Q–operators on implicative Łukasiewicz algebras

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Abstract

In this paper QI–algebras are defined as a generalization of UI–algebras. The notion of UI–algebra was introduced by the author in *Algebras implicativas de Lukasiewicz* $(n+1)$*–valuadas con diversas operaciones adicionales*, Doctoral Thesis, Univ. Nac. del Sur (1989), Bahía Blanca, Argentina, as an algebraic counterpart of a fragment of the monadic infinite–valued Lukasiewicz functional propositional calculus. Universal quantifiers, as a particular case of Q–operators are introduced on I_3–algebras with first element and consequently the variety $\boldsymbol{MI_3^0}$ of monadic 3–valued implicative Lukasiewicz algebras are defined. MI_3^0–algebras are an algebraic counterpart of the 3–valued Lukasiewicz functional calculus, different from that given by L. Monteiro in *Algebras de Lukasiewicz trivalentes monádicas*, Notas de Lógica Matemática, 32, Univ. Nac. del Sur, Bahía Blanca, 1974.

Key words. BCK–algebras, Lukasiewicz algebras, Wasjberg algebras.

AMS subject classification (1991). 03G25, 03G20, 06D30, 06F35

Introduction

In [8], L. Monteiro proposes the notion of existential quantifier on a 3–valued Lukasiewicz algebra $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ as an application $\exists : A \longrightarrow A$ satisfying:

(1) $\exists 0 = 0$,

(2) $\exists x \wedge x = x$,

(3) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,

(4) $\exists \nabla x = \nabla \exists x$,

(5) $\exists \Delta x = \Delta \exists x$, where Δ is an abbreviation for $\sim \nabla \sim$,

and he defines monadic 3–valued Lukasiewicz algebras as 3–valued Lukasiewicz algebras endowed with an additional operation which is an existential quantifier. He also defines the notion of universal quantifier by means of the formula $\forall x = \sim \exists \sim x$, and he proves that $\forall$ verifies the identities

(6) $\forall 1 = 1$,

(7) $\forall x \vee x = x$,

(8) $\forall(x \vee \forall y) = \forall x \vee \forall y$,

(9) $\forall \nabla x = \nabla \forall x$,

(10) $\forall \Delta x = \Delta \forall x$.

Note that if 3–valued Lukasiewicz algebras with a unary operation $\forall$ verifying (6),. . .,(10) are considered, then monadic 3–valued Lukasiewicz algebras can be obtained defining the existential quantifier by the formula $\exists x = \sim \forall \sim x$.

We shall denote by $\boldsymbol{L_3}$ and $\boldsymbol{ML_3}$ the varieties of 3–valued Lukasiewicz algebras and monadic 3–valued Lukasiewicz algebras respectively. A more detailed treatment of theses algebras can be found in [4, 8, 9, 10].

A. Monteiro and L. Iturrioz [7], define a universal quantifier on a Tarski algebra A as an application $\forall : A \longrightarrow A$ satisfying the properties:

(11) $\forall 1 = 1$,

(12) $\forall x \vee x = x$, where $x \vee y$ is an abbreviation for $(x \rightarrow y) \rightarrow y$,

(13) $\forall(x \vee \forall y) = \forall x \vee \forall y$,

(14) $\forall(x \rightarrow y) \rightarrow (\forall x \rightarrow \forall y) = 1$.

This notion arises from the logical notion of universal quantifier on classical implicative propositional calculus.

Consequently, a monadic Tarski algebra is an algebra $\langle A, \rightarrow, \forall, 1\rangle$ of type $(2, 1, 0)$ such that $\langle A, \rightarrow, 1\rangle$ is a Tarski algebra and $\forall$ is a universal quantifier on A.

Later, Y. Komori [3] generalizes the notion of Tarski algebra and he defines the variety of C–algebras, which we re-name $\boldsymbol{I}$ following A. Monteiro, as algebras $\langle A, \rightarrow, 1\rangle$ of type $(2, 0)$ fulfilling the following identities:

(I1) $1 \rightarrow x = x$,

(I2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,

(I3) $(x \to y) \to y = (y \to x) \to x$.

A detailed study of I–algebras can be found in [1, 2, 3, 5, 11].

On the other hand, $(n+1)$–valued I–algebras with n positive integer are an important subvariety of $\boldsymbol{I}$ which we shall denote by $\boldsymbol{I}_{n+1}$. An I_{n+1}–algebra is an I–algebra which satisfies

(I4) $((x^n \to y) \to x) \to x = 1$,

where $x^0 \to y = y$, $x^{n+1} \to y = x \to (x^n \to y)$, for all positive integer n.

The main definitions and results of varieties $\boldsymbol{I}$ and $\boldsymbol{I}_{n+1}$ needed in this paper are summarized in Section 1. In Section 2, the definition of a Q-operator on an I–algebra is given. Besides, the subvarieties $\boldsymbol{QI}_{n+1}$, $\boldsymbol{UI}$ and $\boldsymbol{UI}_{n+1}$ of $\boldsymbol{QI}$ are considered. Finally, the semisimplicity of UI_{n+1}–algebras is proved and subdirectly irreducible algebras are characterized. In Section 3, the notion of universal quantifier on an I_3–algebra with first element is defined and the variety of MI_3^0–algebras is introduced. From Theorem 3.1 we can state that MI_3^0–algebras are an algebraic counterpart of monadic 3-valued Lukasiewicz functional propositional calculus. The results obtained in this section cannot be generalized for $n \geq 5$, since in this case the Lukasiewicz implication cannot be defined in terms of the primitive operations of $(n+1)$–valued Lukasiewicz algebras considered by G. C. Moisil.

1 Preliminaries

Next we give the necessary background on varieties $\boldsymbol{I}$ and $\boldsymbol{I}_{n+1}$needed later.

Lemma 1.1 ([5]) *In any $A \in \boldsymbol{I}$ it holds:*

(I5) $x \to (y \to x) = 1$,

(I6) $x \to 1 = 1$,

(I7) $x \to x = 1$,

(I8) *the relation $x \leq y$ if and only if $x \to y = 1$ is an order on A, and $x \leq 1$ for all $x \in A$,* (I9) $(A,\leq)$ *is a join semilattice and the element* $x \vee y = (x \to y) \to y$ *is the supremum of the elements* x *and* y,

(I10) $x \leq y$ *implies* $y \to z \leq x \to z$,

(I11) $x \to (y \to z) = y \to (x \to z)$,

(I12) $x \leq y$ *implies* $z \to x \leq z \to y$.

Definition 1.1 *Let* $A \in \boldsymbol{I}$. $D \subseteq A$ *is a deductive system (d.s.) of* A, *if it satisfies:*

(D1) $1 \in D$,

(D2) *if* $x, x \to y \in D$ *then* $y \in D$.

We shall denote by $\mathcal{D}(A)$ the set of all d.s. of A.

Remark 1.1 *It is known* [5] *that:*

(i) *If* $A \in \boldsymbol{I}$, $D \in \mathcal{D}(A)$ *and* $R(D) = \{(x,y) \in A^2 : x \to y \in D, y \to x \in D\}$, *then* $R(D) \in Con_I(A)$ *where* $Con_I(A)$ *is the set of all* I*–congruences on* A. *If* $R \in Con_I(A)$ *and* x_R *denotes the equivalence class of* x, $x \in A$, *then* $1_{R(D)} = D$.

(ii) *If* $R \in Con_I(A)$, *then there exists a unique* $D \in \mathcal{D}(A)$ *such that* $R = R(D)$ *and* $D = 1_R$.

For $A \in \boldsymbol{I}_{n+1}$ we write $x \succ y$ instead of $x^n \to y$.

Lemma 1.2 ([1]) *In any* $A \in \boldsymbol{I}_{n+1}$ *it holds:*

(I13) $x \succ (y \succ x) = 1$,

(I14) $x \succ (y \succ z) = (x \succ y) \succ (x \succ z)$,

(I15) $1 \succ x = x$,

I16) $((x \succ y) \succ x) \succ x = 1$.

Lemma 1.3 ([1]) *Let* $A \in \boldsymbol{I}_{n+1}$. *Then the following conditions are equivalent:*

(i) $D \in \mathcal{D}(A)$,

(ii) D *verifies* (D1), *and* (D′2) *if* $x, x \succ y \in D$, *then* $y \in D$.

Remark 1.2 *Let* $A \in \boldsymbol{I}_{n+1}$. *In* [1] *it is proved that:*

(i) *if* A *is non trivial, then* A *is a subdirect product of simple* I_{n+1}*–algebras,*

(ii) *the following conditions are equivalent:*

(1) A *is simple,*

(2) $A \simeq S$, $|S| > 1$, S *is a subalgebra of* $C'_{n+1} = \langle C_{n+1}, \rightarrow, 1\rangle$, *where* $C_{n+1} = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ *and* $x \rightarrow y = min\ \{1, 1 - x + y\}$.

2 $\boldsymbol{QI}$–Algebras

Definition 2.1 *Let* $A \in \boldsymbol{I}$. *A function* $\forall : A \longrightarrow A$ *is a* Q*–operator on* A *if it satisfies:*

(Q1) $\forall 1 = 1$,

(Q2) $\forall x \leq x$,

(Q3) $\forall(x \vee \forall y) = \forall x \vee \forall y$,

(Q4) $\forall(x \rightarrow y) \leq \forall x \rightarrow \forall y$.

Definition 2.2 *A* QI*–algebra is an algebra* $\langle A, \rightarrow, \forall, 1\rangle$ *of type* $(2, 1, 0)$, *where* $\langle A, \rightarrow, 1\rangle$ *is an* I*–algebra and* $\forall$ *is a* Q*–operator on* A.

The variety of these algebras will be denoted by $\boldsymbol{QI}$.

Lemma 2.1 *In any* $A \in \boldsymbol{QI}$ *it holds:*

(Q5) $\forall\forall x = \forall x$,

(Q6) $x \leq y$ *implies* $\forall x \leq \forall y$,

(Q7) $\forall(\forall x \rightarrow y) \leq \forall x \rightarrow \forall y$,

(Q8) $\forall(\forall x \rightarrow \forall y) \leq \forall(\forall x \rightarrow y)$.

Proof. It follows from I1, ... , I3, Q1, ..., Q4. □

Definition 2.3 *Let* $A \in \boldsymbol{QI}$. $D \in \mathcal{D}(A)$ *is a monadic d.s. (m.d.s.) of* A *if it verifies*

(D3) $x \in D$ *implies* $\forall x \in D$.

$\mathcal{D}_M(A)$ will denote the set all of m.s.d. of A.

Lemma 2.2 $Con_{QI}(A) = \{R(D) : D \in \mathcal{D}_M(A)\}$.

Proof. Let $R \in Con_{QI}(A)$. Hence, $R \in Con_I(A)$ and by Remark 1.1 (ii) there exists $D \in \mathcal{D}(A)$ such that $R = R(D)$. Furthermore, D is a monadic d.s.. Indeed, if $x \in D$, then $(1, x) \in R$ and so $(\forall 1, \forall x) \in R$. From the latter assertion, Remark 1.1(i), Q1 and I1 we have that $\forall x \in D$. □

A QI–algebra is said to be a QI_{n+1}–algebra it verifies (14). The variety of these algebras wil be denoted by $\boldsymbol{QI_{n+1}}$.

In any $A \in \boldsymbol{QI_{n+1}}$ we define $x \mapsto y = \forall x \succ y$ for all $x, y \in A$.

Lemma 2.3 *In* $\boldsymbol{QI_{n+1}}$ *holds:*

(Q9) $\forall x \to \forall(\forall x \to y) \leq x \mapsto \forall y$,

(Q10) $1 \mapsto x = x$,

(Q11) $x \mapsto x = 1$,

(Q12) $x \mapsto (y \mapsto x) = 1$,

(Q13) $x \mapsto (y \mapsto z) \leq (x \mapsto y) \mapsto (x \mapsto z)$,

(Q14) $x \leq y$ *implies* $x \mapsto y = 1$,

(Q15) $(x \mapsto (y \mapsto z)) \mapsto ((x \mapsto y) \mapsto (x \mapsto z)) = 1$,

(Q16) $x \mapsto (x \to y) = x \mapsto y$,

(Q17) $x \mapsto \forall x = 1$,

(Q18) $(x \mapsto \forall y) \to \forall x = \forall x$.

Lemma 2.4 *Let* $A \in \boldsymbol{QI_{n+1}}$. *Then the following conditions are equivalent:*

(i) $D \in \mathcal{D}_M(A)$,

(ii) $D \subseteq A$ *verifies* D1 *and* (D″2) *if* $x, x \mapsto y \in D$, *then* $y \in D$.

Proof. (i) $\Longrightarrow$ (ii): Trivial.
(ii) $\Longrightarrow$ (i): (D2) Let $x, x \rightarrow y \in D$. From Q12 and D1 we have that $(x \rightarrow y) \mapsto (x \mapsto (x \rightarrow y)) = 1 \in D$. Hence, by D″2 $x \mapsto (x \rightarrow y) \in D$ and so from Q16 and D″2 we get $y \in D$.
(D3) Let $x \in D$. Then from Q17, D1 and D″2 we have that $\forall x \in D$. □

Taking into account Q10, Q12, Q15 and the results of A. Monteiro [6], it follows that semisimple QI_{n+1}–algebras are the subvariety of $\boldsymbol{QI_{n+1}}$ determined by the identity

(Q19) $((x \mapsto y) \mapsto x) \mapsto x = 1$.

Definition 2.4 *Let $A \in \boldsymbol{I}$. A* U*–operator is a* Q*–operator on A which verifies the identity:*

(U1) $\forall(\forall x \rightarrow \forall y) = \forall x \rightarrow \forall y$.

Definition 2.5 *A UI–algebra is an algebra $\langle A, \rightarrow, \forall, 1\rangle$ of type $(2,1,0)$ such that $\langle A, \rightarrow, 1\rangle \in I$ and $\forall$ is a* U*–operator on A.*

We shall denote by $\boldsymbol{UI}$ the variety of these algebras.

Example 2.1 *Let $C_3^I = \langle C_3, \rightarrow, 1\rangle$ be the I_3–algebra considered in Remark* 1.2 (ii) *and $S = \{0,1\}$ be a I_3–subalgebra of C_3^I. If $A = S \times C_3$ is the product algebra and $\forall : A \longrightarrow A$ is defined as follows*

x	$\forall x$
$(0,0)$	$(0,0)$
$(1,0)$	$(0,0)$
$(0,\frac{1}{2})$	$(0,\frac{1}{2})$
$(1,\frac{1}{2})$	$(0,\frac{1}{2})$
$(0,1)$	$(0,\frac{1}{2})$
$(1,1)$	$(1,1)$

then, $\forall$ is a Q*–operator on A, but it is not a* U*–operator since* $\forall(\forall(0,\frac{1}{2}) \rightarrow \forall(0,0)) \neq \forall(0,\frac{1}{2}) \rightarrow \forall(0,0)$.

An algebra $\langle A, \rightarrow, \forall, 1\rangle$ of type $(2,1,0)$ is said to be a UI_{n+1}–algebra if $\langle A, \rightarrow, 1\rangle \in \boldsymbol{I_{n+1}}$ and $\forall$ is a U–operator on A. $\boldsymbol{UI_{n+1}}$ will denote the variety of these algebras.

Lemma 2.5 *The following identities hold in any* $A \in \boldsymbol{UI}_{n+1}$*:*

(U2) $\forall(\forall x \to y) = \forall x \to \forall y$,

(U3) $((x \mapsto y) \mapsto x) \mapsto x = 1$.

Proof.

(U2) (1) $\forall x \to \forall y \leq \forall x \to y$, [Q2, I12]

(2) $\forall x \to \forall y \leq \forall(\forall x \to y)$, [(1), Q6, U1]

(3) $\forall(\forall x \to y) = \forall x \to \forall y$. [(2), Q7]

(U3) (1) $(x \mapsto y) \mapsto x = \forall(\forall x \succ y) \succ x$

$= \forall((\forall x)^n \to y) \succ x$

$= ((\forall x)^n \to \forall y) \succ x$ [U2]

$= (\forall x \succ \forall y) \succ x$,

(2) $\forall((x \mapsto y) \mapsto x) = \forall((\forall x \succ \forall y) \succ x))$ [(1)]

$= (\forall x \succ \forall y) \succ \forall x$, [U1, U2]

(3) $((x \mapsto y) \mapsto x) \mapsto x = \forall((x \mapsto y) \mapsto x) \succ x$

$= ((\forall x \succ \forall y) \succ \forall x) \succ \forall x$, [(2)]

$= 1$. [(I16)] □

From Lemma 2.5 we have

Theorem 2.1 *Every non trivial* UI_{n+1}*-algebra* A *is semisimple.*

Examples 2.1

(i) *Let* $\langle A, \to, 1\rangle \in \boldsymbol{I}$. *If we define* $\forall x = x$ *for all* $x \in A$, *then* $\mathcal{U}(A) = \langle A, \to, \forall, 1\rangle \in \boldsymbol{UI}$.

(ii) *Let* $C_3^I = \langle C_3, \to, 1\rangle$. *If we define* $\forall 1 = 1$ *and* $\forall 0 = \forall \frac{1}{2} = 0$, *then* $V_3 = \langle C_3, \to, \forall, 1\rangle \in \boldsymbol{UI_3}$.

Lemma 2.6 *Let* $A \in \boldsymbol{UI}_{n+1}$. *If* S *is a non trivial* UI_{n+1}*-subalgebra of* A, *then the following properties hold:*

(i) $\mathcal{D}_M(S) = \{D \cap S : D \in \mathcal{D}_M(A)\}$,

(ii) $E_M(S) = \{M \cap S : M \in E_M(A),\ S \not\subseteq M\}$.

Theorem 2.2 *In any* $A \in \boldsymbol{UI}_{n+1}$ *it holds:*

(i) $\forall(A)$ *is a* UI_{n+1}*-subalgebra of* A,

(ii) *if* A *is simple, then* $\forall(A)$ *is isomorphic to a subalgebra of* $\mathcal{U}(C_{n+1})$.

Proof.

(i) It follows from Q5 and U1.

(ii) By (i) and Lemma 2.6 it follows that $\forall(A)$ is a simple UI_{n+1}-algebra. Since $\forall x = x$ for all $x \in \forall(A)$, we have that $\forall(A)$ is a simple I_{n+1}-algebra. Then, from Remark 1.2, we conclude the proof. □

Lemma 2.7 *If* A *is a simple* UI_{n+1}*-algebra, then* A *has first element.*

Proof. By Theorem 2.2 (ii) it follows that $\forall(A)$ has first element 0. Hence, by Q2 we have that $0 \leq x$ for all $x \in A$. □

Remark 2.1 *The variety* $\boldsymbol{QI}_{n+1}$ *does not verify Theorem 2.2 and Lemma 2.7. Indeed, let us consider the* I_3*-algebra* $A = C_3 \times C_3$ *and the* I_3*-subalgebra* $S = A \setminus \{(0,0)\}$ *of* A. *If we define* $\forall : S \longrightarrow S$ *as follows*

x	$\forall x$
$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 0)$
$(0, \frac{1}{2})$	$(0, \frac{1}{2})$
$(1, 0)$	$(\frac{1}{2}, 0)$
$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
$(0, 1)$	$(0, \frac{1}{2})$
$(1, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
$(\frac{1}{2}, 1)$	$(\frac{1}{2}, \frac{1}{2})$
$(1, 1)$	$(1, 1)$

then $\langle S, \rightarrow, \forall, 1\rangle$ *is a simple* QI_3*-algebra without first element and* $\forall(S)$ *is not isomorphic to a chain of* $\mathcal{U}(C_3)$.

3 Monadic 3–valued Implicative Łukasiewicz Algebras

Definition 3.1 *An I_3^0–algebra is an algebra $\langle A, \rightarrow, 0, 1\rangle$ of type $(2,0,0)$ such that $\langle A, \rightarrow, 1\rangle \in \boldsymbol{I_3}$ and it verifies the identity $0 \rightarrow x = 1$.*

We shall denote by $\boldsymbol{I_3^0}$ the variety of these algebras.

Remark 3.1 *The following results show the relationship between the varieties $\boldsymbol{I_3^0}$ and $\boldsymbol{L_3}$ [5]:*

(i) *If $A \in \boldsymbol{I_3^0}$, then defining $\sim x = x \rightarrow 0$, $\nabla x = \sim x \rightarrow x$ and $x \wedge y = \sim(\sim x \vee \sim y)$ it follows that $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle \in \boldsymbol{L_3}$. Furthermore, it holds that $\Delta x = \sim(x \rightarrow \sim x)$.*

(ii) *If $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle \in \boldsymbol{L_3}$, then defining $0 = \sim 1$ and $x \rightarrow y = (\nabla \sim x \vee y) \wedge (\nabla y \vee \sim x)$ for all $x, y \in A$, it follows that $\langle A, \rightarrow, 0, 1\rangle \in \boldsymbol{I_3^0}$.*

$\boldsymbol{UI_3^0}$ is the variety of algebras $\langle A, \rightarrow, \forall, 0, 1\rangle$ such that $\langle A, \rightarrow, 0, 1\rangle \in \boldsymbol{I_3^0}$ and $\forall$ is a U–operator on A.

Lemma 3.1 *The following properties hold in any $A \in \boldsymbol{UI_3^0}$:*

(P1) $\forall 0 = 0$,

(P2) $\forall \sim \forall x = \sim \forall x$,

(P3) $\Delta \forall x = \sim \forall(\forall x \rightarrow \sim \forall x)$,

(P4) $\forall \Delta \forall x = \Delta \forall x$,

(P5) $\Delta \forall x \leq \forall \Delta x$,

(P6) $\nabla \forall x = \forall \nabla \forall x$,

(P7) $\nabla \forall x \leq \forall \nabla x$,

(P8) $\nabla \forall \Delta x = \forall \Delta x$,

(P9) $\Delta \forall \Delta x = \forall \Delta x$,

(P10) $\forall \Delta x \leq \Delta \forall x$,

(P11) $\forall \Delta x = \Delta \forall x$,

where $\sim, \nabla$ and Δ are the operations described in Remark 3.1 (i).

Proof.

(P1) It follows from Q2.

(P2) $\forall \sim \forall x = \forall(\forall x \to 0)$

$= \forall(\forall x \to \forall 0)$ [P1]

$= \forall x \to 0$ [U1, P1]

$= \sim \forall x.$

(P3) $\triangle \forall x = \sim (\forall x \to \sim \forall x)$

$= \sim (\forall x \to \forall \sim \forall x)$ [P2]

$= \sim \forall(\forall x \to \forall \sim \forall x)$ [U1]

$= \sim \forall(\forall x \to \sim \forall x).$ [P2]

(P4) $\forall \triangle \forall x = \forall \sim \forall(\forall x \to \sim \forall x)$ [P3]

$= \sim \forall(\forall x \to \sim \forall x)$ [P2]

$= \triangle \forall x.$ [P3]

(P5) (1) $\forall x \leq x,$ [Q2]

(2) $\triangle \forall x \leq \triangle x,$ [(1), Remark 3.1(i)]

(3) $\triangle \forall x \leq \forall \triangle x.$ [(2), Q6, P4]

(P6) $\nabla \forall x = \sim \forall x \to \forall x = \forall \sim \forall x \to \forall x$ [P2]

$= \forall(\forall \sim \forall x \to \forall x)$ [U1]

$= \forall \nabla \forall x.$ [P2]

(P7) (1) $\forall x \leq x,$ [Q2]

(2) $\nabla \forall x \leq \nabla x,$ [(1), Remark 3.1(i)]

(3) $\nabla \forall x \leq \forall \nabla x.$ [(2), Q6, P6]

(P8) (1) $\nabla \forall \triangle x \leq \forall \nabla \triangle x,$ [P7]

(2) $\nabla \triangle x = \triangle x,$ [Remark 3.1(i)]

(3) $\nabla \forall \triangle x \leq \forall \triangle x,$ [(1), (2)]

(4) $\nabla\forall\triangle x = \forall\triangle x$. [(3), P7, I8]

(P9) It follows from P8 taking into account that $\alpha = \nabla\alpha$ implies $\alpha = \triangle\alpha$.

(P10) (1) $\triangle x \leq x$, [Remark 3.1(i)]

(2) $\forall\triangle x \leq \forall x$, [(1), Q6]

(3) $\triangle\forall\triangle x \leq \triangle\forall x$, [(2), Remark 3.1(i)]

(4) $\forall\triangle x \leq \triangle\forall x$. [(3), P9]

(P11) It follows from P5, P10 and I8. □

Definition 3.2 *Let $A \in \boldsymbol{I}_3^0$. A universal quantifier on A $\forall$ is a* U*-operator which verifies the identity:*

(M1) $\nabla\forall x = \forall\nabla x$.

The variety of UI_3^0-algebras which verifies (M1) wil be denoted by $\boldsymbol{MI}_3^0$.

Remark 3.2 *Let V_3 be the algebra described in Examples* 2.1 (ii). *V_3 does not verify* (M1). *Indeed,* $\forall\nabla\frac{1}{2} = \forall 1 = 1$, $\nabla\forall\frac{1}{2} = \nabla 0 = 0$, *where* $\sim$ *and* ∇ *are defined as in Remark* 3.1 (i).

Theorem 3.1 *Let* $\langle A, \rightarrow, \forall, 0, 1\rangle$ *be an algebra of type* $(2, 1, 0, 0)$. *Then the following conditions are equivalent:*

(i) $\langle A, \rightarrow, \forall, 0, 1\rangle \in \boldsymbol{MI}_3^0$,

(ii) $\langle A, \wedge, \vee, \sim, \nabla, \forall, 1\rangle \in \boldsymbol{ML}_3$, *where* $\wedge$, $\vee$, $\sim$ *and* ∇ *are defined as in Remark* 3.1 (i).

Proof. (i) $\Longrightarrow$ (ii): It is a consequence of the previous results.
(ii) $\Longrightarrow$ (i): Routine. □

Final Conclusion

From the above results it follows that an axiomatization for monadic 3–valued Lukasiewicz algebras defined by means of the operations $\rightarrow$, $\forall$, ∇, 0 and 1 is:

(L1) $1 \rightarrow x = x$,

(L2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$, (L3) $(x \to y) \to y = (y \to x) \to x$,

(L4) $((x \to (x \to y)) \to x) \to x = 1$,

(L5) $0 \to x = 1$,

(L6) $\forall 1 = 1$,

(L7) $\forall x \vee x = x$, where $a \vee b = (a \to b) \to b$,

(L8) $\forall(x \vee \forall y) = \forall x \vee \forall y$,

(L9) $\forall(x \to y) \to (\forall x \to \forall y) = 1$.

(L10) $\forall(\forall x \to \forall y) = \forall x \to \forall y$,

(L11) $\nabla \forall x = \forall \nabla x$, where $\nabla a = (a \to 0) \to 0$.

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