A Survey On A Certain Unsymmetric Fractional Operator.

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Abstract

We'll construct a testing function space in order to analize distributionally some fractional operator in concomitancy with the unsymmetric Lowndes operator of Kober type. This kind of research was suggested by Adam McBride in $[McB\&R]$.

§1 Introduction.

Throughout this article we assume that $\kappa > 0$, $\alpha > 0$. Moreover, if ξ and σ are reals for x positive we'll write

$$\mathfrak{I}_\kappa^\sigma(\xi,\alpha)\psi(x) = \left(\frac{\kappa}{2}\right)^{1-\alpha} x^{-\alpha-\xi-\sigma}\int_0^x t^{\xi+\sigma}\left(\frac{x-t}{t}\right)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\kappa\sqrt{xt-t^2}\right)\psi(t)dt \tag{1}$$

and

$$K_\kappa(\xi,\alpha)\phi(x) = \left(\frac{\kappa}{2}\right)^{1-\alpha} x^{\xi}\int_x^\infty t^{-\alpha-\xi}\left(\frac{t-x}{x}\right)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\kappa\sqrt{xt-x^2}\right)\phi(t)dt, \tag{2}$$

wherein we assume that ψ and ϕ are functions so that (1) and (2) have a definite meaning and by $J_{\alpha-1}$ we denote the Bessel function of the first kind. The resulting fractional operator in (2) is the well known unsymmetric Lowndes one of Kober type.

The operator (2) was introduced by J. S. Lowndes [L], providing a powerful tool with broad applications in the resolution of some partial differential equations, for instance the generalized biaxially symmetric potential equation, the generalized axially symmetric potential equation in $n+1$ variables, etc.. We remember that the Kober fractional operator is given by

$$K^{\xi,\alpha} = x^\xi \circ W^\alpha \circ x^{-\alpha-\xi}, \tag{3}$$

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where W^α is the classical Weyl integral transform [M], whose general form is

$$W^\nu f(x) = \begin{cases} \int_x^\infty \frac{(t-x)^{\nu-1}}{\Gamma(\nu)} f(t)dt & \text{if } \nu > 0, \\ (-D_x)^n W^{\nu+n} f(x) & \text{if } \nu \leq 0, \ n \in N \text{ and } 0 < \nu + n \leq 1. \end{cases} \tag{4}$$

Let us consider the equation

$$K^{\xi+\alpha,\beta}\phi = \psi, \tag{5}$$

the Lowndes properties

$$K_\kappa(\xi,\alpha) \circ K^{\xi+\alpha,\beta} = K_\kappa(\xi,\alpha+\beta), \tag{6}$$

$$K_\kappa(\xi,\alpha) \circ x^\sigma = x^\sigma \circ K_\kappa(\xi-\sigma,\alpha), \tag{7}$$

and the composition formula

$$K^{\xi,\alpha} \circ K^{\xi+\alpha,\beta} = K^{\xi,\alpha+\beta}. \tag{8}$$

If we write

$$\phi = K^{\xi,-\beta}\psi, \tag{9}$$

from (5), (6), (7) and (9) we find that

$$K_\kappa(\xi,\alpha)\psi = \left(x^\xi \circ K_\kappa(0,\alpha+\beta) \circ W^{-\beta} \circ x^{-\xi+\beta}\right)\psi. \tag{10}$$

The right hand side of equation (10) is defined when $\alpha+\beta > 0$. Hence by taking $\beta = n$, the positive integer for which $0 < \alpha + n \leq 1$, we may extend (2) for $\alpha \leq 0$ by means of

$$K_\kappa(\xi,\alpha)\psi(x) = x^\xi K_\kappa(0,\alpha+n)(-D_x)^n\left(x^{-\xi+n}\psi(x)\right) \tag{11}$$

where as usual by D we denote the current derivation operator.

§2 On Differentiable Properties of $K_\kappa(\xi,\alpha)$.

In this section we introduce a family of homogeneous differential polynomials which arise by simple differentiation. As before, our treatment is purely formal and it will be assumed that all required derivatives and integrals are defined as well as our differentiation under the integral sign arguments are possible. We'll shortly impose conditions that will give a consummate groundwork to the following lines.

Theorem 1 *Let n be a non negative integer. Then*

$$D^n K_\kappa(\xi,\alpha) = \sum_{j=0}^{n} K_\kappa(\xi - j, \alpha + n - j)\, P_{n,j}(x, D) \tag{12}$$

where

$$\begin{aligned} P_{n,0}(x,D) &= \left(-\frac{\kappa^2 x}{2}\right)^n, \\ P_{n,n}(x,D) &= D^n \end{aligned} \tag{13}$$

and

$$P_{n+1,j}(x,D) = -\frac{\kappa^2 x}{2} P_{n,j}(x,D) + D P_{n,j-1}(x,D) \qquad \text{if} \qquad 1 \le j \le n.^1 \tag{14}$$

Proof.

The case $n = 0$ is immediate. Writing (2) in the form

$$K_\kappa(\xi,\alpha)\,\phi(x) = \left(\frac{\kappa}{2}x\right)^{1-\alpha} \int_1^\infty t^{-\alpha-\xi}(t-1)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\kappa x\sqrt{t-1}\right)\phi(tx)dt \tag{15}$$

and making $\theta = \kappa x\sqrt{t-1}$, on using equation (7) we've

$$D_x K_\kappa(\xi,\alpha)\,\phi(x) = 2^{\alpha-1}\int_1^\infty \frac{(t-1)^{\alpha-1}}{t^{\alpha+\xi}} \frac{\partial}{\partial\theta}\left[\theta^{-(\alpha-1)}J_{\alpha-1}(\theta)\right]\frac{\partial\theta}{\partial x}\phi(tx)dt +$$

$$+2^{\alpha-1}\int_1^\infty t^{1-\alpha-\xi}(t-1)^{\alpha-1}\theta^{-(\alpha-1)}J_{\alpha-1}(\theta)\phi^{(1)}(tx)dt \tag{16}$$

$$= -\frac{\kappa^2 x}{2} K_\kappa(\xi-1,\alpha+1)\,\phi(x) + K_\kappa(\xi-1,\alpha)\,\phi^{(1)}(x)$$

$$= K_\kappa(\xi,\alpha+1)\left(-\frac{\kappa^2 x}{2}\phi(x)\right) + K_\kappa(\xi-1,\alpha)\,\phi^{(1)}(x)$$

and the case $n = 1$ follows.

By an induction hypothesis we assume the validity of the result for values $\le n$. On using (13), (14) and (16) we obtain

$$D^{n+1}K_\kappa(\xi,\alpha) = \sum_{j=0}^{n} D_x K_\kappa(\xi - j, \alpha + n - j)\,\phi_{n,j}(x)$$

[1]For the sake of brevity, We'll sometimes write $P_{n,j}(x,D)\phi = \phi_{n,j}$, for $0 \le j \le n$ and suitable functions ϕ.

$$= \sum_{j=0}^{n} \left[-\frac{\kappa^2 x}{2} K_\kappa (\xi - j - 1, \alpha + n - j + 1) \phi_{n,j}(x) + \right. \tag{17}$$

$$\left. + K_\kappa (\xi - j - 1, \alpha + n - j) \phi_{n,j}^{(1)}(x) \right]$$

$$= -\frac{\kappa^2 x}{2} \sum_{j=0}^{n} K_\kappa (\xi - j - 1, \alpha + n - j + 1) \phi_{n,j}(x) +$$

$$+ K_\kappa (\xi - n - 1, \alpha) \phi^{(n+1)}(x)$$

$$+ \sum_{j=1}^{n} K_\kappa (\xi - j, \alpha + n - j + 1) \left[P_{n+1,j} (x, D) + \frac{\kappa^2 x}{2} P_{n,j} (x, D) \right] \phi(x).$$

Now we write

$$P_{n+1,0}(x, D) = \left(-\frac{\kappa^2 x}{2} \right)^{n+1} \qquad \text{and} \qquad P_{n+1,n+1}(x, D) = D^{n+1} \tag{18}$$

and by (7) and (17) we have

$$D^{n+1} K_\kappa (\xi, \alpha) = K_\kappa (\xi, \alpha + n + 1) \left(-\frac{\kappa^2 x}{2} P_{n,0} (x, D) \phi(x) \right) +$$

$$+ K_\kappa (\xi - n - 1, \alpha) P_{n+1,n+1}(x, D) \phi(x) +$$

$$+ \sum_{j=1}^{n} K_\kappa (\xi - j, \alpha + n - j + 1) P_{n+1,j} (x, D) \phi(x)$$

$$= \sum_{j=0}^{n=1} K_\kappa (\xi - j, \alpha + n - j + 1) P_{n+1,j} (x, D) \phi(x)$$

and the result follows.□

Corollary 1 *If n and j are non negative integers such that $0 \leq j \leq n$ then $P_{n,j}$ is an homogeneous differential polynomial of degree j and $n - j$ in the variables D and x respectively.*

Corollary 2 *If n and j are non negative integers such that $0 \leq j \leq n$, the $P_{n,j}$'s are the formal sum of all possible mononials of degree j and $n - j$ in the variables D and x respectively.*

Corollary 3 *Let n, j and k be any possible non negative integers such that $0 \le j \le n$ and $0 \le k \le n-j$. Then*

$$P_{n,j} = \sum_{l=1}^{k} P_{k,l}\ P_{n-k,j-l}.$$

Corollary 4 *If n and j are non negative integers such that $0 \le j \le n$ then there are real numbers $c_{j,l}^{n}$'s such that*

$$P_{n,j} = x^{n-2j} \sum_{l=0}^{j} c_{j,l}^{n}\ x^{l}\ D^{l}.$$

Proof.

In the case $n = 0$ we may take $c_{0,0}^{0} = 1$. Let us assume the result for values $\le n$. In particular, by (18) we have

$$c_{0,0}^{n+1} = \left(-\frac{\kappa^2}{2}\right)^{n+1} \qquad \text{and} \qquad c_{n+1,l}^{n+1} = \delta_{n+1,l} \text{ if } 0 \le l \le n+1,$$

where δ is the classical Kronecker's delta symbol. In the case $1 \le j \le n$, on using Corollary 3 and our inductive hypothesis we write

$$P_{n+1,j} = \sum_{l=1}^{n+1-j} P_{n+1-j,l}\ P_{j,j-l}. \tag{19}$$

Now fixing $0 \le l \le n+1-j$ it follows that

$$P_{n+1-j,l}\ P_{j,j-l} = P_{n+1-j,l} \left[x^{-j+2l} \sum_{h=0}^{j-l} c_{j-l,h}^{j}\ x^{h}\ D^{h} \right]$$

$$= x^{n+1-2j} \sum_{h=0}^{j-l} c_{j-l,h}^{j} \sum_{k=0}^{l} c_{l,k}^{n+1-j} \sum_{i=0}^{\min\{k,h+2l-j\}} \binom{k}{i} \frac{(h+2l-j)!}{(h+2l-j-i)!} x^{h+k-i}\ D^{h+k-i}$$

and by (19) the inductive argument follows.□

Theorem 2 *Let n and j be non negative integers such that $j \le n+1$. Then*

$$P_{n+1,j}\ K_{\kappa}(\xi, \alpha) = \sum_{h=0}^{j} a_{n+1,j}^{h}\ K_{\kappa}(\xi + n + 1 - j - h, \alpha + j - h)\, P_{n+1,h} \tag{20}$$

where $a_{1,0}^{0} = a_{l,l}^{h} = 1$ if $0 \le h \le l$, $1 \le l$ and

$$a_{n+1,j}^{h} = \begin{cases} a_{n,j}^{0} + a_{n,j-1}^{0} & \text{if} \quad h = 0,\ 1 \le j \le n, \\ a_{n,j-1}^{h-1} = a_{n,j}^{h} + a_{n,j-1}^{h} & \text{if} \quad 1 \le h \le j-1, \\ a_{n,j}^{j} & \text{if} \quad h = j. \end{cases} \tag{21}$$

Proof.

By (7) and (13) we've

$$P_{1,0}\ K_\kappa(\xi,\alpha) = K_\kappa(\xi+1,\alpha)\ P_{1,0}$$

and hence $a_{1,0}^0 = 1$. By (13) and (16) we obtain $a_{1,1}^0 = a_{1,1}^1 = 1$.

On using (7), (14) and (16) it follows that

$$P_{2,1}\ K_\kappa(\xi,\alpha) = 2\ K_\kappa(\xi+1,\alpha+1)\ P_{2,0} + K_\kappa(\xi,\alpha)\ P_{2,1},$$

i.e. $a_{2,1}^0 = 2$ and $a_{2,1}^1 = 1$. Moreover

$$P_{2,0}\ K_\kappa(\xi,\alpha) = K_\kappa(\xi+2,\alpha)\ P_{2,0},$$

i.e. $a_{2,0}^0 = 1$ and all the required conditions are fulfilled. Now by (7), (14) and assuming the validity of the result for integer values $\leq n$ we may write

$$\begin{aligned} P_{n+1,j}\ K_\kappa(\xi,\alpha) &= \sum_{l=0}^{j} a_{n,j}^l\ K_\kappa(\xi+n-j+1-l,\alpha+j-l)\ P_{1,0}\ P_{n,l} + \\ &\quad + P_{1,1}\sum_{l=0}^{j-1} a_{n,j-1}^l\ K_\kappa(\xi+n-j+1-l,\alpha+j-1-l)\ P_{n,l} \\ &= a_{n,j}^j\ K_\kappa(\xi+n-2j+1,\alpha)\ P_{1,0}\ P_{n,j} + \\ &\quad + \sum_{l=0}^{j-1}\left(a_{n,j}^l + a_{n,j-1}^l\right) K_\kappa(\xi+n-j+1-l,\alpha+j-l)\ P_{1,0}\ P_{n,l} + \\ &\quad + \sum_{l=1}^{j} a_{n,j-1}^{l-1}\ K_\kappa(\xi+n-j-l+1,\alpha+j-l)\ P_{1,1}\ P_{n,l-1} \\ &= a_{n,j}^j\ K_\kappa(\xi+n-2j+1,\alpha)\ P_{1,0}\ P_{n,j} + \\ &\quad + \left(a_{n,j}^0 + a_{n,j-1}^0\right)\ K_\kappa(\xi+n-j+1,\alpha+j)\ P_{1,0}\ P_{n,0} + \\ &\quad + a_{n,j-1}^{j-1}\ K_\kappa(\xi+n-2j+1,\alpha)\ P_{1,1}\ P_{n,j-1} \\ &\quad + \sum_{l=1}^{j-1} a_{n,j-1}^{l-1}\ K_\kappa(\xi+n-j-l+1,\alpha+j-l)\left(P_{1,0}\ P_{n,l} + P_{1,1}\ P_{n,l-1}\right). \end{aligned}$$

Now, in accordance with (13) and (14) besides the fact $a_{n,j}^j = a_{n,j-1}^{j-1}$ we deduce (20) and the theorem is proved.□

§3 On a certain Fréchet Space.

Definition 1 *Let $\gamma \in R$, $p > 0$, n a non negative integer and $\phi \in C^n(R^+)$.*[2] *We'll write*

$$\Phi_n^\gamma(\phi) = \sup_{0 \le j \le n} \left\{ x^{n+j-\gamma} \ |P_{n,j}(x,D)\phi(x)| : \qquad x > 0 \right\},$$

and

$$\mathfrak{L}_{\gamma,p} = \left\{ \phi \in C^\infty\left(R^+\right) : \qquad Supp\left(\phi\right) \subseteq (0,p] \text{ and } \Phi_n^\gamma(\phi) < \infty, \, n = 0, 1, ... \right\}.$$

Remark 1 *$\mathfrak{L}_{\gamma,p}$ has a natural linear structure. Moreover, we'll consider $\mathfrak{L}_{\gamma,p}$ as the multinormed space endowed with the topology generated by the separating family of seminorms $\{\Phi_n^\gamma\}_{n \ge 0}$.*

Theorem 3 *$\mathfrak{L}_{\gamma,p}$ is a Fréchet space.*

Proof.

Let us consider a Cauchy sequence $\{\phi_k\}_{k \ge 1}$. For all $x > 0$ we have

$$x^{-\gamma} \ |\phi_{k+h}(x) - \phi_k(x)| \le \Phi_0^\gamma\left(\phi_{k+h} - \phi_k\right) \tag{22}$$

and hence $\{x^{-\gamma} \ \phi_k(x)\}_{k \ge 1}$ becomes a uniform convergent sequence. By completeness there is a function φ such that $x^{-\gamma} \ \phi_k(x) \overset{\circ}{\to} \varphi(x)$ uniformly in R^+. In particular, φ becomes a continuous function and $Supp(\varphi) \subseteq (0,p]$. Let $\phi(x) = x^\gamma \ \varphi(x)$ for $x > 0$, and let ζ be any given positive number. There exist $k_\zeta \in N$ such that

$$\Phi_0^\gamma\left(\phi_{k+h} - \phi_k\right) \le \zeta \qquad \text{if} \qquad k \ge k_\zeta, \ h \ge 0. \tag{23}$$

On using (22) and (23) we obtain

$$\left| x^{-\gamma} \ \phi_k(x) - \varphi(x) \right| \le \zeta \qquad \text{if} \qquad k \ge k_\zeta, \ x > 0,$$

i.e.

$$x^{-\gamma} \ |\phi_k(x) - \phi(x)| \le \zeta \qquad \text{if} \qquad k \ge k_\zeta, \ x > 0,$$

i.e.

$$\Phi_0^\gamma\left(\phi_k - \phi\right) \le \zeta \qquad \text{if} \qquad k \ge k_\zeta \tag{24}$$

[2] As usual we'll denote through $C^n(R^+)$, with $0 \le j \le \infty$, for the class of functions of a real positive variable which has continous derivatives up to the order n, with values in a same (no specified) Banach space.

and since ζ was arbitrary $\Phi_0^\gamma(\phi_k - \phi) \to 0$. Moreover

$$\begin{aligned} x^{-\gamma} \; |\phi(x)| &\leq x^{-\gamma} \; |\phi_k(x) - \phi(x)| + x^{-\gamma} \; |\phi_k(x)| \\ &\leq \Phi_0^\gamma(\phi_k - \phi) + \Phi_0^\gamma(\phi_k) \end{aligned}$$

and hence it is clear that $\Phi_0^\gamma(\phi) < \infty$. On the other hand, since $\{\Phi_1^\gamma(\phi_k)\}_{k\geq 1}$ is a Cauchy sequence there are continuous functions ψ, υ, both supported in $(0,p]$, such that $x^{2-\gamma}\; \phi_k(x) \overset{\circ}{\to} \psi(x)$ and $x^{2-\gamma}\; \phi_k^{(1)}(x) \overset{\circ}{\to} \upsilon(x)$ uniformly in R^+. Since

$$\begin{aligned} \left|\psi(x) - x^2\; \varphi(x)\right| &\leq \left|\psi(x) - x^{2-\gamma}\; \phi_k(x)\right| + \left|x^{2-\gamma}\; \phi_k(x) - x^2\; \varphi(x)\right| \\ &\leq \left|\psi(x) - x^{2-\gamma}\; \phi_k(x)\right| + p^2\; \left|x^{-\gamma}\; \phi_k(x) - \varphi(x)\right| \end{aligned}$$

we have $\psi(x) = x^{2-\gamma}\; \phi(x)$ in R^+. Next, choose τ_0 and τ such that $0 < \tau_0 < \tau$. Then

$$(2-\gamma) \int_{\tau_0}^{\tau} x^{1-\gamma}\; \phi_k(x)dx \overset{\circ}{\to} \psi(\tau) - \psi(\tau_0) - \int_{\tau_0}^{\tau} \upsilon(x)dx$$

and so $\psi \in C^1(R^+)$. Thus, $\phi \in C^1(R^+)$ and there result

$$(2-\gamma)\; \tau^{1-\gamma}\; \phi(\tau) = \psi^{(1)}(\tau) - \upsilon(\tau),$$

i.e. $\upsilon(\tau) = \tau^{2-\gamma}\; \phi^{(1)}(\tau)$ in R^+. Now, given an $\varepsilon > 0$ there exist $k_\varepsilon \in N$ such that

$$\Phi_1^\gamma(\phi_{k+h} - \phi_k) \leq \zeta \qquad \text{if} \qquad k \geq k_\varepsilon,\; h \geq 0.$$

For all $x > 0$, $k \geq k_\varepsilon$, $h \geq 0$ we deduce that

$$x^{1-\gamma}\; |P_{1,0}(x,D)(\phi_{k+h} - \phi_k)(x)| = \frac{\kappa^2}{2} x^{2-\gamma}\; |\phi_{k+h}(x) - \phi_k(x)| \leq \varepsilon \quad (25)$$

$$x^{2-\gamma}\; |P_{1,1}(x,D)(\phi_{k+h} - \phi_k)(x)| = x^{2-\gamma}\; \left|\phi_{k+h}^{(1)}(x) - \phi_k^{(1)}(x)\right| \leq \varepsilon$$

and letting $h \to \infty$ we have

$$\begin{aligned} x^{1-\gamma}\; |P_{1,0}(x,D)(\phi_k - \phi)(x)| &\leq \varepsilon \\ x^{2-\gamma}\; |P_{1,1}(x,D)(\phi_k - \phi)(x)| &\leq \varepsilon, \end{aligned}$$

i.e.

$$\Phi_1^\gamma(\phi_k - \phi) \leq \zeta \qquad \text{if} \qquad k \geq k_\varepsilon.$$

Furthermore, on using (25) we obtain

$$\max_{x>0}\left\{\frac{\kappa^2}{2}x^{2-\gamma}\,|\phi(x)|\,,x^{2-\gamma}\,\left|\phi^{(1)}(x)\right|\right\}\leq\varepsilon+\Phi_1^\gamma(\phi_{k_\varepsilon})$$

i.e. $\Phi_1^\gamma(\phi)<\infty$. Now, let us assume that $\{\Phi_{n+1}^\gamma(\phi_k)\}_{k\geq 1}$ is a fundamental sequence, $n\geq 1$, and $x^{n+1-\gamma}\,\phi_k^{(j)}(x)\xrightarrow{\circ}x^{n+1-\gamma}\,\phi^{(j)}(x)$ in R^+ if $0\leq j\leq n$.

On using Corollary 4 with $0\leq j\leq n$ we have

$$x^{-\gamma+n+1+j}\,P_{n+1,j}(x,D)(\phi_k-\phi)(x)=x^{-\gamma+2n+2-j}\sum_{h=0}^{j}c_{j,h}^{n+1}\,x^h\,(\phi_k^{(h)}-\phi^{(h)})(x)\xrightarrow{\circ}0. \tag{26}$$

Now, let μ be a continuous function in R^+ such that

$$x^{-\gamma+2n+2}\,\phi_k^{(n+1)}(x)\xrightarrow{\circ}\mu(x). \tag{27}$$

Once again, we choose τ_0 and τ such that $0<\tau_0<\tau$ and hence

$$\int_{\tau_0}^{\tau}x^{-\gamma+2n+2}\,\phi_k^{(n+1)}(x)dx\xrightarrow{\circ}\int_{\tau_0}^{\tau}\mu(x)dx,$$

i.e.

$$\int_{\tau_0}^{\tau}\mu(x)dx=$$

$$=\tau^{-\gamma+2n+2}\,\phi^{(n)}(\tau)-\tau_0^{-\gamma+2n+2}\,\phi^{(n)}(\tau_0)-(-\gamma+2n+2)\int_{\tau_0}^{\tau}x^{-\gamma+2n+1}\,\phi_k^{(n)}(x)dx.$$

We can immediately conclude that $\phi\in C^{n+1}(R^+)$ and

$$\mu(\tau)=\tau^{-\gamma+2n+2}\,\phi^{(n+1)}(\tau)\text{ if }\tau>0. \tag{28}$$

By (26) - (28) we have $\Phi_{n+1}^\gamma(\phi_k-\phi)\rightarrow 0$. With $k_0\in N$ such that $\Phi_{n+1}^\gamma(\phi_{k_0}-\phi)\leq 1$ we have that

$$\begin{aligned}x^{-\gamma+n+1+j}\,|P_{n+1,j}(x,D)\phi(x)| &\leq \Phi_{n+1}^\gamma(\phi_{k_0}-\phi)+\Phi_{n+1}^\gamma(\phi_{k_0})\\ &\leq 1+\Phi_{n+1}^\gamma(\phi_{k_0})\end{aligned}$$

for all $x>0$, $0\leq j\leq n+1$. Hence

$$\Phi_{n+1}^\gamma(\phi)\leq 1+\Phi_{n+1}^\gamma(\phi_{k_0})<\infty$$

and our assertion follows. □

Remark 2 *If $q > p$, $\gamma > \delta$ then $\mathfrak{L}_{\gamma,p} \subseteq \mathfrak{L}_{\delta,q}$. Moreover, for $\phi \in \mathfrak{L}_{\gamma,p}$ and any non negative integer n we have*

$$\Phi_n^{\gamma}(\phi) \leq p^{\gamma-\delta}\ \Phi_n^{\delta}(\phi)$$

and the topology of $\mathfrak{L}_{\gamma,p}$ is stronger than the induced topology on it by $\mathfrak{L}_{\delta,q}$.

Definition 2 *Let $\eta \in R$, $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ two arbitrary sequences of positive numbers, strict monotone decreasing the first, strict monotone increasing the second, $a_n \to 0$ and $b_n \to \infty$. We'll consider*

$$\mathfrak{L}_{\eta} = \bigcup_{n=1}^{\infty} \mathfrak{L}_{-\eta+a_n,\ b_n},$$

endowed with the structure of a countable - union space in the sense of Gelfand and Shilov [G&S], i.e. a sequence $\{\phi_k\}_{k\geq 1}$ of elements of $\mathfrak{L}_{\eta}$ will converge to an element $\phi \in \mathfrak{L}_{\eta}$ if and only if there exist $n_0 \in N$ such that $\{\phi_k\}_{k\geq 1} \subseteq \mathfrak{L}_{-\eta+a_{n_0},\ b_{n_0}}$, $\phi \in \mathfrak{L}_{-\eta+a_{n_0},\ b_{n_0}}$ and $\phi_k \to \phi$ in $\mathfrak{L}_{-\eta+a_{n_0},\ b_{n_0}}$.

Remark 3 *By Remark 2 we've $\mathfrak{L}_{\eta} \subseteq \mathfrak{L}_{\mu}$ whether $\eta < \mu$. Moreover, it is easy to see that the topological structure does not depend on the choices of the sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$.*

Remark 4 *$\mathfrak{D}(R^+) \subseteq \mathfrak{L}_{\eta}$ and the topology of $\mathfrak{D}(R^+)$ is stronger than the corresponding topology induced by $\mathfrak{L}_{\eta}$.*

Remark 5 *Let η, $\lambda \in R$. Then x^{λ} is a continuous linear operator between $\mathfrak{L}_{\eta}$ and $\mathfrak{L}_{\eta-\lambda}$. Moreover, if $a > 0$, $b > 0$, n is a non negative integer and $\phi \in \mathfrak{L}_{-\eta+a,b}$, using Corollary 4 and its notation we obtain*

$$\Phi_n^{-\eta+\lambda+a}\left(x^{\lambda}\phi\right) \leq \max_{0\leq j\leq n} \sum_{h=0}^{j} \left|c_{j,h}^n\right| \sum_{k=0}^{h} \binom{h}{k} \frac{b^{2n-j-h+k}}{(1+\lambda)_{-k}} \Phi_{h-k}^{-\eta+a}(\phi).$$

Remark 6 *Given $\rho > 0, \lambda \in R$, the operator $\mathfrak{T}_{\rho}\phi(x) = \phi(\rho x)$ is a continuous linear map of $\mathfrak{L}_{\eta}$ into itself. For, let $a > 0$, $b > 0$, n a non negative integer and $\phi \in \mathfrak{L}_{-\eta+a,b}$. In particular, $Supp\left(\mathfrak{T}_{\rho}\phi\right) \subseteq (0, b/\rho]$. On the other hand, we have*

$$\Phi_n^{-\eta+a}\left(\mathfrak{T}_{\rho}\phi\right) \leq \max_{0\leq j\leq n} \left(\frac{b}{\rho}\right)^{2n-j} \sum_{h=0}^{j} \left|c_{j,h}^n\right| \rho^{-h+a-\eta}\ \Phi_h^{-\eta+a}(\phi).$$

Remark 7 *Moreover, with the same notation there becomes $\mathfrak{L}_{\eta} \underrightarrow{D} \mathfrak{L}_{\eta+2}$, since*

$$\Phi_n^{-\eta-2+a}(D\phi) \leq \sum_{h=0}^{j} \left|c_{j,h}^n\right| b^{2n-j-h}\ \Phi_{h+1}^{-\eta+a}(\phi).$$

Remark 8 *$\mathfrak{L}_\eta$ is a testing function space. In fact, we already know that it is a complete countable union space of smooth functions over R^+ and we must only show that its topology is stronger than the induced topology by $\mathcal{E}(R^+)$. To this end let a and b be any positive numbers, $\{\phi_k\}_{k\geq 1} \subseteq \mathfrak{L}_{-\eta+a,b}$ be a sequence such that $\phi_k \to 0$ and K a fixed compact subset of R^+. We may find $c, d \in R^+$ such that $c < x < d$ for every $x \in K$. For a given non negative integer n and $x \in K$ we have*

$$x^{\eta-a+2n} \; |P_{n,n}(x, D)\phi_k(x)| \leq \Phi_n^{-\eta+a}(\phi_k),$$

i.e.

$$\left|\phi_k^{(n)}(x)\right| \leq \max\left\{x^{-\eta+a-2n} : \qquad c \leq x \leq d\right\} \; \Phi_n^{-\eta+a}(\phi_k)$$

i.e. $\phi_k^{(n)}(x) \overset{\circ}{\to} 0$ for every non negative n over any compact subset of R^+.

Remark 9 *The space $\mathfrak{D}(R^+)$ is not dense in $\mathfrak{L}_\eta$. In fact, let b be any positive number, $\gamma > -\eta$ and $\xi \in \mathcal{E}(R^+)$ such that $\xi(x) = 1$ if $0 < x \leq b/2$ and $\xi(x) = 0$ if $x \geq b$. Now, for x positive we define $\phi(x) = x^\gamma \, e^x \, \xi(x)$. On using again Corollary 4 and its notation, for $x > 0$ and any non negative integers n, j such that $j \leq n$ we obtain*

$$x^{n+j-\gamma} \, |P_{n,j}(x, D)\phi(x)| \leq \sum_{l=0}^{j}\sum_{h=0}^{l}\sum_{k=0}^{h} \frac{\left|c_{j,l}^{n}\right|}{(1+\gamma)_{-l+h}} \binom{l}{h}\binom{h}{k} \, x^{2n-j+h} \, e^x \, \left|\xi^{(k)}(x)\right|$$

$$\leq C(n,b) \; e^b \sup_{0\leq k\leq j}\left\{\left|\xi^{(k)}(x)\right| : 0 < x \leq b\right\} \sum_{l=0}^{j}\sum_{h=0}^{l}\sum_{k=0}^{h} \frac{\left|c_{j,l}^{n}\right|}{(1+\gamma)_{-l+h}} \binom{l}{h}\binom{h}{k},$$

i.e.

$$\Phi_n^\gamma(\phi) \leq C(n,b) \; e^b \max_{0\leq k\leq j}\left\{\left|\xi^{(k)}(x)\right| : \frac{b}{2} \leq x \leq b\right\} \sum_{l=0}^{j}\sum_{h=0}^{l}\sum_{k=0}^{h} \frac{\left|c_{j,l}^{n}\right|}{(1+\gamma)_{-l+h}} \binom{l}{h}\binom{h}{k}.$$

Therefore $\phi \in \mathfrak{L}_{\gamma,b}$. Moreover, for any testing function $\varphi \in \mathfrak{D}(R^+)$ we have

$$\Phi_0^\gamma(\phi - \varphi) \geq 1,$$

i.e. our assertion follows.

§4 Fractional Integrals on $\mathfrak{L}'_\eta$.

Theorem 4 *Let $\alpha \geq 1/2$, $\xi \in R$. Then $K_\kappa(\xi, \alpha)$ is a continuous linear integral transformation of $\mathfrak{L}_\eta$ into itself for all $\eta \in R$ such that $\eta > -\xi$.*

Proof.

Let η be under the stated conditions, $\phi \in \mathfrak{L}_\eta$. With the notation of Definition 2 we may find $n_0 \in N$ such that $\phi \in \mathfrak{L}_{-\eta+a_{n_0},\, b_{n_0}}$. Moreover, there exist $n_1 \geq n_0$ such that the number $\sigma = \eta + \xi - a_{n_1}$ is positive and by Remark 2 we also have $\phi \in \mathfrak{L}_{-\eta+a_{n_1},\, b_{n_1}}$.

In particular, $Supp\,(K_\kappa\,(\xi,\alpha)\,\phi) \subseteq (0, b_{n_1}]$. Now, for $x > 0$ and writing $\theta = \kappa x\sqrt{t-1}$ we have

$$|K_\kappa\,(\xi,\alpha)\,\phi(x)| \leq \left(\frac{\kappa x}{2}\right)^{1-\alpha} \int_1^\infty t^{-\alpha-\xi}\,(t-1)^{\frac{\alpha-1}{2}}\;|J_{\alpha-1}\,(\theta)\;\;\phi(tx)|\,dt \quad (29)$$

In general, we'll consider the inequality $[E, J\&L]$

$$\left|\frac{J_{\nu+m}(z)}{z^\nu}\right| \leq C_{\nu,m} \quad (30)$$

which is valid for $z > 0$, $\nu \geq -1/2$, any non negative integer m and wherein the $C_{\nu,m}$'s are positive constants. By (29) and (30) we obtain

$$x^{\eta-a_{n_1}}\;\;|K_\kappa\,(\xi,\alpha)\,\phi(x)| \leq 2^{\alpha-1}C_{\alpha-1,0}\;Be(\alpha,\sigma)\;\Phi_0^{-\eta+a_{n_1}}(\phi)$$

i.e.

$$\Phi_0^{-\eta+a_{n_1}}\,(K_\kappa\,(\xi,\alpha)\,\phi) \leq 2^{\alpha-1}\;C_{\alpha-1,0}\;Be(\alpha,\sigma)\;\Phi_0^{-\eta+a_{n_1}}(\phi).$$

Now, let $n \geq 0$, $0 \leq h \leq j \leq n+1$, $x > 0$. From the estimate

$$|K_\kappa\,(\xi+n+1-j-h, \alpha+j-h)\,P_{n+1,h}(x,D)\phi(x)| \leq$$

$$\leq \left(\frac{\kappa x}{2}\right)^{1-\alpha-j+h} \int_1^\infty \frac{(t-1)^{\frac{\alpha+j-h-1}{2}}}{t^{\alpha+\xi+n+1-2h}}\;\;|J_{\alpha+j-h-1}\,(\theta)\;\;P_{n+1,h}(x,D)\phi(tx)|\,dt$$

$$\leq \frac{\kappa^{h-j}}{2^{h+1-\alpha-j}}\;C_{\alpha-1,j-h+1}\;Be\left(\alpha+\frac{j-h}{2}, \frac{j-3h}{2}+2n+2+\sigma\right)\frac{\Phi_{n+1}^{-\eta+a_{n_1}}(\phi)}{x^{j+n+1+\eta-a_{n_1}}}$$

and applying Theorem 2 we obtain

$$x^{\eta-a_{n_1}+j+n+1}\;\;|P_{n+1,j}\;K_\kappa\,(\xi,\alpha)\,\phi(x)| \leq$$

$$\leq \Phi_{n+1}^{-\eta+a_{n_1}}(\phi)\sum_{h=0}^{j}\frac{a_{n+1,j}^h\;\kappa^{h-j}}{2^{h+1-\alpha-j}}\;C_{\alpha-1,j-h+1}\;Be\left(\alpha+\frac{j-h}{2}, \frac{j-3h}{2}+2n+2+\sigma\right)$$

which yields

$$\Phi_{n+1}^{-\eta+a_{n_1}}\,(K_\kappa\,(\xi,\alpha)\,\phi(x)) \leq C.\Phi_{n+1}^{-\eta+a_{n_1}}(\phi)$$

with

$$C = \max_{0\leq j\leq n+1}\sum_{h=0}^{j}\frac{a_{n+1,j}^h\;\kappa^{h-j}}{2^{h+1-\alpha-j}}\;C_{\alpha-1,j-h+1}\;Be\left(\alpha+\frac{j-h}{2}, \frac{j-3h}{2}+2n+2+\sigma\right),$$

and our assertion follows. □

Remark 10 *For a real number σ we'll consider the formal inner weight product*

$$\langle \phi, \psi \rangle_{x^\sigma dx} = \int_0^\infty \phi(x)\overline{\psi(x)}x^\sigma dx,$$

with respect to which we may consider $\mathfrak{L}_\mu \subseteq \mathfrak{L}_\eta^{'}$ if $\sigma - \eta - \mu > -1$, by means of the relation $\phi \longrightarrow \langle \phi, \psi \rangle_{x^\sigma dx}$.

Remark 11 *Moreover, we've the following formal Parseval relation*

$$\langle \phi, \mathfrak{I}_\kappa^\sigma (\xi, \alpha) \psi \rangle_{x^\sigma dx} = \langle K_\kappa (\xi, \alpha) \phi, \psi \rangle_{x^\sigma dx},$$

which allow us to consider fractional unsymmetric integrals of generalized functions over $\mathfrak{L}_\eta$ in the usual known way. In particular, with respect to the ordinary Lebesgue measure ($\sigma = 0$) and denoting as $\mathfrak{L}_\eta^{'}$ for the space of generalized functions over $\mathfrak{L}_\eta$, for $\phi \in \mathfrak{L}_\eta$ and $\psi \in \mathfrak{L}_\eta^{'}$, $\alpha \geq 1/2$ and $\eta > -\xi$, from the above relation we obtain $\mathfrak{I}_\kappa^0 (\xi, \alpha) \psi \in \mathfrak{L}_\eta^{'}$.

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