theoretical underlays previously detailed. Thus, we pose the following question: What happens with the stability of this particular set -Figure 1- as we introduce the four different synaptic tensors ?

This family is a set of linearly-independent vectors with range equals to ten, but are not a mutually orthogonal memory set. Then only the projection rule warrants the stability of their components. This fact has been verified in one of our numerical experiences. The diagonal coefficients of the projection matrix are smaller than or equal



to one, then pattern stability was also shown to hold when self-coupling was removed. Other simulations confirm that the synaptic tensor constructed by the projection rule with self-coupling decrease its associative memory function with respect to the same system. We obviously exclude the main diagonal. Therefore, the pattern attractivity is altered depending on whether self-coupling is allowed or not allowed under the projection synaptic matrix<sup>10</sup>, while synchronic updating is performed. All asynchronous manipulations make it apparent the absence of spurious limit cycles for systems with self-coupling and the projection synaptic tensor. If the updating is carried out in the asynchronous mode, both random selection or specific order selection neuron updating evolves to the same final state as in the synchronic mode if the initial state and the final state are not involved in a limit length 2 cycle. Possibly the emergence of limit cycles of length 2 for synaptic tensor is obtained by projection rule without self-coupling, while the systems are operated under synchronic regimen. In these cases, the differentiable asynchronous dynamics spread to different local minima.

The Hebbian synaptic tensor does not warrant the pattern stability for this family and the digits 1, 2, 7 and 8 are unstable. The self-coupling cancelling for Hebbian matrix undergoes the stability loss for seven of the ten prototypes. Only the numbers 0,5, and 9 preserve their stability. This kind of behaviour is opposite to that of the projection rule behavior. Therefore, for linearly independent patterns with Hebbian matrix self-coupling whether allowed or forbidden also alter the stability of *a priori* patterns.

In general, synchronic dynamics may lead to attractors which are cycles rather than fixed states<sup>5,9</sup>. For symmetric synaptic tensors such as that given in the present context, the cycles are at most of length 2. For our particular example the lack of self-coupling generated the presence of spurious cycles of length 2 for projection rule synapses<sup>7</sup>, as we have readily probed.

Our simulations make evident that the basins of attraction are sensitive to the details of the dynamics. For instance, asynchronous and synchronic dynamics may define different basins of attraction. The basins may be affected also by the order of updating in the asynchronous mode. This last statement is validated since our modular tools can handle the two differential ways for asynchronous update. One has a random selection for the neuron update and the other permits us to select the choosing order of neuronal selection updating.

All asynchronous manipulations in general could be use as an algorithm that searches for spurious stable states in each particular synaptic tensors and then the algorithmic outputs could be match for synchronic updating performance.

To conclude, the stable state patterns under determined synaptic tensors are stable for both synchronic or asynchronous updating, although their attractive basins are altered. In addition, the attractive basins are affected if in asynchronous operation mode of the neuron order selection of updating is changed.

0	0	0	0	0
0	1	1	1	1
0	1			1
0	1	0		1
0	1	1	1	1

0	0	1	0	0	0
0	0	1	0	0	0
0	0	1	0	0	0
0	0	1	0	0	0
0	0	1	0	0	0
0	0	1	0	0	0

1	1	1	1	1	1
0	0	0	0	1	0
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	0	0	0
1	1	1	1	1	1

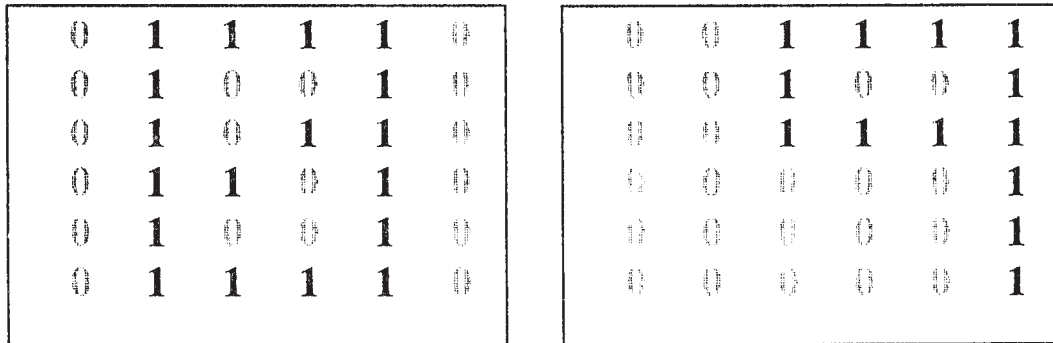
0	1	1	1	1	0
0	0	0	0	1	0
0	0	1	1	1	0
0	0	0	0	1	0
0	0	0	0	1	0
0	1	1	1	1	0

1	0	0	1	0	0
1	0	0	1	0	0
1	1	1	1	0	0
0	0	0	1	0	0
0	0	0	1	0	0
0	0	0	1	0	0

1	1	1	1	0	0
1	0	0	0	0	0
1	1	1	1	0	0
0	0	0	1	0	0
0	0	0	1	0	0
1	1	1	1	0	0

1	0	0	0	0	0
1	0	0	0	0	0
1	0	0	0	0	0
1	1	1	1	0	0
1	0	0	1	0	0
1	1	1	1	0	0

0	1	1	1	1	0
0	0	0	0	1	0
0	0	0	0	1	0
0	0	0	0	1	0
0	0	0	0	1	0
0	0	0	0	1	0



**Figure 1**

## 5. CONCLUDING REMARKS

In this work we have investigated two different deterministic neurodynamic contexts within a unifying approach. Thus, the convergence and stability properties of both the magnetic spin and neurophysiological systems have been established, focusing on the interrelationship between such properties and the synaptic and updating features of the systems. A set of modular representational tools enabling a qualitative assessment of the dynamical characteristics has been implemented to facilitate our comparative analysis.

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