Semantic Models of Modal Logic in n-valued Logics

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Abstract

For the algebraic study of n-valued logics introduced by Lukasiewicz, Moisil introduced in 1941 the so called n-valued Lukasiewicz algebras. These algebras have been extensively studied by various authors, being emphasized by R. Cignoli in his doctoral thesis, Universidad Nacional del Sur, 1970 (see also R.Cignoli, *Moisil Algebras*, Notas de Lógica Matemática, 26, UNS, Bahía Blanca, 1970), under the name of Moisil algebras. In these algebras are introduced operators that are said modal, though some of its properties would not be commonly accepted as properties of a modal operator.

In this work it is introduced a propositional calculus M_n, that is demonstrated complete with respect to the algebras of Moisil. This calculus contains operating s_i whose meaning would be that of to graduate the possibility of a proposition. They are introduced semantic models for modal logic in the sense of Kripke for these logics M_n, what permits to corroborate that, in fact, the operators s_i can be considered modal, and its little usual properties are due to the fact that they are valid in very particular models.

1 Introduction

Moisil algebras of orden n (n an integer≥ 2) were introduced by G. Moisil in 1941 (see, for example,[5]) under the name of n-valued Lukasiewicz algebras. A detailed analysis of Moisil algebras as well as further references may be found in [1] or [2] .

In this paper a propositional calculos M_n is defined as an extension of the intuitionistic implicative calculus by the addition of a De Morgan negation and of certain modal operators s_i wich graduate the posibility of a proposition. This propositional calculus will correspond to the Moisil algebras of orden n in a sense we shall make precise in 3.

In 4, we shall introduce semantical models (in the sense of Kripke [3] and [4]) for the calculus M_n and we shall see that this calculus may have a modal interpretation.

Analogous results for calculi P_n, corresponding to Post algebras of order n, will be indicated in 5.

2 Moisil algebras

Let us now recall the equational definition of Moisil algebras of orden n given in [1] (or [2]).

Definition 2.1 . *A Moisil algebra of order n (n an integer ≥ 2)is a system $(A, 1, \wedge, \vee, \sim, s_1, s_2, \ldots, s_{n-1})$ such that $(A, 1, \wedge, \vee)$ is a distributive lattice with last element 1, and $\sim, s_1, \ldots, s_{n-1}$ are unary operations defined on A satisfying the following conditions (for all $x, y \in A$):*

M1) $\sim\sim x = x$

M2) $\sim (x \vee y) = \sim x \wedge \sim y$

L1i) $s_i(x \vee y) = s_i x \vee s_i y$ for $i = 1, \ldots, n-1$

L2i) $s_i x \vee \sim s_i x = 1$ for $i = 1, \ldots, n-1$

L3ij) $s_i s_j x = s_j x$ for $i = 1, \ldots, n-1$

L4i) $s_i \sim x = \sim s_{n-i} x$ for $i = 1, \ldots, n-1$

L5i) $s_i x \leq s_{i+1} x$ for $i = 1, \ldots, n-2$

L6) $x \leq s_{n-1} x$

L7i) $x \wedge \sim s_i x \wedge s_{i+1} y \leq y$ for $i = 1, \ldots, n-2$

Examples 2.1 .*(see [5]). Let L_n be the chain $0 < 1/n-1 < 2/n-1 < \ldots < n-2/n-1 < 1$, where we define $x \vee y = \max(x,y)$, $x \wedge y = \min(x,y)$, $\sim x = 1-x$,*

$$s_i(n-j/n-1) = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i \geq j \end{cases} \tag{1}$$

Then L_n is a Moisil algebra of order n.

Morover, the following representation theorem holds:

Theorem 2.1 . *Any Moisil algebra of order n is isomorphic with a subdirect product of algebras L_n. ([5] and [2]).*

Every Moisil algebra is a Heyting algebra, and the relative pseudo-complement $\Rightarrow$ can be defined in terms of the operations $\wedge, \vee, \sim, s_i$. In L3) - L5) the operators s_i can be defined in terms of the operations $\Rightarrow, \sim, \wedge$ and $\vee$, but this is not possible for $n \geq 6$ (see [1]).

It is a trivial observation that for L4), part of the operators s_i could be defined in terms of the others.

3 The propositional calculus

We introduce now the calculus M_n. The formulas of this calculus are built up in the usual way from a set $\{p_i\}_{i \in I}$ of propositional variables by means of the connectives $\rightarrow$(implication), $\vee$ (disjunction), $\wedge$ (conjunction), $\sim$ (negation) and s_i for $i = 1, \ldots, n-1$ (modal operators). The connective $\leftrightarrow$(equivalence) is defined as usual by

$$a \leftrightarrow b =_{df} (a \rightarrow b) \wedge (b \rightarrow a) \tag{2}$$

We shall eliminate parenthesis following the usual conventions that the connectives $\sim, s_1, \ldots, s_{n-1}$ binds more strongly than either $\wedge$ or $\vee$, and each of this more strongly than either $\rightarrow$ or $\leftrightarrow$.

The following formulas are the axiom schemas of M_n (where P, Q and R denote formulas):

A1) $P \rightarrow (Q \rightarrow P)$

A2) $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$

A3) $P \to P \vee Q$

A4) $Q \to P \vee Q$

A5) $(P \to R) \to ((Q \to R) \to (P \vee Q \to R))$

A6) $P \wedge Q \to P$

A7) $P \wedge Q \to Q$

A8) $(P \to Q) \to ((P \to R) \to (P \to Q \wedge R))$

A9) $P \leftrightarrow \sim\sim P$

A10i) $s_i(P \vee Q) \to s_i P \vee s_i Q$ for $i = 1, \ldots, n-1$

A11i) $s_i P \vee \sim s_i P$ for $i = 1, \ldots, n-1$

A12ij) $s_i s_j P \leftrightarrow s_j P$ for $i, j = 1, \ldots, n-1$

A13i) $s_i \sim P \leftrightarrow \sim s_{n-1} P$ for $i = 1, \ldots, n-1$

A14i) $s_i P \to s_{i+1} P$ for $i = 1, \ldots, n-2$

A15) $P \to s_{n+1} P$

A16i) $P \wedge \sim s_i P \wedge s_{i+1} Q \to Q$ for $i = 1, \ldots, n-2$

The rules of M_n are the following:

R1) $\frac{P, P \to Q}{Q}$ R2) $\frac{P \to Q}{\sim Q \to \sim P}$ R3i) $\frac{P \to Q}{s_i P \to s_i Q}$ for $i = 1, \ldots, n-1$

According to the observations made after theorem 2.1, the connectives of this calculus are not independent, but we take it in this way for reasons of simplicity.

We shall say that a formula P is a thesis of M_n and we shall denote it by $\vdash_n P$, if it belongs to the least set of formulas wich contains the axioms and is closed under R1), R2) and R3i).

For example, the following formulas are theses of M_n:

18) $\vdash_n \sim (P \vee Q) \leftrightarrow \sim P \vee \sim Q$

19i) $\vdash_n s_i P \vee s_i Q \to s_i(P \vee Q)$ for $i = 1, \ldots, n-1$

By the usual methods it can be constructed the Lindembaum algebra L^n of the calculus M_n, and see that L^n is the free Moisil algebra of order n with c generators, where c is the cardinal number of the set of propositional variables of M_n. L^n is also a characteristic matrix of M_n, that is, $\vdash_n P$ if and only if $|P| = 1$, where $|P|$ is the equivalence class wich contain P. By theorem 2.1, the calculus M_n has the chain L_n as a characteristic matrix, and therefore, this calculus is decidable.

4 Semantical models

We shall now introduce the semantical models for M_n.

Definition 4.1 . *A Moisil model structure of order n is a system $(K, \leq, g_1, g_2, \ldots, g_{n-1})$, where $(K, \leq)$ is an ordered set, K is not empty, and g_i for $i = 1, \ldots, n-1$ are mappings from K into K such that (for all $k, k' \in K$):*

K1) if $k \leq k'$ then $g_0(k') \leq g_0(k)$

K2) $g_0^2 = g_0 g_0 = I$ (I is the identity map from K onto K)

K3i) if $k \leq k'$ then $g_i(k) \leq g_i(k')$ for $i = 1, \ldots, n-1$

K4ij) $g_i g_j = g_i$ for $i = 1, \ldots, n-1$ and $j = 0, 1, \ldots, n-1$

K5i) $g_0 g_i = g_{n-1}$ for $i = 1, \ldots, n-1$

K6i) $g_i(k) \leq g_{i+1}(k)$ for $i = 1, \ldots, n-1$

K7) $k = g_1(k) \cup g_2(k) \cup \ldots \cup g_{n-1}(k)$

Definition 4.2 . *A Moisil model of order n is a system $(K, \leq, g_0, \ldots, g_{n-1}, f)$, where $(K, \leq, g_0, \ldots, g_{n-1})$ is a Moisil model structure of order n and f an application from the cartesian product of K and the set of propositional variables of M_n into a set of two elements: 1(true) and 0(false), such that (we shall denote $f_k(p) = f(k,p)$):*

$$\textit{If } f_k(p) = 1 \textit{ and } k \leq k', \textit{ then } f_{k'}(p) = 1$$

This mapping f may be extended, in a unique way, to an application v from de cartesian product of K and the formulas of M_n into $\{0, 1\}$, wich we shall call valuation, by induction on the number of connectives of a formula, in the following way:

V1) $v_k(p) = f_k(p)$, where p is a propositional variable

V2) $v_k(P \vee Q) = 1$ if and only if either $v_k(P) = 1$ or $v_k(Q) = 1$; otherwise $v_k(P \vee Q) = 0$

V3) $v_k(P \wedge Q) = 1$ if and only if $v_k(P) = v_k(Q) = 1$; otherwise $v_k(P \wedge Q) = 0$

V4) $v_k(P \rightarrow Q) = 1$ if and only if for every k' such that $k \leq k'$, if $v_{k'}(P) = 1$ then $v_{k'}(P) = 1$; otherwise $v_k(P \rightarrow Q) = 0$

V5) $v_k(\sim P) = 1$ if and only if $v_{g_0(k)}(P) = 0$; otherwise $v_k(\sim P) = 0$

V6i) $v_k(s_i P) = 1$ if and only if $v_{g_i(k)} = 1$ for $i = 1, \ldots, n-1$; otherwise $v_k(s_i P) = 0$

Definition 4.3 . *We shall say that a formula P is valid and we shall write $\models_n P$, if for every Moisil model structure of order n $(K, \leq, g_0, \ldots, g_{n-1})$, for every valuation v and for every $k \in K$, $v_k(P) = 1$.*

In the following lemmas, $(K, \leq, g_0, \ldots, g_{n-1}$ is a Moisil model structure of order n and v a valuation on $(K, \leq, g_1, \ldots, g_{n-1})$.

Lemma 4.1 . *For any formula P, if $v_k(P) = 1$ and $k \leq k'$ then $v_{k'}(P) = 1$.*

Proof. By induction on the number of connectives of P. In the induction steps corresponding to $\sim P$ and $s_i P$, it can be used K1) and K3i), respectively. •

Lemma 4.2 . *$v_k(P) = 1$ if and only if $v_{g_{r(k)}}(P) = 1$ for $r = i, \ldots, n-1$, if and only if there exists r such that $1 \leq r \leq i$ and $v_{g_{r(k)}}(P) = 1$.*

Proof. If follows from V6i), lemma 4.1 and K6i). •

From the preceding lemma and according to Kripke's interpretation of the modal operators, it follows that s_1 con be interpreted as a operator of necessity, s_{n-1} as a operator of possibility and s_i for $i = 2, \ldots, n-2$ as certain intermediate grade between necessity and possibility.

The following lemmas will allow us to prove that any Moisil model structure of order n is the cardinal sum (as ordered sets) of chains with at most $n-1$ elements.

Lemma 4.3 . *For all $k \in K$, $g_1(k) \leq k \leq g_{n-1}(k)$.*

Proof. By K7), given k there exists $k' \in K$ and $i(1 \leq i \leq n-1)$ such that $g_i(k') = k$. By K4ij), $g_1(k) = g_1(k')$ and $g_{n-1}(k) = g_{n-1}(k')$.

By K6i) $g_1(k') \leq g_i(k') \leq g_{n-1}(k')$. Hence, $g_1(k) \leq k \leq g_{n-1}(k)$. •

Lemma 4.4 . *For all $k \in K$, either $k \leq g_i(k)$ or $g_{i+1}(k) \leq k$, for $i = 1, \ldots, n-2$.*

Proof. By k7), given k, there exists $k' \in K$ and $j(1 \leq j \leq n-1)$ such that $g_j(k') = k$. By K4ij), $g_i(k) = g_i(k')$ and $g_{i+1}(k) = g_{i+1}(k')$.

By K6i), either $g_j(k') \leq g_i(k')$ or $g_{i+1}(k') \leq g_j(k')$.

Hence, either $k \leq g_i(k)$ or $g_{i+1} \leq k$. •

Lemma 4.5 . *For every $k \in K$, there exists an $i(1 \leq i \leq n-1)$ such that $g_i(k) = k$.*

Proof. By lemma 4.3 , $g_1(k) \leq k$. Let $j = \max\{i | g_i(k) \leq k\}$. If $j = n-1$ then $g_{n-1}(k) \leq k$, and by lemma 4.3 , $g_{n-1}(k) = k$. If $j \leq n-2$ then $g_j(k) \leq k$ and not $g_{j+1}(k) \leq k$, hence by lemma 4.3 , $k \leq g_j(k)$. Thus $g_j(k) = k$. •

Lemma 4.6 . *If $k \leq k'$, then $g_i(k) \leq g_i(k')$ for $i = 1, \ldots, n-1$.*

Proof. By K3i), if $k \leq k'$ then $g_i(k) \leq g_i(k')$. By K1), $g_0(g_i(k')) \leq g_0(g_i(k))$. By K5i), $g_{n-i}(k') \leq g_{n-i}(k)$. By K4ij), $g_i(k') \leq g_i(k)$. Thus, $g_i(k) \leq g_i(k')$. •

Lemma 4.7 . *Every $k \in K$ belongs to one and only one chain with at most $n-1$ elements.*

Proof. By lemma 4.5 , given k, there exists i such that $g_i(k) = k$, hence k belongs to the chain $g_1(k), \ldots, g_{n-1}(k)$, wich have at most $n-1$ elements.

Let us suppose that k belongs to another chain, then there exists k' that either $k \leq k'$ or $k' \leq k$. By lemma 4.5 , there exists j such that $g_j(k') = k'$. By lemma 4.6 , $g_j(k') = g_j(k)$, and then belongs to the chain $g_1(k), \ldots, g_{n-1}(k)$, wich contradicts the assumption. •

Theorem 4.1 . *Every Moisil model structure of order n is the cardinal sum of chains with at most $n-1$ elements.*

Proof. It follows from lemma 4.7 . ●

Lemma 4.8 . *For every $k \in K$, either $g_0(k) \leq k$ or $k \leq g_0(k)$.*

Proof. By lemma 4.5 ,given k, there exists i such that $g_i(k) = k$. By K5i), $g_0(k) = g_{n-i}(k)$. By K6i) either $g_{n-i}(k) \leq g_i(k)$ or $g_i(k) \leq g_{n-i}(k)$. Hence, either $g_0(k) \leq k$ or $k \leq g_0(k)$. ●

Lemma 4.9 . *For every formula P, if $\vdash_n P$ then $\models_n P$.*

Proof. It is easy to see that the axioms A1)–A3) are valid, using lemma 4.1 . The validity of A4) – A16i) offers no difficulty. We only remark that in checking the validity of A15) and A16i), the lemmas 4.3 and 4.4 may be used, respectively. The rules R1), R2) and R3i) preserve validity. Therefore, all theses are valid. ●

We now construct a canonical (or Lindenbaum) model for the calculus M_n wich will verify the following property: $\vdash_n P$ if and only if P is valid in this model.

Let K_L be the set of all prime filters of the Lindenbaum algebra L^n, let $\subseteq$ be the inclusion relation and let us define the following mappings:

$$g_0(k) = -\{\sim x | x \in k\} \text{ (} - \text{ denotes the set theoretical complement)}$$

$$g_i(k) = \{x | s_i x \in K\} \text{ for } i = 1, \ldots, n-1$$

It can be proved that the $g_i(i = 0, \ldots, n-1$ are applications from k into k, wich satisfies the conditions K1)–K7) (see [1]). Hence $(K_L, \subseteq, g_0, ..., g_{n-1})$ is a Moisil model structure of order n. Now, let us consider the model $(K_L, \subseteq, g_0, \ldots, g_{n-1}, f)$, where

$$f_k(p) = 1 \text{ if and only if } |p| \in K$$

Lemma 4.10 . *Let $(K_L, \subseteq, g_0, \ldots, g_{n-1})$ be the canonical model and v the corresponding valuation. Then, for every formula P*

$$v_k(P) = 1 \textit{ if and only if } |P| \in k$$

Proof. By induction on the number of connectives of P. The inductive steps corresponding to $\wedge, \vee, \sim$ and $s_i(i = 1, \ldots, n-1)$ offers no difficulty. Let us see only the inductive step corresponding to $\rightarrow$. It is sufficient to prove that $|P| \rightarrow |Q| \in k$ if and only if for every k' such that $k \subseteq k'$, if $|P| \in k'$ then $|Q| \in k'$. Let k' be such that $k \subseteq k'$. If $|P| \in k'$, since $|P| \rightarrow |Q| \in k'$, we have $|Q| \in k'$. Conversely, let us suppose that for every k' such that $k \subseteq k'$, if $|P| \in k'$ then $|Q| \in k'$, and $|P| \rightarrow |Q| \notin k$. Let r be the filter generated by $K \cup \{|P|\}$, then $|Q| \notin r$. Indeed, if $|Q| \in r$, then there exists $|R| \in k$ such that $|R| \wedge |P| \leq |Q|$ (see [6] ,p. 46), hence $|R| \leq |P| \rightarrow |Q|$ and finally, $|P| \rightarrow |Q| \in k$, wich contradicts the assumption. Let k' be a maximal (prime) filter such that $r \subseteq k'$ and $|Q| \notin k'$, then $k \subseteq k'$ and $|P| \in k'$, hence $|Q| \in k'$, wich is impossible. •

Lemma 4.11 . *For every formula $|P|$, if $|P|$ is valid in the canonical model, then $\vdash_n P$.*

Proof. By hypothesis and lemma 4.10,$|P| \in k$ for all $k \in K_L$. Since the intersection of all prime filters is 1, we have $|P| = 1$. Hence $\vdash_n P$. •

The semantical completeness of the calculus M_n follows from the following theorem:

Theorem 4.2 . *For every formula P, $\vdash_n P$ if and only if $\models_n P$.*

Proof. It follows from the lemmas 4.9 and 4.11 . •

The preceding theorem may be generalized without difficulty to a strong completeness theorem: If Γ is a set of formulas, $\Gamma \vdash_n P$ if and only if $\Gamma \models_n P$.

From the preceding theorem and lemma 4.8 it follows that is a thesis of the calculus M_n the so called Kleene's law:

$$\vdash_n P \wedge \sim P \rightarrow Q \vee \sim Q.$$

5 Post algebras

In [1] is given the following relation between the Moisil algebras of order n and the Post algebra of order n.

A is a Post algebra of order n if and only if A is a Moisil algebra of order n and A contains $n-2$ elements $e_1, \ldots, e_{n-2}$ such that:

P1) $s_i(e_j) = 0$ if $i + j < n$

P2) $s_i(e_j) = 1$ if $i + j \geq n$

A Post calculus of order n (or calculus P_n) may be defined, therefore, adding to the alphabet of the calculus M_n $n-2$ propositional constants $e_1, \ldots, e_{n-2}$ and the following axioms:

A17ij) $s_i(e_j)$ if $i + j \geq n$

A18ij) $\sim s_i(e_j)$ if $i + j < n$

A Post model structure of order n will be a Moisil model structure of order n such that

K8) $g_i(k) \cap g_{i+1}(k) = \emptyset$ for $i = 1, \ldots, n-2$

We add to the valuation the following condition

V7) $v_k(e_j) = 1$ if and only if $k \in g_{n-j}(k) \cup g_{n-j+1}(k) \cup \ldots \cup g_{n-1}(k)$

From theorem 4.1 and condition K8) it follows that every Post model structure of order n is the cardinal sum of chains with exactly $n-1$ elements.

From V7) it follows that the axioms A17ij) and A18ij) are valid in every Post model of order n.

The canonical model can be constructed in the same way and in a analogous way it can be proved the completeness of the calculus P_n.

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