W_{n+1}–algebras with an additional operation

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Abstract

In this paper, we investigate the $(n+1)$-bounded Wajsberg algebras (or W_{n+1}–algebras) with an additional operation which is an automorphism of period k. We characterize the congruences and the subdirectly irreducible algebras. Finally, we determine the structure of the free algebra over a finite set of generators.

1 Preliminaries

In this section, we define and give properties of Wajsberg algebras. Properties of Wajsberg algebras can be found in [3, 4, 5, 6].

Definition 1.1 *An algebra* $\mathcal{A} = (A, \rightarrow, \sim, 1)$ *of type* $(2,1,0)$ *is a Wajsberg algebra (or* W*–algebra) if the following identities are verified:*

(W1) $1 \rightarrow x = x$,

(W2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,

(W3) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

(W4) $(\sim y \rightarrow \sim x) \rightarrow (x \rightarrow y) = 1$.

We shall denote by $\mathbf{W}$ the variety of W–algebras.

It is well known that in any W–algebra the following operations are defined:

(i) $0 = \sim 1$,

(ii) $a \vee b = (a \rightarrow b) \rightarrow b$,

(iii) $a \wedge b = \sim (\sim a \vee \sim b)$,

(iv) $a + b = \sim b \rightarrow a$,

(v) $0 \cdot a = 0$, $(n+1) \cdot a = n \cdot a + a$ for all positive integer n.

Lemma 1.1 ([4, 5]) *For any $A \in \mathbf{W}$ it holds:*

(W5) $x \to x = 1$,

(W6) $(A, \vee, \wedge, \sim, 0, 1)$ *is a Kleene algebra where* $x \leq y$ *if and only if* $x \to y = 1$,

(W7) $x \to (y \to x) = 1$,

(W8) $x \to 0 = \sim x$,

(W9) $x \leq y$ *implies* $z \to x \leq z \to y$,

(W10) $x \leq y \to z$ *implies* $y \leq x \to z$,

(W11) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

Let $[0, 1]$ be the unitarian interval of the totally ordered additive group of real numbers, then $R[1] = ([0, 1], \to, \sim, 1)$ is a W-algebra where the operations $\to$ and $\sim$ are defined by the formulas

$$x \to y = min\ \{1, 1 - x + y\} \text{ and } \sim x = 1 - x.$$

It is known (see [4]) that $\mathbf{W}$ is generated by $R[1]$. On the other hand, for every positive integer n, $C_{n+1} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ is a subalgebra of $R[1]$ and C_{n+1} is a subalgebra of C_{m+1} if and only if n divides m (or for short n/m).

If $A \in \mathbf{W}$, $B(A)$ denotes the set of Boolean elements of A and if A is finite, $At(B(A))$ denotes the set of atoms of $B(A)$.

Lemma 1.2 ([4, 5]) *Let $A \in \mathbf{W}$. For all $a \in A$, the following conditions are equivalent:*

(i) $a \in B(A)$,

(ii) $a \wedge \sim a = 0$,

(iii) $2 \cdot a = a$,

(iv) $a \to b = \sim a \vee b$, *for all* $b \in A$.

Definition 1.2 *Let $A \in \mathbf{W}$. A subset $D \subseteq A$ is a deductive system (d.s.) if it verifies:*

(D1) $1 \in D$,

(D2) $a, a \to b \in D$ *imply* $b \in D$.

In every finite W-algebra, the d.s. are principal filters generated by the elements of $B(A)$ and the maximal d.s. are principal filters generated by the elements of $At(B(A))$.

Let $A \in \mathbf{W}$ and $a, b \in A$, $[a, b]$ denotes the set $\{x \in A : a \leq x \leq b\}$.

Theorem 1.1 ([5]) *Let $A \in \mathbf{W}$ be finite. Then $A \simeq \prod_{a \in At(B(A))} [0, a]$, where $[0, a] \simeq C_{r+1}$ if $|[0, a]| = r + 1$. ($D \simeq E$ denotes isomorphic algebras)*

Definition 1.3 *A W–algebra A is a W_{n+1}–algebra, if the following identity holds:*

(W12) $\sim a \vee n \cdot a = 1$.

We shall denote by $\mathbf{W_{n+1}}$ the variety of W_{n+1}–algebras.

Theorem 1.2 ([5]) *Let $A \in \mathbf{W_{n+1}}$. The following conditions are equivalent:*

(i) *A is simple,*

(ii) *$A \simeq C_{r+1}$, for some r, $1 \leq r \leq n$.*

Let $A \in \mathbf{W}$ and $D \subseteq A$. D is a Stone filter if it is a lattice filter and for all $d \in D$ there is $a \in D \cap B(A)$ such that $a \leq d$. For any $A \in \mathbf{W_{n+1}}$, $D \subseteq A$ is a d.s. if and only if it is a Stone filter. Besides, the sets of completly irreducible, irreducible, prime and maximal d.s. of A coincide, and we will denote them as $\mathcal{E}(A)$.

Theorem 1.3 ([5]) *If $A \in \mathbf{W}$ has more than one element, then A is a subdirect product of $\{A/M\}_{M \in \mathcal{E}(A)}$, where $A/M \simeq C_{i+1}$ for any i, $1 \leq i \leq n$. Furthermore, if A is finite, then A is a direct product of $\{C_{i+1}^{m_{i+1}}\}_{1 \leq i \leq n}$, where $m_{i+1} = |M_{i+1}|$ and $M_{i+1} = \{D \in \mathcal{E}(A) : |A/D| = i + 1\}$.*

If A is finite, Theorem 1.3 and Theorem 1.1 coincide.

2 k–cyclic W_{n+1}–algebras

In this section, we introduce the definition of k–cyclic W_{n+1}–algebras, and we give properties of these algebras. We determine the simple k–cyclic W_{n+1}–algebras and we prove that these algebras coincide with the subdirectly irreducible ones.

Definition 2.1 *Let n and k be positive integers. An algebra $(A, \rightarrow, \sim, \tau, 1)$ of type $(2, 1, 1, 0)$ is a k–cyclic W_{n+1}–algebra (or $W_{n+1,k}$–algebra) if $A = (A, \rightarrow, \sim, 1)$ is a W_{n+1}–algebra and $\tau : A \rightarrow A$ is an automorphism such that $\tau^k(x) = x$ for all $x \in A$.*

The variety of k–cyclic W_{n+1}–algebras will be denoted by $\mathbf{W_{n+1,k}}$. As usual the elements of $\mathbf{W_{n+1,k}}$ will be denoted simply by A o by (A, τ) in case we want to specify the automorphism. In what follows, $I(A)$ will be the set of the elements of A invariant by τ.

Example 2.1 Let n and k be positive integers. For each integer t, such that $1 \leq t \leq n$, let $\mathcal{C}_{t+1,k} = (C_{t+1,k}, \tau)$, where $C_{t+1,k}$ is the product of k copies of C_{t+1}, and τ is defined by

$$\tau(x)(i) = \begin{cases} x(i-1), & \text{if } i \neq 1 \\ x(k), & \text{if } i = 1 \end{cases}$$

It is easy to see that $\mathcal{C}_{t+1,k} \in \mathbf{W}_{n+1,k}$. If $j \geq k$, then

$$\tau^j(x)(i) = \tau^r(x)(i) = \begin{cases} x(i-r), & \text{if } i > r \\ x(i-r+k), & \text{if } i \leq r \end{cases},$$

where $j = qk + r$, $0 \leq r < k$.

Definition 2.2 *Let $(A, \tau) \in \mathbf{W}_{n+1,k}$. A kernel of A is a subset $N \subseteq A$ such that:*

(i) *N is a d.s.,*

(ii) *N verifies* (D3): *if $x \in N$ then $\tau(x) \in N$.*

We shall denote by $\mathcal{N}(A)$ and $Con(A)$ the sets of kernels and congruences of A respectively.

Lemma 2.1 *If $(A, \tau) \in \mathbf{W}_{n+1,k}$, then $Con(A) = \{R(N) \subseteq A \times A : N \in \mathcal{N}(A)\}$, where $R(N) = \{(x, y) \in A \times A : x \to y, y \to x \in N\}$.*

If A is an algebra with more than one element, the family of proper kernels of A is a upper inductive, therefore every proper kernel of A is contained in a maximal one.

Lemma 2.2 *Let $(A, \tau) \in \mathbf{W}_{n+1,k}$. For any $N \in \mathcal{N}(A)$ it holds:*

(C3) *$\tau(N) = N$,*

(C4) *If X is a subset of A such that $N \subseteq X$, then $N \subseteq \tau(X)$.*

Lemma 2.3 *In any $(A, \tau) \in \mathbf{W}_{n+1,k}$ it holds:*

(i) *if D if a maximal d.s. of A, then $N = D \cap \tau(D) \cap \ldots \cap \tau^{k-1}(D)$ is a maximal kernel of A,*

(ii) *if N is a maximal kernel of A, then there is a maximal d.s. D of A such that $N = D \cap \tau(D) \cap \ldots \cap \tau^{k-1}(D)$.*

Proof. (i): Cleary N is a d.s. of A. Besides, if $x \in N$, then $x \in \tau^j(D)$ for every j, $0 \leq j \leq k-1$. Hence $\tau(x) \in \tau^{j+1}(D)$ for every j, $0 \leq j \leq k-1$ and so $N \in \mathcal{N}(A)$. Now, suppose that there is $M \in \mathcal{N}(A)$ such that $N \subseteq M$ and $M \neq A$. Let $D' \in \mathcal{E}(A)$ be such that $M \subseteq D'$. As D' is prime, there is j, $0 \leq j \leq k-1$, such that $\tau^j(D) \subseteq D'$ and so $D \subseteq \tau^{k-j}(D')$. Since D is maximal, it follows that $D = \tau^{k-j}(D')$. Then, $M \subseteq D' = \tau^j(D)$ and by (C4) we have $M \subseteq N$.
(ii): As N is a d.s. of A, there is $D \in \mathcal{E}(A)$ such that $N \subseteq D$. From (C4) $N \subseteq D \cap \tau(D) \cap \ldots \cap \tau^{k-1}(D)$ and taking into account that both are maximal kernels, the equality holds. □

We will say that a maximal d.s. D *is of period* d, if d is the least positive integer such that $\tau^d(D) = D$. In this case, we will say that the maximal kernel $N = D \cap \tau(D) \cap \ldots \cap \tau^{d-1}(D)$ *is of period* d (or d-kernel for short).

Remark 2.1 *It is easy to check that:*

(i) *if D is a maximal d.s. of period d, then d/k,*

(ii) *if $N = D \cap \tau(D) \cap \ldots \cap \tau^{d-1}(D)$ is a d-kernel of A, then D, $\tau(D)$, $\ldots$,$\tau^{d-1}(D)$ are the unique maximal d.s. which contains N.*

Lemma 2.4 *If $A \in \mathbf{W_{n+1,k}}$ has more than one element, then the intersection of all maximal kernels of A is $\{1\}$.*

Proof. Let $\{N_i\}_{i \in I}$ be the family of all maximal kernels of A. For each $i \in I$ there is a maximal d.s. D_i of period k_i such that $N_i \subseteq D_i$. It follows at once from Lemma 2.3 and Remark 2.1 (ii) that $\mathcal{E}(A) = \{D_i, \tau(D_i), \ldots, \tau^{k_i-1}(D_i)\}_{i \in I}$. Hence, $\bigcap_{i \in I} N_i = \bigcap_{i \in I} D_i \cap \tau(D_i) \cap \ldots \cap \tau^{k_i-1}(D_i) = \bigcap\{D : D \in \mathcal{E}(A)\} = \{1\}$. □

Theorem 2.1 *Any $A \in \mathbf{W_{n+1,k}}$ with more than one element is a subdirect product of the family $\{A/N : N$ is a maximal kernel of $A\}$.*

Proof. It follows from Lemma 2.4 and from a well - known result of universal algebra. □

Lemma 2.5 *If $A \in \mathbf{W_{n+1,k}}$, then the following properties are satisfied:*

(C5) $[a, 1]$ *is a kernel if and only if $a \in B(A) \cap I(A)$.*

(C6) $[a, 1]$ *is a maximal kernel if only if a is an atom of the Boolean algebra $B(A) \cap I(A)$.*

(C7) *If $a \in At(B(A))$, then $a \vee \tau(a) \vee \ldots \vee \tau^{k-1}(a)$ is an atom of $B(A) \cap I(A)$.*

It is clear that if (A,τ) is a finite algebra, every kernel is a principal filter. Besides, if $\Pi(A)$ is the poset of prime elements of A and $C \subseteq \Pi(A)$ is a connected component of A, then $\tau(C)$ so is.

Theorem 2.2 *Let $A \in \mathbf{W}_{n+1,k}$ be finite non trivial and $\{N_i\}_{1\leq i\leq r}$ be the set of maximal kernels of A. Then A is a direct product of the family $\{A/N_i\}_{1\leq i\leq r}$.*

Proof. By Theorem 2.1 there is monomorphism $h : A \to \prod_{i=1}^{r} A/N_i$ such that for each $a \in A$, $h(a) = (q_{N_i}(a))_{1\leq i\leq r}$, where $q_{N_i} : A \to A/N_i$ is the canonical homomorphism.

Let $y \in \prod_{i=1}^{r} A/N_i$, from (C6) for each i, $1 \leq i \leq r$ there is $b_i \in At(B(A) \cap I(A))$ such that (1) $(x_i, y(i)) \in R(N_i)$, where $N_i = [b_i, 1]$ and $x_i = y(i) \wedge b_i$.

Let $x = \bigvee_{i=1}^{r} x_i$, then $x_i \to x = 1 \in N_i$. By (W11) it follows that $x \to x_i = \bigwedge_{i=1}^{r} (x_j \to x_i)$. From (W8), (W9) and Lemma 1.2 for each j, $j \neq i$, we have $x_j \leq b_j \leq\sim b_i \leq b_i \to x_i$ and by (W10) $x_j \to x_i \in N_i$. Therefore, (2) $(x, x_i) \in R(N_i)$, so (1) and (2) we obtain $y = h(x)$. □

Lemma 2.6 *If $A \in \mathbf{W}_{n+1,k}$, then the following conditions are equivalent:*

(i) *A is simple,*

(ii) *$B(A) \cap I(A) = \{0, 1\}$.*

Proof. (i) $\Rightarrow$ (ii): It follows from (C5).
(ii) $\Rightarrow$ (i): Let N be a proper kernel of A and $x \in N$. As N is a Stone filter of A, there is $s \in N \cap B(A)$ such that $s \leq x$. If $b = s \wedge \tau(s) \wedge \ldots \wedge \tau^{k-1}(s)$, then $b \in B(A) \cap I(A) \cap N = \{1\}$ and so $x = 1$. Therefore, $N = \{1\}$. □

Theorem 2.3 *$C_{t+1,k}$ is simple, for all t, $1 \leq t \leq n$.*

Proof. If $x \in B(C_{t+1,k}) \cap I(C_{t+1,k})$, then $\tau^j(x)(i) = x(i)$ for all i, j, $1 \leq i \leq k$, $1 \leq j \leq k-1$. Besides, $\tau^j(x)(k) = x(k-j)$, hence $x(k) = x(k-j)$ for all j, $1 \leq j \leq k-1$, that is, $x(i) = x(j)$ for all $i, j \in \{1, \ldots, k\}$. Let $a \in C_{t+1}$ be such that $x(i) = a$, $1 \leq i \leq k$. Since $x \in B(C_{t+1,k})$ from Lemma 1.2 $x \wedge \sim x = 0$, then $a \wedge \sim a = 0$, that is $a \in B(C_{t+1}) = \{0, 1\}$. Therefore, $x = 0$ o $x = 1$ and by Lemma 2.6, $C_{t+1,k}$ is simple. □

From Lemma 2.6 it is immediate that any subalgebra of a simple algebra A is simple. Then, the subalgebras of $C_{t+1,k}$, $1 \leq t \leq n$ are simple.

Lemma 2.7 *Any simple $W_{n+1,k}$-algebra is finite.*

Proof. Let $A \in \mathbf{W}_{n+1,k}$ be a simple algebra. Then $\{1\}$ is a maximal kernels of A and there is $D \in \mathcal{E}(A)$ such that $\{1\} = D \cap \tau(D) \cap \ldots \cap \tau^{k-1}(D)$. If D is of period d, then from Remark 2.1 (ii) we have $\mathcal{E}(A) = \{D, \tau(D), \ldots, \tau^{d-1}(D)\}$ and by Theorem 1.3, A is finite. □

Theorem 2.4 *Let $A \in \mathbf{W}_{n+1,k}$ be a simple algebra. Then $A \simeq \mathcal{C}_{t+1,d}$, where $1 \leq t \leq n$ and d/k.*

Proof. By Lemma 2.7 A is finite and $\mathcal{E}(A) = \{M, \tau(M), \ldots, \tau^{d-1}(M)\}$ with d/k. From Theorem 1.1 we have that $\Pi(A)$ is the cardinal sum of chains with at most n elements. On the other hand all the chains of $\Pi(A)$ have the same cardinal number. Indeed, let C_r, $C_s \subseteq \Pi(A)$ be two chains such that $|C_r| = r$, $|C_s| = s$ and $r \neq s$. Let u_r and u_s be the last elements of C_r and C_s respectively. Hence, $u_r, u_s \in At(B(A))$ and $D = [u_r, 1]$ is a maximal d.s. of A. Let α_r be the least positive integer such that $\tau^{\alpha_r}(u_r) = u_r$. Then, α_r/k and $N = [(u_r \vee \tau(u_r) \vee \ldots \vee \tau^{\alpha_r - 1}(u_r), 1]$ is a maximal kernel of A. Since A is simple, $u_r \vee \tau(u_r) \vee \ldots \vee \tau^{\alpha_r-1}(u_r) = 1$. Hence, there exist j, $0 \leq j \leq \alpha_r - 1$ such that $u_s \leq \tau^j(u_r)$, which is a contradiction.

Therefore, for all $D \in \mathcal{E}(A)$, we have that $A/D \simeq C_{t+1}$ for some t, $1 \leq t \leq n$. By Theorem 1.3, $\varphi : A \to \prod_{D \in \mathcal{E}(A)} A/D$ defined by , $\varphi(x) = (q_M(x), q_{\tau(M)}(x), \ldots, q_{\tau^{d-1}(M)}(x))$ for each $x \in A$, is a W–isomorphism. Identifying isomorphic algebras, we have that there is a W–isomorphism between A and $C_{t+1}^{\mathcal{E}(A)}$. Let us consider the automorphism τ defined on $C_{t+1}^{\mathcal{E}(A)}$ as in Example 2.1. To prove that φ is a $W_{n+1,k}$–isomorphism, we need to show that for every j, $0 \leq j \leq d-1$, $q_{\tau^j(M)}(x) = q_{\tau^{j+1}(M)(\tau(x))}$. Let $h : A/\tau^j(M) \to A/\tau^{j+1}(M)$ be defined by $h(q_{\tau^j(M)}(x)) = q_{\tau^{j+1}(M)}(\tau(x))$. As τ is a W–automorphism such that $\tau^d = Id$, it is easy to prove that h is an order-isomorphism and this completes the proof. □

Let $A \in \mathbf{W}_{n+1,k}$. We will denote by $\mathcal{M}(A)$ the set of maximal kernels of A.

Clearly $\{M_{i+1,d}\}_{1 \leq i \leq n, d/k}$, where $M_{i+1,d} = \{N \in \mathcal{M}(A) : A/N \simeq C_{i+1,d}\}$, $1 \leq i \leq n$ and d/k, is a partition of $\mathcal{M}(A)$ and so $\mathcal{M}(A) = \bigcup_{i=1}^{n} \bigcup_{d/k} M_{i+1,d}$.

Then from Theorems 2.1 and 2.2 we have Theorem 2.5.

Theorem 2.5 *If $A \in \mathbf{W}_{n+1,k}$ has more than one element, then there is a monomorphism $\varphi : A \to \prod_{i=1}^{n} \prod_{d/k} (\mathcal{C}_{i+1,d})^{M_{i+1,d}}$. If A is finite, then φ is an isomorphism.*

3 Free algebras

In this section, we describe the free $W_{n+1,k}$–algebra with a finite set of generators.

Throughout this section $\mathcal{L} = \mathcal{L}(n+1, k, r)$ denotes the $W_{n+1,k}$–algebra with a set G

of free generators, $|G| = r$, where r is a positive integer.

To prove the principal result of this section we need to describe the subalgebras of $C_{n+1,k}$.

We will denote by $D(r)$ and $M(r)$ the sets of positive divisors and maximal positive divisors of a positive integer r, respectively. As $C_{n+1,k}$ is polynomially equivalent to a Post algebra of order $n+1$ (see [6]), the subalgebras of $C_{n+1,k}$ are $S(A,d) = \{f \in C_{n+1,k} : f(i) \in A$ for every $i, 1 \leq i \leq k$ and $f(i) = f(j)$ if $n/(i-j)\}$ where A is a subalgebra of C_{n+1} and $d \in D(k)$ (see [1]). Besides, $S(A_1,d_1) \subseteq S(A_2,d_2)$ if and only if $A_1 \subseteq A_2$ and d_1/d_2. Hence, the subalgebras of $C_{n+1,k}$ are the algebras $C_{p+1,d}$ with $p \in D(n)$ and $d \in D(k)$. On the other hand, $C_{p_1+1,d_1} \subseteq C_{p_2+1,d_2}$ if and only if p_1/p_2 and d_1/d_2. It is clear that C_{p_1+1,d_1} is proper subset of C_{p_2+1,d_2} if p_1 is a proper divisor of p_2 or d_1 is a proper divisor of d_2.

The maximal subalgebras of $C_{n+1,k}$ are the algebras $C_{p+1,k}$ with $p \in M(n)$ or $C_{n+1,d}$ with $d \in M(k)$. Therefore, the number of maximal subalgebras of $C_{n+1,k}$ is $|M(n)| + |M(k)|$.

We will denote by $\bigotimes_{i=1}^{r} b_i$ and $\bigodot_{i=1}^{r} b_i$ the greatest common divisor and the least common multiple of the numbers $b_1, \ldots, b_r$, respectively.

It is easy to see that $\bigcap_{i=1}^{r} C_{p_i+1;d_i} = S(A,d)$, where $A = \bigcap_{i=1}^{r} C_{p_i+1}$ and $d = \bigotimes_{i=1}^{r} d_i$.

Lemma 3.1 $\bigcap_{i=1}^{r} C_{p_i+1} = C_{p+1}$, *where* $p = \bigotimes_{i=1}^{r} p_i$.

Proof. It is simple to see that $C_{p+1} \subseteq \bigcap_{i=1}^{r} C_{p_i+1}$. Let $p = \bigotimes_{i=1}^{r} p_i$ and $x \in \bigcap_{i=1}^{r} C_{p_i+1}$ such that $x = \frac{h_i}{p_i}$, $1 \leq i \leq r$, $0 \leq h_i \leq p_i$. Then $p \cdot x = p \cdot \frac{h_1}{p_1} = p \cdot \frac{h_2}{p_2} = \ldots = p \cdot \frac{h_r}{p_r}$ and $p_1 \cdot p_2 \cdot \ldots \cdot p_r = p \cdot \bigodot_{i=1}^{r} p_i$. Hence, for each j, $1 \leq j \leq r$ we have $p_j = p \cdot \frac{\bigodot_{i=1}^{r} p_i}{\prod_{i \neq j} p_i}$. Therefore, $p \cdot x = \frac{h_1 \cdot \prod_{i\neq 1} p_i}{\bigodot_{i=1}^{r} p_i} = \ldots = \frac{h_r \cdot \prod_{i \neq r} p_i}{\bigodot_{i=1}^{r} p_i}$. Let $a = h_1 \cdot \prod_{i \neq 1} p_i = \ldots = h_r \cdot \prod_{i \neq r} p_r$. As p_i/a for each i, $1 \leq i \leq r$. we have that $\bigodot_{i=1}^{r} p_i/a$ and so px is a integer, i.e. $x \in C_{p+1}$. □

Corollary 3.1 $\bigcap_{i=1}^{r} C_{p_i+1,d_i} = C_{a+1,b}$, *where* $a = \bigotimes_{i=1}^{r} p_i$ *and* $b = \bigotimes_{i=1}^{r} d_i$.

Let $(A, \tau) \in \mathbf{W_{n+1,k}}$ and $X \subseteq A$. We will denote by $[X]$ the $W_{n+1,k}$-subalgebra of A generated by X.

By [2], the number of elements of $\mathcal{L}(n+1,k,r)$ coincide with the numbers of elements of the free W_{n+1}–algebra with kr generators. On the other hand, by [5] we have that $\mathbf{W_{n+1}}$ is locally finite and therefore $\mathcal{L}(n+1,k,r)$ is finite. Then by Theorem 2.5 we have

$$\mathcal{L} = \mathcal{L}(n+1,k,r) \simeq \prod_{i=1}^{n} \prod_{d\in D(k)} (\mathcal{C}_{i+1,d})^{M_{i+1,d}}.$$

In a similar way to [1] we prove that

$$|M_{i+1,d}| = \mu(i,d,r) = \frac{|Epi(\mathcal{L}, \mathcal{C}_{i+1,d})|}{d}. \quad (1)$$

Now, let us consider

$F(i,d,r) = C_{i+1,d}^{G}$,

$F'(i,d,r) = \{f \in F(i,d,r) : f(G) \not\subseteq S \text{ for all maximal subalgebra } S \text{ of } C_{i+1,d}\}$,

$F''(i',d,r) = \{f \in F(i,d,r) : f(G) \subseteq C_{i'+1,d},\ i' \in M(i)\}$,

$F''(i,d',r) = \{f \in F(i,d,r) : f(G) \subseteq C_{i+1,d'}; d' \in M(d)\}$.

Then,

$$|F(i,d,r)| = (i+1)^{dr}. \quad (2)$$

Let $f : G \to C_{n+1,k}$ and $C_{i+1,d}$ a subalgebra of $C_{n+1,k}$. Then $[f(G)] = \mathcal{C}_{i+1,d}$ if and only if $f(G) \subseteq C_{i+1,d}$ and $f(G) \not\subseteq S$ for all maximal subalgebra S of $C_{i+1,d}$. Hence, we have

$$\delta(i,d,r) = |F'(i,d,r)| = |Epi(\mathcal{L}, \mathcal{C}_{i+1,d})|. \quad (3)$$

On the other hand,

$$F'(i,d,r) = F(i,d,r) \setminus \left(\bigcup_{i'\in M(i)} F''(i',d,r) \cup \bigcup_{d'\in M(d)} F''(i,d',r) \right),$$

therefore,

$$\begin{aligned} |F'(i,d,r)| &= |F(i,d,r)| - \left| \bigcup_{i'\in M(i)} F''(i',d,r) \right| - \left| \bigcup_{d'\in M(d)} F''(i,d',r) \right| + \\ &\quad \left| \bigcup_{i'\in M(i)} F''(i',d,r) \cap \bigcup_{d'\in M(d)} F''(i,d',r) \right|. \end{aligned} \quad (4)$$

Let

$$\alpha(i,d,r) = \left| \bigcup_{i' \in M(i)} F''(i',d,r) \right|,$$

$$\beta(i,d,r) = \left| \bigcup_{d' \in M(d)} F''(i,d',r) \right|,$$

$$\gamma(i,d,r) = \left| \bigcup_{i' \in M(i)} F''(i',d,r) \cap \bigcup_{d' \in M(d)} F''(i,d',r) \right|.$$

For all finite set $\mathcal{J}$, holds

$$\left| \bigcup_{j \in \mathcal{J}} A_j \right| = \sum_{X \subseteq \mathcal{J}, X \neq \emptyset} (-1)^{|X|-1} \left| \bigcap_{j \in X} A_j \right|. \tag{5}$$

Then, by (5) and Corllary 3.1, it follows that

$$\begin{aligned}
\alpha(i,d,r) &= \sum_{X \subseteq M(i), X \neq \emptyset} (-1)^{|X|-1} \left| \bigcap_{j \in X} F''(j,d,r) \right|, \\
\bigcap_{j \in X} F''(j,d,r) &= \left\{ f \in F(i,d,r) : f(G) \subseteq C_{j'+1,d}; j' = \bigotimes_{j \in X} j \right\}, \\
\beta(i,d,r) &= \sum_{Y \subseteq M(d), Y \neq \emptyset} (-1)^{|Y|-1} \left| \bigcap_{j \in Y} F''(i,j,r) \right|, \\
\bigcap_{j \in Y} F''(i,j,r) &= \left\{ f \in F(i,d,r) : f(G) \subseteq C_{i+1,j'}; j' = \bigotimes_{j \in Y} j \right\},
\end{aligned}$$

thus,

$$\begin{aligned}
\left| \bigcap_{j \in X} F''(j,d,r) \right| &= \left(\bigotimes_{j \in X} j + 1 \right)^{dr}, \\
\alpha(i,d,r) &= \sum_{X \subseteq M(i), X \neq \emptyset} (-1)^{|X|-1} \left(\bigotimes_{j \in X} j + 1 \right)^{dr}, \qquad (6) \\
\left| \bigcap_{j \in Y} F''(i,j,r) \right| &= (i+1)^{r \cdot \bigotimes_{j \in Y} j},
\end{aligned}$$

$$\beta(i,d,r) = \sum_{Y\subseteq M(d), Y\neq\emptyset} (-1)^{|Y|-1}(i+1)^{r\cdot \bigotimes_{j\in Y} j}. \tag{7}$$

To compute $\gamma(i,d,r)$ let us observe that

$$\bigcup_{i'\in M(i)} F''(i',d,r) \cap \bigcup_{d'\in M(d)} F''(i,d',r) = \bigcup_{i'\in M(i), d'\in M(d)} (F''(i',d,r)\cap F''(i,d',r)),$$

$$F''(i',d,r)\cap F''(i,d',r) = F''(i',d',r) = \{f \in F(i,d,r) : f(G) \subseteq C_{i'+1,d'}\}.$$

Therefore,

$$\gamma(i,d,r) = \sum_{Z\subseteq M(i)\times M(d), Z\neq\emptyset} (-1)^{|Z|-1} \left| \bigcap_{Z\ni(i',d')} F''(i',d',r)\right|.$$

Let us consider

$$Z_1 = \{i' \in M(i) : (i',d') \in Z \text{ for some } d' \in M(d)\},$$

$$Z_2 = \{d' \in M(d) : (i',d') \in Z \text{ for some } i' \in M(i)\}.$$

Then,

$$\bigcap_{(i',d')\in Z} F''(i',d',r) = \left\{ f \in F(i,d,r) : f(G) \subseteq C_{s+1,t}, s = \bigotimes_{i'\in Z_1} i', t = \bigotimes_{d'\in Z_2} d' \right\},$$

$$\left|\bigcap_{(i',d')\in Z} F''(i',d',r)\right| = \left(\bigotimes_{i'\in Z_1} i'+1\right)^{r\cdot \bigotimes_{d'\in Z_2} d'},$$

$$\gamma(i,d,r) = \sum_{Z\subseteq M(i)\times M(d), Z\neq\emptyset} (-1)^{|Z|-1} \left(\bigotimes_{i'\in Z_1} i'+1\right)^{r\cdot \bigotimes_{d'\in Z_2} d'}. \tag{8}$$

From (2), (4), (6), (7) and (8), we have

$$\delta(i,d,r) = (i+1)^{dr} - \alpha(i,d,r) - \beta(i,d,r) + \gamma(i,d,r).$$

So, we have proved the following theorem:

Theorem 3.1 *$\mathcal{L}(n+1,k,r)$ is isomorphic to $\prod_{i=1}^{n} \prod_{d\in D(k)} C_{i+1,d}^{\mu}$, where*

(i) $\mu = \frac{\delta}{d}$,

(ii) $\delta = (i+1)^{dr} - \alpha - \beta + \gamma$,

(iii) $\alpha = \sum_{X \subseteq M(i), X \neq \emptyset} (-1)^{|X|-1} \left(\bigotimes_{j \in X} j + 1 \right)^{dr}$,

(iv) $\beta = \sum_{Y \subseteq M(d), Y \neq \emptyset} (-1)^{|Y|-1} (i+1)^{r \cdot \bigotimes_{j \in Y} j}$,

(v) $\gamma = \sum_{Z \subseteq M(i) \times M(d), Z \neq \emptyset} (-1)^{|Z|-1} \left(\bigotimes_{i' \in Z_1} i' + 1 \right)^{r \cdot \bigotimes_{d' \in Z_2} d'}$.

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