DECOMPOSABLE SURFACES WITH VANISHING EQUIAFFINE SCALAR CURVATURE

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Introduction

The so-called Equiaffine Theorema Egregium establishes an important relation among the main scalar invariants of Affine Hypersurface Geometry. It states, in abreviated terms, that : " If L is the unimodular affine mean curvature, J the Pick invariant, and R the Riemannian scalar curvature, then these three quantities are related by the equation R = L + J " , [**1, 2, 6, 7**] .

The case where these three invariants are simultaneously constant has been treated in full generality, but only for 2-surfaces, by Dillen, Martinez, Milán, García Santos and Vrancken in [**2**] .

On the other hand, one can consider the problem of classifying, both locally and globally, the so-called Subclass of Hypersurfaces of Decomposable Type where those invariants are individually constant. We started to do this in previous works [**4, 5, 6**] . Precisely in this last reference the problem was set in terms of analyzing the solutions of differential equations of the following kinds :

$$(*) \qquad \frac{y^{iv}}{(y'')^{r}} - s\,\frac{(y''')^{2}}{(y'')^{r+1}} = K = \textit{constant}\,,$$

where the constant values of r and s have a dimensional meaning , which vary according to the problem to be considered. The method of work was that of Qualitative Analysis, which was already used also in [**3**] .

Four cases of current geometrical interest were considered, regarding the constant values of r and s :

1) $r = 2,\ s = \frac{2n+3}{n+2}$, (vanishing unimodular affine mean curvature);

2) $r = 2 - \frac{1}{n+2}$, $s = \frac{2n+3}{n+2}$, (constant nonvanishing unimodular affine mean curvature);

3) $r = 2,\ s = \frac{7n+10}{4(n+2)}$, (vanishing scalar curvature);

4) $r = 2 - \frac{1}{n+2}$, $s = \frac{7n+10}{4(n+2)}$, (constant nonvanishing scalar curvature).

These cases arose when applying a standard method for calculating a first integral of the equation considered: namely $(y')^2=z$, which led to get the further equation $\frac{1}{2}$ y dz$-$[s z + K y^{r+1}] dy = 0. A first integration of the latter, under the additional condition for y to satisfy y > 0 , had $\mu = y^{-(1+2s)}$ as integrating factor, leading to

$$\frac{1}{2} y^{-2s} dz - [s z y^{-(1+2s)} + K y^{r-2s}] dy = 0 ,$$

which is an exact equation.

In order to integrate the latter we had to separate two *analytical cases*:

A) Comprising the above *geometrical cases* 1) , 2) , 4) , and 3) with n > 2 .
(Observe that here $r+1-2s > 0$) .

B) Corresponding to the remaining, limiting case 3) with n = 2 .
(Now, we have $r+1-2s = 0$) .

In the first case, the solution was written $\frac{1}{2} y^{-2s} z - \frac{1}{r+1-2s} K y^{r+1-2s} = \frac{1}{2} C$, and from this on the further analysis practiced in full detail **[6]** .

It is the objective of the presen paper to treat the remaining case B), where we have $r = 2$, $s = \frac{3}{2}$, and hence the solution can be written $\frac{1}{2} y^{-3} z - K \log y = \frac{1}{2} C$. It follows that we get in this case

$$y' = y^{3/2} \left(C + 2K \log y \right)^{1/2} .$$

This is the equation to be analyzed from now on.

1. Qualitative analysis for the inverse function

Type I : C = 0 , K = 0 : It is easy to sece that this is the most simple of solutions, namely:

(1.1) $$f^k(t^k) = (t^k)^2 , \text{ (parabolic type) } .$$

Type II : C = 0 , K < 0 : Here the differential equation becomes

(1.2) $$y' \doteq y^{3/2} (2K \log y)^{1/2} ,$$

which requires, in order to be integrated, qualitative analysis, to be performed ahead. Observe, too, that in this situation it also makes sense that the constant be strictly greater than zero, **K > 0** .

Type III : C > 0 , K = 0 : Here it is again possible to integrate explicitly. We obtain

(1.3) $$f^k (t^k) = - \log t^k .$$

For the remaining of cases, it is more feasible to look at x as a function of y . For this purpose we consider the equation in its equivalent form

(1.4) $$\frac{dx}{dy} = \frac{1}{y^{3/2}\left(C + 2K \log y \right)^{1/2}} .$$

By fixing a point (x_o , y_o), with $y_o > 0$, as initial condition, we can describe a solution to the latter equation by

(1.5) $$x = x_o + \int_{y_o}^{y} \frac{dt}{t^{3/2}\left(C + 2K \log t \right)^{1/2}} .$$

We shall use the last expressions in order to accomplish our goal of analyzing the behavior of solutions for the remaining of cases.

Let us observe, firstly, that the local behavior shall depend mainly on the first two derivatives of x, with respect to y. So we now calculate the second one, and put together with the first, in the following equations

(1.6) $$\frac{dx}{dy} = y^{-3/2}\left(C + 2K \log y \right)^{-1/2} ,$$
$$\frac{d^2x}{dy^2} = \frac{-3 K \log y - \frac{3 C}{2} - K}{y^{5/2}\left(C + 2 K \log y \right)^{3/2}} .$$

Secondly, we observe that the global behavior of the solutions shall also depend on the analysis in the neighborhood of singular points, as well as on the questions of convergence, or divergence, of the integral in (1.5). This is needed in order to determine when the surface is geometrically complete [6].

Type IV : $C > 0$, $K > 0$: We consider, then, equation (1.5) and observe that, since we must also have $C + 2K \log y > 0$, it follows that $y_o > \bar{y}_o := e^{-C/(2K)} > 0$. Hence, there exists $t_o \ (> y_o)$ such that $C + 2K \log t > 1$, for every $t > t_o$. It follows that we can write the estimate

$$\int_{t_o}^{y} \frac{dt}{t^{3/2}(C + 2 K \log t)^{1/2}} < \int_{t_o}^{y} \frac{dt}{t^{3/2}} ,$$

and since the last integral is convergent, we get that there exists $x_{11} \ (> x_o)$ such that

(1.7) $$\lim_{y \to \infty} x (y) = x_{11} .$$

On the other hand, by making the substitution $0 < v = C + 2K \log t$, we can write the inequality

$$0 < \int_{y}^{y_o} \frac{dt}{t^{3/2}(C + 2 K \log t)^{1/2}} = 2 K \int_{s}^{s_o} \frac{dv}{e^{(v - C)/4K} v^{1/2}} < \frac{2 K}{e^{-C/4K}} \int_{s}^{s_o} \frac{dv}{v^{1/2}}$$

valid for every y such that $\bar{y}_o < y < y_o$. Convergence of the last integral implies that there exists $\overline{x}_{11}$ ($< x_o$) such that

$$\lim_{y \to \overline{y}_o} x(y) = \overline{x}_{11} . \tag{1.8}$$

Finally, it follows from (1.6) , that

$$\frac{dx}{dy} > 0 , \quad \frac{dx^2}{dy^2} < 0 , \quad \forall \; y > \overline{y}_o . \tag{1.9}$$

Type V : C > 0 , K < 0 : We consider again (1.5) and observe that, since $0 < y_o < \overline{y}_o := e^{-C/2K}$ (> 1) , we get , for $y > y_o$ the estimate

$$\int_{y_o}^{y} \frac{dt}{t^{3/2} (C + 2K \log t)^{1/2}} < \frac{1}{y_o^{1/2}} \int_{y_o}^{y} \frac{dt}{t (C + 2K \log t)^{1/2}} .$$

The last integral is easily seen to be convergent. Hence, it follows that there exists $x_{21} > 0$ such that

$$\lim_{y \to \overline{y}_o} x(y) = x_{21} \tag{1.10}$$

On the other hand, given $\epsilon > 0$, there exists t_o such that $t(C+2K \log t) < \epsilon$ for every $t < t_o$. Hence, for $y < t_o$ we can write

$$\int_{y}^{t_o} \frac{dt}{t^{3/2} (C + 2K \log t)^{1/2}} > \frac{1}{\epsilon^{1/2}} \int_{y}^{t_o} \frac{dt}{t} ,$$

and since the last integral diverges to $+\infty$ as y converges to 0 , it follows that

$$\lim_{y \to 0} x(y) = -\infty . \tag{1.11}$$

With regards to the first two derivatives we obtain that

$$\frac{dx}{dy} > 0 , \qquad \frac{d^2x}{dy^2} = \begin{cases} < 0 , & \text{in } (0 , \check{y}_o) ; \\ = 0 , & \text{at } \check{y}_o ; \\ > 0 , & \text{in } (\check{y}_o , \overline{y}_o) ; \end{cases} \tag{1.12}$$

where $0 < \check{y}_o := e^{-(\frac{1}{3} + \frac{C}{2K})} = \overline{y}_o \, e^{-1/3} < \overline{y}_o$.

Type VI : C < 0 , K < 0 : Let us consider again (1.5) and observe that, since $0 < y_o < \overline{y}_o := e^{-C/2K}$ (< 1) , we can proceed in a similar fashion as we did with **Type V: C > 0 ; K < 0 ,** to obtain that there exists $x_{31} > 0$ such that

$$\lim_{y \to \overline{y}_o} x(y) = x_{31} . \tag{1.13}$$

Besides, as before

$$(1.14) \qquad \lim_{y \to 0} x(y) = -\infty ,$$

and

$$(1.15) \qquad \frac{dx}{dy} > 0 , \qquad \frac{d^2x}{dy^2} = \begin{cases} < 0 , & \text{in } (0, \check{y}_o) ; \\ = 0 , & \text{at } \check{y}_o ; \\ > 0 , & \text{in } (\check{y}_o , \bar{y}_o) ; \end{cases}$$

where $0 < \check{y}_o := e^{-(\frac{1}{3} + \frac{C}{2K})} = \bar{y}_o \, e^{-1/3} < \bar{y}_o$.

2. Qualitative analysis and further integration of the direct function

Let us consider the non-linear, ordinary differential equation $y' = y^{3/2} \left(C + 2K \log y \right)^{1/2}$, set up in the Introduction, which we now rewrite as

$$(2.1) \qquad \frac{dy}{dx} = g'(x) = [g(x)]^{3/2} \left(C + 2K \log g(x) \right)^{1/2} .$$

We shall use this expression in order to analyze the diverse cases .

Type IV : C > 0 , K > 0 : We obtain, as a consequence of (1.7), (1.8) and (1.9) that the function $y = g(x)$, where $g : (\bar{x}_{11} , x_{11}) \to R$, is characterized by the three following conditions

$$(2.2) \qquad \lim_{x \to x_{11}} g(x) = +\infty .$$

$$(2.3) \qquad \lim_{x \to \bar{x}_{11}} g(x) = \bar{y}_o := e^{-C/2K} .$$

$$(2.4) \qquad \frac{dg}{dx} > 0 , \quad \frac{d^2g}{dx^2} > 0 , \quad \forall \ x \in (\bar{x}_{11} , x_{11}) .$$

A typical integral of the function g can be written

$$(2.5) \qquad G(x) := \tilde{y}_o + \int_{\tilde{x}_o}^{x} g(t) \, dt ,$$

for some point $(\tilde{x}_o , \tilde{y}_o)$, with $\tilde{x}_o \in (\bar{x}_{11} , x_{11})$.

By (2.1) , we can write the latter as

$$(2.6) \qquad G(x) = \tilde{y}_o + \int_{\tilde{x}_o}^{x} \frac{g'(t) \, dt}{[g(t)]^{1/2} \left(C + 2K \log g(t) \right)^{1/2}} ,$$

Integration by parts furnishes

$$G(x) = \tilde{y}_o + 2 \left(\frac{[g(t)]^{1/2}}{\left(C + 2K \log g(t) \right)^{1/2}} \right)_{\tilde{x}_o}^{x} +$$

$$(2.7) \qquad + 2K \int_{\tilde{x}_o}^{x} \frac{g'(t)\,dt}{[g(t)]^{1/2}\Big(C + 2K \log g(t)\Big)^{3/2}} .$$

Hence, it follows that

$$(2.8) \qquad \lim_{x \to x_{11}} G(x) = +\infty .$$

Also, from (2.3) and (2.5)

$$(2.9) \qquad \lim_{x \to \overline{x}_{11}} G(x) = \overline{G}_{11} \in R ,$$

and, by using (2.4),

$$(2.10) \qquad G'(x) = g(x) > 0 , \; G''(x) = g'(x) > 0 , \; \forall \; x \in (\overline{x}_{11}, x_{11}) .$$

We need one further integration and write

$$(2.11) \qquad F(x) = \hat{y}_o + \int_{\hat{x}_o}^{x} G(t)\,dt,$$

with $\hat{x}_o \in (\overline{x}_{11}, x_{11})$.

It is immediate to obtain, from (2.8), (2.9) and (2.10) above, that the first two derivatives of F behave as follows:

$$(2.12) \qquad \begin{aligned} &F'(x) = G(x) \text{ ranges from } \overline{G}_{11} \text{ to } +\infty , \\ &F''(x) = G'(x) > 0 , \text{ for every } x \in (\overline{x}_{11}, x_{11}) . \end{aligned}$$

It is easy to see that

$$(2.13) \qquad \lim_{x \to \overline{x}_{11}} F(x) = \overline{F}_{11} \in R .$$

In order to complete the study on the global behavior of F we need to determine $\lim_{x \to x_{11}} F(x)$, and to achieve this we have to analyze in more detail G(x) near that limit point. We consider the second term in equation (2.7) and observe, too, the differential equation (2.1) in order to write

$$\int_{\tilde{x}_o}^{x} \frac{[g(t)]^{1/2}\,dt}{\Big(C + 2K \log g(t)\Big)^{1/2}} = \int_{\tilde{x}_o}^{x} \frac{g'(t)\,dt}{[g(t)]\Big(C + 2K \log g(t)\Big)} =$$
$$= \frac{1}{2K}\Big(\log (C + 2K \log g(x) - \log (C + 2K \log g(\tilde{x}_o) \Big) .$$

From this and (2.2) we get, finally,

$$(2.14) \qquad \lim_{x \to x_{11}} F(x) = +\infty .$$

Type V : C > 0 , K < O : According to (1.10) , (1.11) and (1.12) , we have for the direct function $y = g(x)$, with $g : (-\infty, x_{21}) \to R$, the three characterizing conditions described next

(2.15) $$\lim_{x \to x_{21}} g(x) = \bar{y}_o = e^{-\frac{C}{2K}} .$$

(2.16) $$\lim_{x \to -\infty} g(x) = 0 .$$

(1.17) $$\frac{dg}{dx} > 0 , \quad \frac{d^2g}{dx^2} = \begin{cases} > 0 \text{ in } (-\infty, \check{x}_o), \\ = 0 \text{ at } \check{x}_o = g^{-1}(\check{y}_o), \\ < 0 \text{ in } (\check{x}_o, x_{21}), \end{cases}$$

where $0 < \check{y}_o := e^{-(\frac{1}{3} + \frac{C}{2K})} = \bar{y}_o \, e^{-1/3} < \bar{y}_o$.

Again, a typical integral of the function g can be written

(2.18) $$G(x) := \check{y}_o + \int_{\check{x}_o}^{x} g(t)\,dt , \quad \check{x}_o \in (-\infty, x_{21}) .$$

By (2.1) , we write

(2.19) $$G(x) = \check{y}_o + \int_{\check{x}_o}^{x} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(C + 2K \log g(t)\right)^{1/2}} .$$

It is obvious that

(2.20) $$\lim_{x \to x_{21}} G(x) = \bar{G}_{21} \in R ,$$

and also that

(2.21) $$G'(x) = g(x) > 0 , \quad G''(x) = g'(x) > 0 , \quad \forall\, x \in (-\infty, x_{21}) .$$

Next, we observe that there exists $t_{21} \in (-\infty, x_{21})$ such that $C + 2K \log g(t) > 1$ for every $t < t_{21}$. From this we obtain the estimate

$$0 < \int_{x}^{t_{21}} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(C + 2K \log g(t)\right)^{1/2}} < \int_{x}^{t_{21}} \frac{g'(t)\,dt}{[g(t)]^{1/2}} =$$
$$= 2\left([g(t_{21})]^{1/2} - [g(x)]^{1/2}\right) ,$$

and this implies that

(2.22) $$\lim_{x \to -\infty} G(x) = \bar{G}_{-\infty} \in R .$$

We define now

$$(2.23) \qquad F(x) = \hat{y}_o + \int_{\hat{x}_o}^{x} G(t)\,dt, \quad \hat{x}_o \in (-\infty, x_{21}).$$

It is immediate to obtain, from (2.20), (2.21) and (2.22) above, that the first two derivatives of F behave as follows:

$$(2.24) \qquad \begin{array}{l} F'(x) = G(x) \text{ always increasing, ranges from } \overline{G}_{-\infty} \text{ to } \overline{G}_{21}, \\ F''(x) = G'(x) = g(x) > 0, \text{ for every } x \in (-\infty, x_{21}). \end{array}$$

As it is again obvious that

$$(2.25) \qquad \lim_{x \to x_{21}} F(x) = F_{21} \in R.$$

And once again, it is equally easy to compute $\lim_{x \to -\infty} F(x)$ in the cases where $\overline{G}_{-\infty}$ is > 0 or < 0. In the remaining, limiting case $\overline{G}_{-\infty} = 0$, we firstly write

$$(2.26) \qquad G(x) = \int_{-\infty}^{x} g(t)\,dt = \int_{-\infty}^{x} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(C + 2K \log g(t)\right)^{1/2}},$$

and choose t_{22} such that $C + 2K \log g(t) < 3K \log g(t)$, for every $t < t_{22}$.

Hence,

$$\int_{x}^{t_{22}} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(C + 2K \log g(t)\right)^{1/2}} > \frac{1}{(-3K)^{1/2}} \int_{x}^{t_{22}} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(-\log g(t)\right)^{1/2}} =$$

$$= \frac{1}{(-3K)^{1/2}} \left\{ \left(2[g(t)]^{1/2}\left(-\log g(t)\right)^{-1/2}\right)_{x}^{t_{22}} - \left(2[g(t)]^{1/2}\left(-\log g(t)\right)^{-3/2}\right)_{x}^{t_{22}} + \right.$$

$$\left. + 3 \int_{x}^{t_{22}} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(-\log g(t)\right)^{5/2}} \right\},$$

where the last equality is obtained by integrating twice by parts.

Then we can write that, for every $x < t_{22}$,

$$(2.27) \qquad G(x) > \frac{2}{(-3K)^{1/2}} [g(x)]^{1/2} \left(\left(-\log g(x)\right)^{-1/2} - \left(-\log g(x)\right)^{-3/2}\right),$$

hence,

$$\int_x^{t_{22}} G(t)\,dt > \frac{2}{-3K}\int_x^{t_{22}} \frac{[g(t)]^{1/2}\left(\left(-\log g(t)\right)^{-1/2} - \left(-\log g(t)\right)^{-3/2}\right) g'(t)\,dt}{[g(t)]^{3/2}\left(-\log g(t)\right)^{1/2}} =$$

$$= -\frac{2}{3K}\int_x^{t_{22}} \left(\left(-\log g(t)\right)^{-1} - \left(-\log g(t)\right)^{-2}\right)\frac{g'(t)}{g(t)}\,dt =$$

$$= -\frac{2}{3K}\left(-\log\left(-\log g(t)\right) - \left(-\log g(t)\right)^{-1}\right)\Bigg|_x^{t_{22}} ,$$

and since this last diverges to $+\infty$ as x diverges to $-\infty$, it follows that

(2.28) $$\lim_{x\to-\infty} F(x) = -\infty .$$

Type VI : C < 0 , K < 0 : We treat consider here the function $y = g(x)$, $g : (-\infty, x_{31}) \to R$, characterized by the following conditions, which are easily obtained from (1.13), (1.14), and (1.15),

(2.29) $$\lim_{x\to x_{31}} g(x) = \bar{y}_o = e^{-\frac{C}{2K}} .$$

(2.30) $$\lim_{x\to-\infty} g(x) = 0 .$$

(2.31) $$\frac{dg}{dx} > 0 , \quad \frac{d^2g}{dx^2} = \begin{cases} > 0 \text{ in } (-\infty, \check{x}_o), \\ = 0 \text{ at } \check{x}_o = g^{-1}(\check{y}_o), \\ < 0 \text{ in } (\check{x}_o, x_{31}), \end{cases}$$

where $0 < \check{y}_o := e^{-\left(\frac{1}{3} + \frac{C}{2K}\right)} = \bar{y}_o\, e^{-1/3} < \bar{y}_o$.

Again, a typical integral of the function g can be written

(2.32) $$G(x) := \tilde{y}_o + \int_{\tilde{x}_o}^{x} g(t)\,dt , \quad \tilde{x}_o \in (-\infty, x_{31}) .$$

By (2.1) , we can write

(2.33) $$G(x) = \tilde{y}_o + \int_{\tilde{x}_o}^{x} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(C + 2K\log g(t)\right)^{1/2}} .$$

It is obvious that

(2.34) $$\lim_{x\to x_{31}} G(x) = \bar{G}_{31} \in R ,$$

and also that

(2.35) $$G'(x) = g(x) > 0 , \; G''(x) = g'(x) > 0 , \; \forall\, x \in (-\infty, x_{31}) .$$

Next, we observe that there exists $t_{31} \in (-\infty, x_{31})$ such that $C + 2K \log g(t) > 1$ for every $t < t_{31}$, from this we obtain the estimate

$$0 < \int_x^{t_{31}} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(C + 2K\log g(t)\right)^{1/2}} < \int_x^{t_{31}} \frac{g'(t)\,dt}{[g(t)]^{1/2}} =$$

$$= 2\left([g(t_{31})]^{1/2} - [g(x)]^{1/2}\right),$$

and the latter implies that

$$(2.36) \qquad \lim_{x \to -\infty} G(x) = \bar{\bar{G}}_{-\infty} \in \mathbb{R}.$$

We define next

$$(2.37) \qquad F(x) = \hat{y}_o + \int_{\hat{x}_o}^{x} G(t)\,dt, \quad \hat{x}_o \in (-\infty, x_{31}).$$

It is immediate to obtain, from (2.34), (2.35) and (2.36) above, that the first two derivatives of F behave as follows:

$$(2.38) \qquad \begin{aligned} &F'(x) = G(x) \text{ always increasing, ranges from } \bar{\bar{G}}_{-\infty} \text{ to } \bar{G}_{31}, \\ &F''(x) = G'(x) = g(x) > 0, \text{ for every } x \in (-\infty, x_{31}). \end{aligned}$$

It is again obvious that

$$(2.39) \qquad \lim_{x \to x_{31}} F(x) = F_{31} \in \mathbb{R}.$$

While it is equally easy to compute $\lim_{x \to -\infty} F(x)$ in the cases where $\bar{\bar{G}}_{-\infty}$ is > 0 or < 0. In the remaining, limiting case $\bar{\bar{G}}_{-\infty} = 0$, we firstly write

$$(2.40) \qquad G(x) = \int_{-\infty}^{x} g(t)\,dt = \int_{-\infty}^{x} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(C + 2K\log g(t)\right)^{1/2}},$$

and observe that $C + 2K \log g(t) < 2K \log g(t)$, for every $t < x_{31}$.

Hence,

$$\int_x^{x_{31}} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(C + 2K\log g(t)\right)^{1/2}} > \frac{1}{(-2K)^{1/2}} \int_x^{x_{31}} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(-\log g(t)\right)^{1/2}} =$$

$$= \frac{1}{(-2K)^{1/2}} \left\{ \left(2[g(t)]^{1/2}\left(-\log g(t)\right)^{-1/2}\right)_x^{x_{31}} - \left(2[g(t)]^{1/2}\left(-\log g(t)\right)^{-3/2}\right)_x^{x_{31}} + \right.$$

$$\left. + 2\int_x^{x_{31}} \frac{g'(t)\,dt}{[g(t)]^{1/2}\left(-\log g(t)\right)^{5/2}} \right\},$$

where the last equality is obtained by integrating twice by parts .

Then we can write that, for every $x < x_{31}$,

$$G(x) > \frac{2}{(-2K)^{1/2}} [g(x)]^{1/2} \left(\left(-\log g(x)\right)^{-1/2} - \left(-\log g(x)\right)^{-3/2} \right),$$

and, hence,

$$\int_x^{x_{31}} G(t)\,dt > \frac{1}{-K} \int_x^{x_{31}} \frac{[g(t)]^{1/2} \left(\left(-\log g(t)\right)^{-1/2} - \left(-\log g(t)\right)^{-3/2} \right) g'(t)\,dt}{[g(t)]^{3/2} \left(-\log g(t)\right)^{1/2}} =$$

$$= -\frac{1}{K} \int_x^{x_{31}} \left(\left(-\log g(t)\right)^{-1} - \left(-\log g(t)\right)^{-2} \right) \frac{g'(t)}{g(t)}\,dt =$$

$$= -\frac{1}{K} \left(-\log\left(-\log g(t)\right) - \left(-\log g(t)\right)^{-2} \right)\Bigg|_x^{x_{31}} ,$$

and since this last diverges to $+\infty$ as x diverges to $-\infty$, it follows that

(2.41) $$\lim_{x \to -\infty} F(x) = -\infty .$$

This finishes the analysis of the function F in this case and, therefore, in all of possible cases.

3. The classifying result.

Theorem. *Let* $X : M^2 \to E^3$ *be a nondegenerate surface of decomposable type with vanishing unimodular affine scalar curvature . Then, each of its two components* f^1 , f^2 , *must be of one of the* ***original types*** **I** ***through*** **VI,** *whose properties are enumerated below, or the corresponding three more kinds of types that are obtained from those original types by suitable reflections in the* x- *and* y-*axis . All of the solutions belonging to the original types share the common feature that their second derivatives,* $(f^k)'' = y > 0$, *satisfy, in each case, the classifying, non-linear, ordinary differential equation*

$$y' = y^{3/2} \left(C + 2K \log y \right)^{1/2} .$$

Type I : C = 0 , K = 0 .

$f^k : R \to R$, *given by* $f^k(t^k) = (t^k)^2$, *(parabolic type)* .

Type III : C > 0 , K = 0 .

$f^k : (-\infty, 0) \to R$, *defined by* $f^k(t^k) = -\log(-t^k)$, (logarithmic type) .

Type IV : C > 0 , K > 0 .

f^k *defined on a finite open interval, i.e.* $f_k : (\bar{t}^k_{11}, t^k_{11}) \to \mathbb{R}$, *such that*

$$\lim_{t^k \to \bar{t}^k_{11}} f^k(t^k) = \bar{f}^k_{11} \in \mathbb{R}, \quad \lim_{t^k \to t^k_{11}} f^k(t^k) = +\infty,$$

$$\lim_{t^k \to \bar{t}^k_{11}} (f^k)'(t^k) = (\bar{f}^k_{11})' \in \mathbb{R}, \quad \lim_{t^k \to t^k_{11}} (f^k)'(t^k) = +\infty,$$

$$\lim_{t^k \to \bar{t}^k_{11}} (f^k)''(t^k) = e^{-C/2K}, \quad \lim_{t^k \to t^k_{11}} (f^k)''(t^k) = +\infty,$$

$$(f^k)'''(t^k) > 0, \quad (f^k)^{iv}(t^k) > 0, \quad \textit{for every } t^k \in (\bar{t}^k_{11}, t^k_{11}).$$

Type V : C > 0 , K < 0 .

f^k *defined on a semi-infinite interval ,* $f^k : (-\infty, t^k_{21}) \to \mathbb{R}$, *such that*

$$\lim_{t^k \to -\infty} f^k(t^k) = \begin{cases} +\infty, & \text{if } (\bar{f}^k)'_{-\infty} < 0, \\ -\infty, & \text{if } (\bar{f}^k)'_{-\infty} \geq 0, \end{cases} \qquad \lim_{t^k \to t^k_{21}} f^k(t^k) = f^k_{21} \in \mathbb{R},$$

$$\lim_{t^k \to -\infty} (f^k)'(t^k) = (\bar{f}^k)'_{-\infty} \in \mathbb{R}, \quad \lim_{t^k \to t^k_{21}} (f_k)'(t^k) = (\bar{f}^k)'_{21} \in \mathbb{R},$$

$$\lim_{t^k \to -\infty} (f^k)''(t^k) = 0, \quad \lim_{t^k \to t^k_{21}} (f^k)''(t^k) = e^{-C/2K},$$

$$(f^k)'''(t^k) > 0, \quad \text{for every } t^k \in (-\infty, t^k_{21}),$$

$$(f^k)^{iv}(t^k) > 0 \text{ for } t^k \in (-\infty, \check{t}^k_o),$$

$$(f^k)^{iv}(\check{t}^k_o) = 0,$$

$$(f^k)^{iv}(t^k) < 0 \quad \text{for } t^k \in (\check{t}^k_o, t^k_{21}),$$

where $\check{t}^k_o = \left((f^k)''\right)^{-1}\left(e^{-\left(\frac{1}{3}+\frac{C}{2K}\right)}\right)$.

Type VI : C < 0 , K < 0 .

$f^k : (-\infty , t^k_{31}) \to R$, *with the following properties*

$$\lim_{t^k \to -\infty} f^k (t^k) = \begin{cases} +\infty , & \text{if } (\bar{\bar{f}}^k)'_{-\infty} < 0 , \\ -\infty , & \text{if } (\bar{\bar{f}}^k)'_{-\infty} \geq 0 , \end{cases} \qquad \lim_{t^k \to t^k_{31}} f^k (t^k) = f^k_{31} \in R ,$$

$$\lim_{t^k \to -\infty} (f^k)' (t^k) = (\bar{\bar{f}}^k)'_{-\infty} \in R , \qquad \lim_{t^k \to t^k_{31}} (f^k)' (t^k) = (\bar{\bar{f}}^k_{31})' \in R ,$$

$$\lim_{t^k \to -\infty} (f^k)'' (t^k) = 0 , \qquad \lim_{t^k \to t^k_{31}} (f^k)'' (t^k) = (f^k_{31})'' := e^{-C/2K} ,$$

$$(f^k)''' (t^k) > 0 , \text{ for every } t^k \in (-\infty , t^k_{31}) ,$$

$$(f^k)^{iv} (t^k) > 0 \text{ for } t^k \in (-\infty , \check{t}^k_o) ,$$

$$(f^k)^{iv} (\check{t}^k_o) = 0 ,$$

$$(f^k)^{iv} (t^k) < 0 \quad \text{for } t^k \in (\check{t}^k_o , t^k_{31}) ,$$

where $\check{t}^k_o = \left((f^k)'' \right)^{-1} \left(e^{-\left(\frac{1}{3} + \frac{C}{2K} \right)} \right)$.

Type II : C = 0 , K < 0 .

The component function f^k *satisfies in this instance the conditions corresponding to* **Types V** *and/or* **VI** *above, as limiting case, by taking in either or those* $C = 0$. *Moreover, it also makes sense now to consider a second* **Type II : C = 0 , K > 0 .** *For this latter type the properties of the component function* f^k *are described by* **Type IV** , *by taking again* $C = 0$.

Proof: it is to be to be applied here the analysis performed in §§ 1 and 2, with regards to equation ($*$) mentioned in the Introduction , *geometrical case* 3) for dimension $n = 2$, i.e. $r = 2$, $s = 3/2$. The equations that apply in this limiting case are : (1.1) for **type I** ; (1.3) for **type III** ; (2.2) , (2.3) , (2.4) , (2.8) , (2.9) , (2.10) , (2.12) , (2.13) , (2.14) , for **type IV** ; (2.15) ,

(2.16) , (2.17) , (2.20) , (2.21) , (2.22) , (2.24) , (2.25) , (2.28) , for **type V** ; and (2.29) , (2.30) , (2.31) , (2.34) , (2.35) , (2.36) , (2.38) , (2.39) , and (2.41) for **type VI.** Finally, the observation concerning **type II** follows from equation (1.2) and the corresponding analysis for **types IV , V ,** and **VI** .

BIBLIOGRAPHY

1. W. Blaschke, *Vorlesungen über Differentialgeometrie.* II, Springer-Verlag, Berlin, 1923.
2. F. Dillen, A. Martínez, F. Milán, F. García Santos and L. Vrancken, *On the Pick invariant, the affine mean curvature and the Gauss curvature of affine surfaces,* Results in Math. **20**, (1991), 622-642.
3. S. Gigena , *Constant affine mean curvature hypersurfaces of decomposable type,* Proceedings of Symposia in Pure Mathematics, American Mathematical Society **54**, (1993), Part 3, 289-316.
4. ______, *El invariante de Pick para hipersuperficies descomponibles,* Math. Notae **37**, (1993/94), 87-104.
5. ______, *Hypersurface Geometry and Related Invariants in a Real Vector Space,* Cuadernos del Instituto de Matemática Beppo Levi No. 28, Rosario, 1996.
6. ______, *Ordinary Differential Equations in Affine Geometry,* Le Matematiche, Vol. LI, (1996) .
7. A.-M. Li, U. Simon, G. Zhao, *Global Affine Differential Geometry of Hypersurfaces,* W. de Gruyter, Berlin, New York, 1993.

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