Implicative modal semilattices

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Abstract

In this paper implicative modal semilattices (or ims–algebras) are introduced as a generalization of 4–valued modal algebras defined by A. Monteiro. Furthermore, congruences are determined, the semisimplicity of ims–algebras is proved and finally, simple algebras are characterized.

1 Introduction

In 1978, A. Monteiro introduced 4–valued modal algebras as algebras $\langle A, \vee, \wedge, \sim, \nabla, 0, 1\rangle$ such that $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra and ∇ is a unary operation on A which verifies certain identities (see [2], [3], [4], [7], [8]). He suspected that these algebras were algebraic models of certain 4–valued modal propositional calculus. Later, J. Font and M. Rius as well as A. V. Figallo and A. Ziliani proved this assumption. However, as it can be observed in [6] and [5], these calculi present great technical complexity.

On the other hand, in 4–valued modal algebras it can be defined an implication operation $\rightarrow$ by the formula $p \rightarrow q = \nabla \sim p \vee q$ from which congruences can be described in terms of deductive systems (see [7], [3]).

Besides, it is easily verified that $\{\wedge, \rightarrow, \nabla, 0\}$ is an independient set of connectives in these algebras, in the sense that no-one is definable from the others. This fact leads us to consider the connectives $\{\wedge, \rightarrow, \nabla, 0, 1\}$, although the constant 1 is not really necessary (see condition A1 of Definition 1.1). Consequently, a new class of algebras $\langle A, \wedge, \rightarrow, \nabla, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ called implicative modal semilattices (or ims–algebras, for short) is studied in this paper. Congruences are determined, the semisimplicity of this variety is proved and finally, simple algebras are characterized.

In a future paper, a propositional calculus for which ims–algebras are adequated algebraic models in the sense of [6] will be presented.

Definition 1.1 *An implicative modal semilattice (or* ims*–algebra) is an algebra* $\langle A, \wedge, \rightarrow, \nabla, 0, 1\rangle$ *of type* $(2, 2, 1, 0, 0)$ *where* $\langle A, \wedge, 0, 1\rangle$ *is a semilattice with first element* 0 *and last element* 1 *which satisfies the following identities:*

(A1) $x \rightarrow x = 1$,

(A2) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(A3) $(x \rightarrow y) \rightarrow x = x$,

(A4) $(x \wedge y) \rightarrow z = x \rightarrow (y \rightarrow z)$,

(A5) $x \rightarrow (y \wedge z) = (x \rightarrow z) \wedge (x \rightarrow y)$,

(A6) $(x \rightarrow y) \wedge y = y$,

(A7) $\nabla 0 = 0$,

(A8) $\nabla(\nabla x \wedge y) = \nabla x \wedge \nabla y$,

(A9) $\nabla(x \rightarrow y) = x \rightarrow \nabla y$,

(A10) $(\nabla x \rightarrow \nabla(x \wedge y)) \rightarrow x = (\nabla x \rightarrow \nabla(x \wedge y)) \rightarrow (x \wedge ((x \rightarrow y) \rightarrow y))$,

(A11) $(x \rightarrow y) \rightarrow ((\nabla x \rightarrow \nabla(x \wedge y)) \rightarrow (\nabla(x \wedge z) \rightarrow \nabla(x \wedge y \wedge z))) = 1$.

We shall denote by **IMS** the variety of these algebras.

The following example will play an important role in Section 3.

Example 1.1 *For each cardinal number* α, *let* $B_\alpha = \{0, 1\} \cup A$, *where* $|A| = \alpha$, *and such that* $(B_\alpha, \leq)$ *is an ordered set where* $\leq = Id_{B_\alpha} \cup \{(0, 1)\} \cup \{(0, a), (a, 1)\}_{a \in A}$. *If we define* $\nabla 0 = 0$, $\nabla x = 1$ *for all* $x \neq 0$, *and*

$$x \rightarrow y = \begin{cases} 1, & si\ x \neq 1 \\ y & si\ x = 1 \end{cases},$$

then it is simple to show that $\langle B_\alpha, \wedge, \rightarrow, \nabla, 0, 1\rangle \in$ **IMS**.

Lemma 1.1 *In* IMS *holds the following identities:*

(A12) $1 \rightarrow x = x$, (A13) $x \rightarrow 1 = 1$,

(A14) $x \rightarrow (y \rightarrow x) = 1$, (A15) $\nabla 1 = 1$,

(A16) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$, (A17) $x \wedge \nabla x = x$.

Proof. It is routine. □

2 Congruences

Now, we are going to determine **IMS**–congruences following a similar technique to the one applied in [3] to describe the conguences in 4–valued modal algebras.

Definition 2.1 *Let* $A \in$ **IMS**. $D \subseteq A$ *is a deductive system (d.s) if it verifies:*

(D1) $1 \in D$,

(D2) $x,\ x \rightarrow y \in D$, *imply* $y \in D$.

$\mathcal{D}(A)$ will denote the set of all deductive systems of A.

Remark 2.1 *Let* $A \in$ IMS. *It is easy to see that:*

(i) *If* $D \in \mathcal{D}(A)$, *then* D *is a filter of* A. *Furthermore, if* $x \in D$, *then* $\nabla x \in D$ *and* $z \rightarrow x \in D$ *for all* $z \in D$.

(ii) *If* $a \in A$, *then the d.s.* $[a)$ *generated by* a *is* $[a) = \{x \in A : a \leq x\}$ ([10]).

Theorem 2.1 *Let* $A \in$ IMS.

(i) *If* $D \in \mathcal{D}(A)$, *then the relation*

$$R(D) = \{(x, y) \in A \times A : \text{(h1) } x \rightarrow y,\ \ \text{(h2) } y \rightarrow x,\ \ \text{(h3) } \nabla x \rightarrow \nabla(x \wedge y),\ \ \text{(h4) } \nabla y \rightarrow \nabla(x \wedge y) \in D\}$$

is a congruence on A. *Moreover,* $[1]_{R(D)} = D$ *where* $[x]_{R(D)}$ *denotes the equivalence class of* x.

(ii) *If θ is a congruence on A, then $[1]_\theta$ is a d.s. of A and $R([1]_\theta) = \theta$.*

We shall obtain the proof of this theorem as a consequence of the following lemmas and corollaries.

Lemma 2.1 *If $A \in$ **IMS**, it verifies the following properties:*

(A18) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,

(A19) $((x \to y) \to x) \to x = 1$,

(A20) $x \leq y$, implies $x \to y = 1$, $\nabla x \to \nabla(x \wedge y) = 1$,

(A21) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,

(A22) $x \leq y$, implies $\nabla x \leq \nabla y$,

(A23) $\nabla(x \wedge y) \to \nabla x = 1$,

(A24) $(x \to y) \to (\nabla(x \wedge y) \to (\nabla(z \wedge x) \to \nabla(z \wedge y))) = 1$.

Proof. We only check

(A22) It is a consequence of the hypothesis, A17 and A8.

(A24) From A22, A20, A13 and A2

$$\begin{aligned}(1)\ 1 &= \nabla(x \wedge y \wedge z) \to \nabla(y \wedge z)\\ &= ((x \to y) \to (\nabla(x \wedge z) \to (\nabla(x \wedge y) \to \nabla(x \wedge y \wedge z))))\\ &\quad \to ((x \to y) \to (\nabla(x \wedge z) \to (\nabla(x \wedge y) \to \nabla(y \wedge z)))).\end{aligned}$$

On the other hand, from A11, A16, A2, A23 and A12

$$\begin{aligned}(2)\ 1 &= (x \to y) \to (\nabla(x \wedge z) \to ((\nabla x \to \nabla(x \wedge y)) \to \nabla(x \wedge y \wedge z)))\\ &= (x \to y) \to (((\nabla(x \wedge z) \to \nabla x) \to (\nabla(x \wedge z) \to \nabla(x \wedge y)))\\ &\quad \to (\nabla(x \wedge z) \to \nabla(x \wedge y \wedge z)))\\ &= (x \to y) \to ((\nabla(x \wedge z) \to (\nabla(x \wedge y)) \to (\nabla(x \wedge z) \to \nabla(x \wedge y \wedge z)))\\ &= (x \to y) \to (\nabla(x \wedge z) \to (\nabla(x \wedge y) \to \nabla(x \wedge y \wedge z))).\end{aligned}$$

By (1), (2), A12 and A16 we get A24. □

Corollary 2.1 *The relation $R(D)$ of Theorem 2.1 (i) is an equivalence relation on A.*

Proof. We only prove the transitivity of $R(D)$.
Suppose that

(1) $(x, y) \in R(D)$, (2) $(y, z) \in R(D)$,

then by A21 it is easy to see that

(3) $x \rightarrow z \in D$, (4) $z \rightarrow x \in D$.

From (1) (h3)

(5) $\nabla x \rightarrow \nabla(x \wedge y) \in D$.

From A24 and (2) (h1) we obtain

$$\nabla(y \wedge z) \rightarrow (\nabla(x \wedge y) \rightarrow \nabla(x \wedge z)) \in D,$$

then by Remark 2.1 (i) and successive application of A2 we get

$$(\nabla y \rightarrow \nabla(y \wedge z)) \rightarrow ((\nabla y \rightarrow \nabla(x \wedge y)) \rightarrow (\nabla y \rightarrow \nabla(x \wedge z))) \in D,$$

hence from (2) (h3) and (1) (h4)

$$\nabla y \rightarrow \nabla(x \wedge z) \in D,$$

and so, by Remark 2.1 (i) and A2 we have

(6) $(\nabla x \rightarrow \nabla y) \rightarrow (\nabla x \rightarrow \nabla(x \wedge z)) \in D$.

From A23, Remark 2.1 (i) and A2

$$(\nabla x \rightarrow \nabla(x \wedge y)) \rightarrow (\nabla x \rightarrow \nabla y) \in D,$$

and so by (5) we obtain

(7) $\nabla x \rightarrow \nabla y \in D$.

From (6) and (7)

(8) $\nabla x \to \nabla(x \wedge z) \in D$.

Similar arguments show that

(9) $\nabla z \to \nabla(x \wedge z) \in D$.

Finally, from (3), (4), (8) and (9) we have that $(x, z) \in R(D)$. □

Lemma 2.2 *The following identities are verified in* IMS.

(A25) $\nabla\nabla x = \nabla x$,

(A26) $(x \to y) \to ((z \to x) \to (z \to y)) = 1$,

(A27) $(y \to x) \to ((\nabla x \to \nabla y) \to (\nabla(x \to z) \to \nabla(y \to z))) = 1$,

(A28) $\nabla z \to \nabla((x \to z) \wedge (y \to z)) = 1$,

(A29) $x \to y = 1$, $\nabla x \to \nabla(x \wedge y) = 1$ imply $x \leq y$,

(A30) $\nabla((x \to z) \wedge (y \to z)) = \nabla(x \to z) \wedge \nabla(y \to z)$,

(A31) $(x \to y) \to ((\nabla x \to \nabla(x \wedge y)) \to (\nabla(x \wedge z) \to \nabla(x \wedge z \wedge y))) = 1$.

Proof.

(A27) $(y \to x) \to ((\nabla x \to \nabla y) \to (\nabla(x \to z) \to \nabla(y \to z)))$

$= (y \to x) \to ((\nabla x \to \nabla y) \to ((x \to \nabla z) \to (y \to \nabla z)))$ [A9]

$= (\nabla x \to \nabla y) \to ((y \to x) \to (y \to ((x \to \nabla z) \to \nabla z)))$ [A16]

$= (\nabla x \to \nabla y) \to (y \to (x \to ((x \to \nabla z) \to \nabla z)))$ [A2]

$= (\nabla x \to \nabla y) \to (y \to ((x \to \nabla z) \to (x \to \nabla z)))$ [A16]

$= 1.$ [A1, A13]

(A29) It is a consequence of A10 and A12.

(A30) From A17 we have

(1) $(x \to z) \wedge (y \to z) \leq \nabla(x \to z) \wedge \nabla(y \to z)$,

and then

$$
\begin{aligned}
(2)\quad & \nabla((x \to z) \wedge (y \to z)) \leq \nabla(\nabla(x \to z) \wedge \nabla(y \to z)) && [\text{A22}] \\
& = \nabla\nabla(x \to z) \wedge \nabla(y \to z) && [\text{A8}] \\
& = \nabla(x \to z) \wedge \nabla(y \to z). && [\text{A25}]
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(3)\quad & (\nabla(x \to z) \wedge \nabla(y \to z)) \to \nabla((x \to z) \wedge (y \to z)) && \\
& = \nabla(x \to z) \to (\nabla(y \to z) \to \nabla((x \to z) \wedge (y \to z))) && [\text{A4}] \\
& = \nabla((x \to \nabla z) \to ((y \to \nabla z) \to ((x \to z) \wedge (y \to z))) && [\text{A9}] \\
& = \nabla(((x \to \nabla z) \to ((y \to \nabla z) \to (x \to z))) && \\
& \quad \wedge((x \to \nabla z) \to ((y \to \nabla z) \to (y \to z)))) && [\text{A5}] \\
& = \nabla(((y \to \nabla z) \to ((x \to \nabla z) \to (x \to z))) && \\
& \quad \wedge((x \to \nabla z) \to ((y \to \nabla z) \to (y \to z)))) && [\text{A16}] \\
& = \nabla((\nabla z \to ((y \to \nabla z) \to (x \to z))) && \\
& \quad \wedge(\nabla z \to ((x \to \nabla z) \to (y \to z)))) && [\text{A2, A16}] \\
& = \nabla((\nabla z \to (x \to z)) \wedge (\nabla z \to (y \to z))) && [\text{A2, A12, A14}] \\
& = \nabla(\nabla z \to ((x \to z) \wedge (y \to z))) && [\text{A5}] \\
& = \nabla z \to \nabla((x \to z) \wedge (y \to z)) && [\text{A9}] \\
& = 1. && [\text{A28}]
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
(4)\quad & \nabla(\nabla(x \to z) \wedge \nabla(y \to z)) \to && \\
& \quad \nabla((\nabla(x \to z) \wedge \nabla(y \to z)) \wedge \nabla((x \to z) \wedge (y \to z))) && \\
& = \nabla(\nabla(x \to z) \wedge \nabla(y \to z)) \to \nabla(\nabla((x \to z) \wedge (y \to z))) && [(2)] \\
& = ((\nabla x \to z) \wedge \nabla(y \to z)) \to \nabla((x \to z) \wedge (y \to z)) && [\text{A8, A25}] \\
& = 1. && [(3)]
\end{aligned}
$$

From (3), (4) and A29 it follows that

$$
(5)\quad \nabla(x \to z) \wedge \nabla(y \to z) \leq \nabla((x \to z) \wedge (y \to z)).
$$

Hence, by (2) and (5) we obtain A30.

(A31) $1 = (x \to y) \to (\nabla(x \wedge y) \to (\nabla((x \wedge z) \wedge x) \to \nabla((x \wedge z) \wedge y)))$ [A24]

$= (x \to y) \to (\nabla(x \wedge z) \to (\nabla(x \wedge y) \to \nabla(x \wedge z \wedge y)))$ [A16]

$= (x \to y) \to ((\nabla(x \wedge z) \to \nabla(x \wedge y)) \to$

$(\nabla(x \wedge z) \to \nabla(x \wedge z \wedge y)))$ [A2]

$= (x \to y) \to ((1 \to (\nabla(x \wedge z) \to \nabla(x \wedge y))) \to$

$(\nabla(x \wedge z) \to \nabla(x \wedge z \wedge y)))$ [A12]

$= (x \to y) \to (((\nabla(x \wedge z) \to \nabla x) \to (\nabla(x \wedge z) \to \nabla(x \wedge y))) \to$

$(\nabla(x \wedge z) \to \nabla(x \wedge z \wedge y)))$ [A23]

$= (x \to y) \to (\nabla(x \wedge z) \to ((\nabla x \to \nabla(x \wedge y)) \to \nabla(x \wedge z \wedge y)))$ [A2]

$= (x \to y) \to ((\nabla x \to \nabla(x \wedge y)) \to (\nabla(x \wedge z) \to \nabla(x \wedge z \wedge y)))$. [A16] □

Corollary 2.2 *$R(D)$ is compatible with ∇.*

Proof. Suppose that

(1) $(x, y) \in R(D)$.

From A2, A23 and A13 we have

$$(\nabla x \to \nabla(x \wedge y)) \to (\nabla x \to \nabla y) \in D,$$

and then by (1) (h3)

(2) $\nabla x \to \nabla y \in D$.

Similarly, we obtain

(3) $\nabla y \to \nabla x \in D$.

On the other hand,

$$\nabla\nabla x \to \nabla(\nabla x \wedge \nabla y) = \nabla x \to (\nabla x \wedge \nabla y) \quad \text{[A8, A25]}$$

$$= (\nabla x \to \nabla y) \wedge (\nabla x \to \nabla x) \quad [A5]$$

$$= \nabla x \to \nabla y. \quad [A1]$$

therefore by (2) it follows that

(4) $\nabla\nabla x \to \nabla(\nabla x \wedge \nabla y) \in D$.

Analogously, we get

(5) $\nabla\nabla y \to \nabla(\nabla x \wedge \nabla y) \in D$. □

Corollary 2.3 *$R(D)$ is compatible with* $\to$.

Proof. We shall prove that if

(1) $(x, y) \in R(D)$, then

(i) $(x \to z,\ y \to z) \in R(D)$,
(ii) $(z \to x,\ z \to y) \in R(D)$,

for all $z \in A$.

(i) : From A21 and (1) (h1) we have

(2) $(y \to z) \to (x \to z) \in D$.

Similarly, we obtain

(3) $(x \to z) \to (y \to z) \in D$.

On the other hand, from A23, A13, A2 and (1) (h3)

$$\nabla x \to \nabla y \in D,$$

then taking into account A27 and (1) (h2)

$$\nabla(x \to z) \to \nabla(y \to z) \in D,$$

so by A30, A5 and A1 we have

(4) $\nabla(x \to z) \to \nabla((x \to z) \wedge (y \to z)) \in D$.

In the same way we obtain

(5) $\nabla(y \to z) \to \nabla((y \to z) \wedge (x \to z)) \in D$.

(ii) : By A26 and (1) (h2)

(6) $(z \to x) \to (z \to y) \in D$.

Analogously, we get

(7) $(z \to y) \to (z \to x) \in D$.

From (1) (h4), Remark 2.1 (i), A2, A9 and A5 we have

(8) $\nabla(z \to x) \to \nabla((z \to x) \wedge (z \to y)) \in D$.

Similar arguments show that

(9) $\nabla(z \to y) \to \nabla((z \to x) \wedge (z \to y)) \in D$. □

Corollary 2.4 *$R(D)$ is compatible with $\wedge$.*

Proof. We are going to demonstrate that if

(1) $(x, y) \in R(D)$, then

(i) $(x \wedge z,\ y \wedge z) \in R(D)$,
(ii) $(z \wedge x,\ z \wedge y) \in R(D)$,

for all $z \in A$.

(i) : Taking into account A5, A4, A16, (1) (h1) and Remark 2.1 (i) we have

(1) $(x \wedge z) \to (y \wedge z) \in D$.

Similarly, we have

(2) $(y \wedge z) \to (x \wedge z) \in D$.

On the other hand, by A31, (1) (h2) and (h3)

(3) $\nabla(x \wedge z) \rightarrow \nabla((x \wedge z) \wedge (y \wedge z)) \in D$.

Analogously, we have

(4) $\nabla(y \wedge z) \rightarrow \nabla((x \wedge z) \wedge (y \wedge z)) \in D$.

(ii) : It is a consequence of (i) and the commutativity of $\wedge$. □

Corollary 2.5 $[1]_{R(D)} = D$.

Proof. It is a consequence of A1, A12, A13, A15 and Remark 2.1. □

Finally, since it is not difficult to check Theorem 2.1 (ii) we omit the proof.

3 Simple ims–algebras

Our next task will be to characterize the simple algebras.

It is easy to see that the family of all deductive systems of an ims–algebra A ordered by set inclusion, is upper inductive. Then by Zorn's Lemma any proper d.s. of A is contained in a maximal d.s..

From [9] taking into account that ims–congruences of an algebra A are determined by the d.s. of A and the operator $\rightarrow$ verifies A14, A18, A19 and A12, we can state that any proper d.s. of A is the intersection of maximal deductive systems of A. From here, $\{1\}$ is the intersection of all maximals d.s. of A. Then, by a well known result of universal algebra this variety is semisimple, i.e. any non trivial ims–algebra is a subdirect product of simple ims–algebras.

On the other hand, if $A \in$ **IMS** by a Birkhoff's theorem [1], we have that

(1) A is simple iff $\{1\}$ is the unique maximal proper d.s. of A.

(2) $D \in \mathcal{D}(A)$ is maximal iff A/D is simple.

The main result of this section is the following theorem

Theorem 3.1 *Let A be a non trivial ims–algebra. Then the following conditions are equivalent:*

(i) *A is simple,*

(ii) *The operations on A are defined as follows:*

(a) $x \wedge y = 0$ *for all* $x, y \in A \setminus \{0,1\}$, $x \neq y$,

(b) $x \rightarrow y = 1$ *for all* $x, y \in A$, $x \neq 1$,

(c) $\nabla x = 1$ *for all* $x \in A$, $x \neq 0$.

Proof. (i) $\Longrightarrow$ (ii): Note first that (1) $x \rightarrow y = 0$ implies $x = 1$, $y = 0$. Indeed, by A6 $y = (x \rightarrow y) \wedge y = 0 \wedge y = 0$. On the other hand, taking into account Remark 2.1 (ii), if we consider $[x)$, by (i) we have that $[x) = A$ or $[x) = \{1\}$. If $[x) = A$, then $x = 0$ and so $x \rightarrow y = 1$ which contradicts the hypothesis. Therefore, $[x) = \{1\}$ and hence $x = 1$.

(a) Let $x, y \in A \setminus \{0,1\}$, $x \neq y$ and suppose that $x \wedge y \neq 0$. Then $[x \wedge y) \neq A$ and so by (i) $[x \wedge y) = \{1\}$. Therefore, $x = y = 1$, a contradiction.

(b) Let $x, y \in A$, $x \neq 1$ and suppose that $x \rightarrow y \neq 1$. Then by (i) $[x \rightarrow y) = A$, and so $x \rightarrow y = 0$. Hence by (1) $x = 1$, a contradiction.

(c) Let $x \in A$, $x \neq 0$. If $[\nabla x) = A$, then $x = 0$ which contradicts the hypothesis. Therefore, $[\nabla x) = \{1\}$ and so $\nabla x = 1$.

(ii) $\Longrightarrow$ (i): Let $D \in \mathcal{D}(A)$ and $D \neq A$. Suppose that $x \in D$ and $x \neq 1$. By (b) $x \rightarrow 0 \in D$ and so $0 \in D$, which is a contradiction. □

Remark 3.1 *If $A \in$ **IMS** is simple, then by Theorem 3.1 we have that $(A, \leq)$ is an ordered set where $\leq = Id_A \cup \{(0,1)\} \cup \{(0,a),(a,1)\}_{a \in A \setminus \{0,1\}}$.*

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