PARACYCLIC COMPLEXES ARISING FROM α-DERIVATIONS

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1. INTRODUCTION

Let A be a k-algebra with 1, $\alpha: A \to_{\downarrow} A$ an automorphism of algebras. In [R-S(2)] we describe a construction of the graded α-differential algebra $\Omega_k^\alpha(A)$. Now we define Karoubi's operator κ for α-differential non-commutative forms, and study some of its properties.

This operator allow us to construct a parachain complex for $\Omega_k^\alpha(A)$. This may be the first step to get a mixed complex for $\Omega_k^\alpha(A)$, in order to define the α-cyclic homology of A.

2. PARACHAIN COMPLEXES.

Definition 2.1. *A parachain complex (V, b, B) is a graded k-module $V = \bigoplus_{i \in N} V_i$ with two operators $b: V_i \to V_{i-1}$, $B: V_i \to V_{i+1}$ such that*

(1) $b^2 = B^2 = 0$
(2) *the operator $T = 1 - (bB + Bb)$ is invertible.*

It may be easily checked that T commutes with b and B. When T is the identity, the two differentials b and B commute. Such a parachain complex is called a *mixed complex*.

Example 2.2.
Let $(\overline{C}_*(A), b)$ be the normalized Hochschild complex given by $\overline{C}_n(A) = A^{\otimes(n+1)}/D_n$, where D_n is spanned by the elements $a_0 \otimes \cdots \otimes a_n$ such that $a_i = 1$ for some i with $1 \leq i \leq n$, and

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}$$

Let $B: \overline{C}_n(A) \to \overline{C}_{n+1}(A)$ be the operator given by

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{in} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}$$

Now $(\overline{C}_*(A), b, B)$ is a mixed complex, and the cyclic homology of A is

$$HC_*(A) = H_*(Tot(\overline{C}_*(A), b, B))$$

Definition 2.3. *A bi-parachain complex is a N^2-graded k-module $\bigoplus_{(i,j) \in N^2} V_{i,j}$ with operators $b: V_{i,j} \to V_{i-1,j}$, $\overline{b}: V_{i,j} \to V_{i,j-1}$ $B: V_{i,j} \to V_{i+1,j}$, $\overline{B}: V_{i,j} \to V_{i,j+1}$ such that*

(1) $b^2 = \overline{b}^2 = B^2 = \overline{B}^2 = 0$
(2) *the operators $T = 1 - (bB + Bb)$ and $\overline{T} = 1 - (\overline{b}\overline{B} + \overline{B}\overline{b})$ are invertible*
(3) *b and B commute in the graded sense with $\overline{b}$ and $\overline{B}$.*

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Proposition 2.4. *There is a functor $V \to Tot(V)$ from bi-parachain complexes to parachain complexes, where $Tot(V)$ is:*

$$Tot_n(V) = \sum_{i+j=n} V_{i,j}$$

$$Tot(b) = b + \overline{b}, \qquad Tot(B) = \overline{B} + \overline{T}B \qquad \text{and} \qquad Tot(T) = T\overline{T}$$

So, when $Tot(T) = 1$, $(Tot(V), Tot(b), Tot(B))$ is a mixed complex.

Proof. It follows immediately that $Tot(b)^2 = Tot(B)^2 = 0$. Now,

$$\begin{aligned} Tot(T) &= 1 - (Tot(b)Tot(B) + Tot(B)Tot(b)) \\ &= 1 - (b + \overline{b})(\overline{B} + \overline{T}B) - (\overline{B} + \overline{T}B)(b + \overline{b}) \\ &= 1 - (bB + Bb)\overline{T} - (\overline{b}\overline{B} + \overline{B}\overline{b}) \\ &= 1 - (1 - T)\overline{T} - (1 - \overline{T}) \\ &= T\overline{T}. \quad \square \end{aligned}$$

The definition and proposition above may be generalized, getting multi-parachain complexes, and a functor from multi-parachain complexes to parachain complexes.

So, as the above proposition shows, the construction of parachain complexes may be the first step to get mixed complexes.

Example 2.5.
Let A be a k-algebra, and G a finite group acting on A by automorphisms.
Take $V_{p,q} = k[G^{p+1}] \otimes A^{\otimes q+1}$. Define the operators:

$$\begin{aligned} d_i &: V_{p,q} \to V_{p,q-1} & \overline{d}_i &: V_{p,q} \to V_{p-1,q} \\ t &: V_{p,q} \to V_{p,q} & \overline{t} &: V_{p,q} \to V_{p,q} \\ s &: V_{p,q} \to V_{p,q+1} & \overline{s} &: V_{p,q} \to V_{p+1,q} \end{aligned}$$

respectively by:

$$\begin{aligned} d_i(g_0, \ldots, g_p; a_0, \ldots, a_q) &= (g_0, \ldots, g_p; a_0, \ldots, a_i a_{i+1}, \ldots, a_q) \quad (0 \le i \le q-1) \\ d_q(g_0, \ldots, g_p; a_0, \ldots, a_q) &= (g_0, \ldots, g_p; ((g_0 g_1 \cdots g_p)^{-1} a_q) a_0, \ldots, , a_{q-1}) \\ \overline{d}_i(g_0, \ldots, g_p; a_0, \ldots, a_q) &= (g_0, \ldots, g_i g_{i+1}, \ldots, g_p; a_0, \ldots, a_q) \quad (0 \le i \le p-1) \\ \overline{d}_p(g_0, \ldots, g_p; a_0, \ldots, a_q) &= (g_p g_0, \ldots, g_{p-1}; g_p(a_0), \ldots, g_p(a_q)) \\ t(g_0, \ldots, g_p; a_0, \ldots, a_q) &= (g_0, \ldots, g_p; (g_0 g_1 \cdots g_p)^{-1}(a_q), a_0, \ldots, a_{q-1}) \\ \overline{t}(g_0, \ldots, g_p; a_0, \ldots, a_q) &= (g_p, g_0, \ldots, g_{p-1}; g_p(a_0), \ldots, g_p(a_q)) \\ s(g_0, \ldots, g_p; a_0, \ldots, a_q) &= (g_0, \ldots, g_p; 1, a_0, \ldots, a_q) \\ \overline{s}(g_0, \ldots, g_p; a_0, \ldots, a_q) &= (1, g_0, \ldots, g_p; a_0, \ldots, a_q) \end{aligned}$$

Take $b = \sum_{i=0}^{q} d_i$, $\overline{b} = \sum_{i=0}^{p} \overline{d}_i$, $B = (1-t)s\sum_{i=0}^{q}(-1)^{iq}t^i$, $\overline{B} = (1-\overline{t})\overline{s}\sum_{i=0}^{p}(-1)^{ip}\overline{t}^i$, $T = t^{p+1}$ and $\overline{T} = \overline{t}^{q+1}$.

Then $(V, b, B, \overline{b}, \overline{B})$ is a bi-parachain complex.

As $T\overline{T} = 1$, in this case we have that $(Tot(V), Tot(b), Tot(B))$ is a mixed complex.

3.α-DIFFERENTIAL FORMS AND THE KAROUBI OPERATOR κ.

Let A be an associative k-algebra with 1 and $\alpha: A \to A$ an automorphism of algebras. An α-derivation of A into an A-bimodule M is a k-linear map, $d_\alpha: A \to M$, such that

$$d_\alpha(ab) = \alpha(a)d_\alpha(b) + d_\alpha(a)b \qquad \text{for } a, b \in A.$$

In [R-S(2)] we describe the construction of $\Omega_k^\alpha(A)$. A and $A \otimes A^{op}$ are considered A-bimodules with the structures defined respectively by $a \circ x \circ b = ax\alpha(b)$, and $a \circ (x \otimes y) \circ b = ax \otimes yb$. Now $\Omega_k^\alpha(A) = I_\alpha = \mathrm{Ker}(A \otimes A^{op} \xrightarrow{\mu_\alpha} A)$, where $\mu_\alpha(a \otimes b) = a\alpha(b)$, and $d_\alpha: A \to \Omega_k^\alpha(A)$ is defined by $d_\alpha(a) = 1 \otimes a - \alpha(a) \otimes 1$. $\Omega_k^\alpha(A)$ is an A-bimodule, and μ_α is a morphism of A-bimodules.

The pair $(d_\alpha, \Omega_k^\alpha(A))$ is characterized by the following universal property:
Let δ be an α-derivation of A with values in an A-bimodule M, then there exists a unique homomorphism of bimodules $i_\delta: \Omega_k^\alpha(A) \to M$ such that $\delta = i_\delta \circ d_\alpha$.

Setting $\Omega^{0,\alpha}(A) = A$, $\Omega^{1,\alpha}(A) = \Omega_k^\alpha(A)$, and $\Omega^{n,\alpha}(A) = \Omega^{1,\alpha}(A) \otimes_A \cdots \otimes_A \Omega^{1,\alpha}(A)$, $\Omega^\alpha(A) = \bigoplus \Omega^{n,\alpha}(A)$ is naturally a graded algebra, and there is a unique α-differential d_α on $\Omega^\alpha(A)$ extending the derivation $d_\alpha: \Omega^{0,\alpha}(A) \to \Omega^{1,\alpha}(A)$. The graded α-differential algebra $\Omega^\alpha(A)$ is characterized by the following universal property:
Let $\phi: A \to \Omega'$ be an homomorphism of algebras with units where Ω' is a graded α-differential algebra, then there is a unique homomorphism of graded α-differential algebras $\widehat{\phi}: \Omega^\alpha(A) \to \Omega'$ which extends ϕ.

Lemma 3.1. *The map $x \otimes \overline{y} \to x \otimes y - x\alpha(y) \otimes 1$ is an isomorphism of left A-modules*

$$A \otimes \overline{A} \xrightarrow{\cong} \Omega_k^\alpha(A)$$

Proof. One first remarks that $x \otimes y - x\alpha(y) \otimes 1$ depends only on the class of y in $\overline{A}$, so the map is well defined, and its image is in I_α. The quotient of $A \otimes A^{op}$ by the relations $x \otimes y - x\alpha(y) \otimes 1$ maps isomorphically to A (with inverse map given by $x \to$ class of $x \otimes 1$). Therefore the kernel of this factor map is isomorphic to the kernel of μ_α. □

Let us introduce the following usual notation: $x \otimes \overline{y}$ (or equivalently $x \otimes y - x\alpha(y) \otimes 1$) is written $xd_\alpha(y)$.
I_α is an A-bimodule because it is a sub-A-bimodule of $A \otimes A^{op}$. So, by the isomorphism shown in the previous Lemma, $A \otimes \overline{A}$ becomes an A-bimodule. The left module structure is simply

$$a(xd_\alpha(y)) = axd_\alpha(y)$$

The right module structure is

$$(xd_\alpha(y))b = xd_\alpha(yb) - x\alpha(y)d_\alpha(b)$$

because, in I_α,

$$\begin{aligned}(x \otimes y - x\alpha(y) \otimes 1)b &= x \otimes yb - x\alpha(y) \otimes b \\ &= (x \otimes yb - x\alpha(yb) \otimes 1) - (x\alpha(y) \otimes b - x\alpha(yb) \otimes 1)\end{aligned}$$

So we have the classical formula

$$d_\alpha(yb) = \alpha(y)d_\alpha(b) + d_\alpha(y)b$$

Now,

$$\Omega^{n,\alpha}(A) = I_\alpha \otimes_A \cdots \otimes_A I_\alpha = A \otimes \overline{A}^n$$

with the identification

$$a_0 d_\alpha(a_1)\dots d_\alpha(a_n) = a_0 \otimes \cdots \otimes a_n$$

and the operator $d_\alpha\colon \Omega^{n,\alpha}(A) \to \Omega^{n+1,\alpha}(A)$ is given by

$$d_\alpha(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n$$

At this time it is interesting to remark a difference with the case A commutative and $\alpha = id$, where the cohomology of the complex $\Omega^*(A), d)$ is trivial (by Poincaré's Lemma). When $\alpha \neq id$ this fact not necessarily holds, take for example $A = k[t]$ where k is a field, $\mathrm{car}(k) = 0$, q an n-root of 1.

The product in the α-differential graded algebra $(\Omega^\alpha(A), d_\alpha)$ is performed by using the rules of d_α, for instance,

$$\begin{aligned}(1 \otimes x)(y \otimes z) &= d_\alpha(x)(y d_\alpha(z)) = (d_\alpha(x) y) d_\alpha(z)\\ &= (d_\alpha(xy) - \alpha(x) d_\alpha(y)) d_\alpha(z)\\ &= 1 \otimes xy \otimes z - \alpha(x) \otimes y \otimes z\end{aligned}$$

Now we will use the identification $\Omega^{n,\alpha}(A) = A \otimes \overline{A}^n$, so the α-Hochschild homology $HH_{\alpha,*}(A) = HH_{\alpha,*}(A,A)$ is the homology of the complex

$$\dots \to \Omega^{2,\alpha}(A) \xrightarrow{b} \Omega^{1,\alpha}(A) \xrightarrow{b} A \to 0$$

with

$$\begin{aligned}&b(a_0 \otimes a_1 \otimes \cdots \otimes a_n)\\ &= (a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i (a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)\\ &+ (-1)^n (\alpha(a_n) a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})\end{aligned}$$

Let the Karoubi operator $\kappa\colon \Omega^\alpha(A) \to \Omega^\alpha(A)$ be the degree zero operator given by

$$\begin{aligned}&\kappa(\omega d_\alpha(a_n)) = \kappa(a_0 d_\alpha(a_1)\dots d_\alpha(a_n))\\ &= (-1)^{|\omega|}\left(d_\alpha(a_n)\omega + d_\alpha\left((\alpha - id)(a_n) a_0\right) d_\alpha(a_1)\dots d_\alpha(a_{n-1})\right)\\ &= (-1)^{n+1}\left(d_\alpha(\alpha(a_n) a_0) d_\alpha(a_1)\dots d_\alpha(a_{n-1}) - \alpha(a_n) d_\alpha(a_0) d_\alpha(a_1)\dots d_\alpha(a_{n-1})\right)\end{aligned}$$

Lemma 3.2.

(1) $b d_\alpha + d_\alpha b = 1 - \kappa$
(2) $b\kappa = \kappa b$ *and* $d_\alpha \kappa = \kappa d_\alpha$

Proof. (1) follows easily by direct computation, and (2) follows immediately by (1). □

Let us define $\alpha\colon \Omega^{n,\alpha}(A) \to \Omega^{n,\alpha}(A)$ by

$$\alpha(a_0 d_\alpha(a_1)\dots d_\alpha(a_n)) = \alpha(a_0) d_\alpha(\alpha(a_1))\dots d_\alpha(\alpha(a_n))$$

Lemma 3.3. *On $\Omega^{n,\alpha}(A)$, we have the identities:*

(1) $\kappa^{n+1} d_\alpha = d_\alpha\ \alpha$
(2) $\kappa^n = \alpha + b\kappa^n d_\alpha$
(3) $\kappa^{n+1} = \alpha - d_\alpha b\alpha$
(4) κ *is invertible*

Proof.

(1) Using the identification $\Omega^{n,\alpha}(A) = A \otimes \overline{A}^n$, we have

$$\begin{aligned}\kappa(a_0 \otimes \cdots \otimes a_{n+1}) = (-1)^{n+1}(\alpha(a_{n+1}) \otimes a_0 \otimes \cdots \otimes a_n)\\ + (-1)^n(1 \otimes \alpha(a_{n+1})a_0 \otimes a_1 \otimes \cdots \otimes a_n)\end{aligned}$$

Now,

$$\kappa(1 \otimes a_0 \otimes \cdots \otimes a_n) = (-1)^n(1 \otimes \alpha(a_n) \otimes a_0 \otimes \cdots \otimes a_{n-1})$$

showing that $\kappa^{n+1} d_\alpha = d_\alpha\ \alpha$ on $\Omega^{n,\alpha}(A)$.

(2) A direct computation shows that

$$\begin{aligned}\kappa^n d_\alpha(a_0 \otimes \cdots \otimes a_n) &= \kappa^n(1 \otimes a_0 \otimes \cdots \otimes a_n)\\ &= (-1)^n(1 \otimes \alpha(a_1) \otimes \cdots \otimes \alpha(a_n) \otimes a_0)\end{aligned}$$

and

$$\begin{aligned}\kappa^n(a_0 \otimes \cdots \otimes a_n) = (-1)^n[&(\alpha(a_1) \otimes \cdots \otimes \alpha(a_n) \otimes a_0)\\ &+ \sum_{i=1}^{n-1}(-1)^i(1 \otimes \alpha(a_1) \otimes \cdots \otimes \alpha(a_i a_{i+1}) \otimes \cdots \otimes \alpha(a_n) \otimes a_0)\\ &+ (-1)^n(1 \otimes \alpha(a_1) \otimes \cdots \otimes \alpha(a_n)a_0)]\end{aligned}$$

Now, it is immediate that $\kappa^n = \alpha + b\kappa^n d_\alpha$.

(3) By (1) and (2), we have that

$$\begin{aligned}\kappa^{n+1} &= \kappa\ \kappa^n = \kappa(\alpha + b\kappa^n d_\alpha)\\ &= \kappa\ \alpha + b\kappa^{n+1} d_\alpha = (\kappa + bd_\alpha)\ \alpha\\ &= (1 - d_\alpha b)\ \alpha\end{aligned}$$

(4) The polynomial $(\kappa^n - \alpha)(\kappa^{n+1} - \alpha)$ has constant term α^2, which is invertible, and

$$(\kappa^n - \alpha)(\kappa^{n+1} - \alpha) = (b\kappa^n d_\alpha)(-d_\alpha b\ \alpha) = 0$$

So κ is invertible. $\square$

We define the Connes operator B on $\Omega^{n,\alpha}(A)$ by

$$B = \sum_{j=0}^{n} \kappa^j d_\alpha$$

Proposition 3.4. *$(\Omega^\alpha(A), B, b)$ is a "parachain complex" (see [G-J]).*

Proof. We can compute $\kappa^{n(n+1)}$ in two ways. First, using (2) and (1), we have

$$\kappa^{n(n+1)} - \alpha^{n+1} = \sum_{j=0}^{n} \alpha^{n-j}\kappa^{nj}(\kappa^n - \alpha) = \sum_{j=0}^{n} \alpha^{n-j} b\kappa^{nj+n} d_\alpha$$
$$= \alpha^n bB$$

On the other hand, using (3) and (1), we have

$$\kappa^{n(n+1)} - \alpha^n = \sum_{j=0}^{n-1} \alpha^{n-1-j}\kappa^{(n+1)j}(\kappa^{n+1} - \alpha) = -\sum_{j=0}^{n-1} \alpha^{n-1-j}\kappa^{(n+1)j} d_\alpha b\ \alpha$$
$$= -\alpha^n Bb$$

Thus we obtain

$$\kappa^{n(n+1)} = \alpha^{n+1} + \alpha^n bB = \alpha^n - \alpha^n Bb$$

So

$$bB + Bb = 1 - \alpha \quad \square$$

Remark. The above proposition is a technical result. However, in some cases (for example if α is given by the action of a group on A), we'll have the possibility of constructing a bi-parachain complex V such that $(Tot(V), Tot(b), Tot(B))$ is a mixed complex.

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