Composition and Factorization of Diffusions on Principal Fiber Bundles

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Abstract

Let $P(M,G)$ be a principal fiber bundle, $K \in \Gamma_G(\tau P)$ and $A \in \mathcal{G}^{(2)}$. If Y and g are independent diffusions with infinitesimal generators K and A respectively, then $Z = Y \cdot g$ is a diffusion in P with infinitesimal generator $K + A^*$.

We state the following result about the factorization of L-diffusions in principal fiber bundles. Let $\mathbf{H}$ be a 2-connection [2], $K \in \Gamma(\tau M)$ and $A \in \mathcal{G}^{(2)}$. Let Z be a $\mathbf{H}K + A^*$-diffusion. Then there is an unique pair (Y, g) such that: *i)* Y is a $\mathbf{H}K$-diffusion and g is a A-diffusion, *ii)* Y and g are independent, and *iii)* $Z = Y \cdot g$.

The composition and factorization of stochastic processes in principal fiber bundles was study by many authors for different class of processes ([3] Brownian motions, [1] martingales and [7] diffusions).

The purpose of this work is to study the composition and the factorization of diffusions in the context of Schwartz geometry.

This paper is organized as follows, in 1. we prepare some notions concerning with Schwartz geometry, 2-connections and diffusions. We prove the following result about composition of independent diffusions, that generalized a result of M. Liao [7, Proposition 5]. Let $K \in \Gamma_G(\tau P)$ and $A \in \mathcal{G}^{(2)}$, if Y and g are independent diffusions with infinitesimal generators K and A respectively, then $Z = Y \cdot g$ is a diffusion in P with infinitesimal generator $K + A^*$. We have the following Corollary. Let $\mathbf{H}$ be a 2-connection of

$P(M,G)$, $K \in \Gamma(\tau M)$ and $A \in \mathcal{G}^{(2)}$. If X and g are diffusions in M and G with infinitesimal generators K and A respectively, then $Z = HX \cdot g$ is a diffusion in P with infinitesimal generator $\mathbf{H}K + A^*$.

In 2. we prove that in a principal fiber bundle every diffusion with infinitesimal generator $\mathbf{H}K$ is a horizontal lift by $\mathbf{H}$. We state the following result about the factorization of L-diffusions in principal fiber bundles. Let $P(M,G)$ be a principal fiber bundle, $\mathbf{H}$ a 2-connection, $K \in \Gamma(\tau M)$ and $A \in \mathcal{G}^{(2)}$. Let Z be a $\mathbf{H}K + A^*$-diffusion. Then there is an unique pair (Y, g) such that: *i)* Y is a $\mathbf{H}K$-diffusion and g is a A-diffusion, *ii)* Y and g are independent, and *iii)* $Z = Y \cdot g$.

1 Schwartz Geometry, 2-Connections and Diffusions

In this paper, all manifolds are finite dimensional, σ-compact and of class $\mathcal{C}^\infty$. As to manifolds and stochastic differential geometry, we shall freely concepts and notations of Kobayashi-Nomizu [6], Emery [4], Protter [10].

Now, we recall some fundamental facts about Schwartz second order geometry ([8], [9], [4], [11]) and diffusion theory.

If x is a point in a manifold M, the second order tangent space to M at x, denoted $\tau_x M$, is the vector space of all differential operators on M, at x, of order at most two, with no constant term. If $dim M = n$, $\tau_x M$ has $n + \frac{1}{2}n(n+1)$ dimensions; using a local coordinate system (U, x^i) around x, every $L \in \tau_x M$ can be written in a unique way as

$$L = a^i D_i + a^{ij} D_{ij} \qquad \text{with } a^{ij} = a^{ji}$$

where $D_i = \dfrac{\partial}{\partial x^i}$ and $D_{ij} = \dfrac{\partial}{\partial x^i \partial x^j}$ are differential operators at x (we use here and in other expressions in coordinates the convention of summing over the repeated indices). The elements of $\tau_x M$ are called second-order tangent vectors (or tangent vectors of order two) at x; the elements of the dual vector space $\tau_x^* M$ are called second order forms at x.

The disjoint union $\tau M = \bigcup_{x \in M} \tau_x M$ (respectively $\tau^* M = \bigcup_{x \in M} \tau_x^* M$) is canonically endowed with a vector bundle structure over M, called the second order tangent fiber bundle (respectively second order cotangent fiber bundle) of M.

We denote by $\Gamma(\tau M)$ the space of second order operator on M, that is, the space of sections of τM. For $L \in \Gamma(\tau M)$ let QL stand for the "square field operator" associated to L. It is given by

$$QL(f,g) = \frac{1}{2}[L(fg) - fL(g) - gL(f)]$$

where $f, g \in \mathcal{C}^\infty(M)$. In a coordinate system (U, x^i) around $x \in M$ this operator is given by

$$QL(f,g) = a^{ij} D_i f D_j g$$

in case $L = a^i D_i + a^{ij} D_{ij}$.

We have a morphism of vector bundles $Q : \tau M \longrightarrow TM \odot TM$ defined by

$$Q_x(L = a^i D_i + a^{ij} D_{ij}) = a^{ij} D_i \odot D_j$$

Definition 1.1 *Let M and N be manifolds and pick $x \in M$ and $y \in N$. A linear mapping $f : \tau_x M \longrightarrow \tau_y N$ is called a Schwartz morphism if*

i) $f(T_x M) \subset T_y N$ and

ii) for every $L \in \tau_x M$ we have that $Q(fL) = (f \otimes f)(QL)$.

It is known [4] that $f : \tau_x M \longrightarrow \tau_y N$ is a Schwartz morphism if and only if there exists a smooth mapping $\phi : M \longrightarrow N$ with $\phi(x) = y$ such that $f = \phi_*(x)$.

Let $P(M,G)$ be a principal fiber bundle where P is a bundle space, M is a base space and G is a structure group. Let $\mathcal{G}^{(2)}$ be the space of sections of $\tau(G)$ invariant to left. We recall that there is a homomorphism $* : \mathcal{G}^{(2)} \longrightarrow \Gamma(\tau P)$ defined by the right action of G on P: if $A \in \mathcal{G}^{(2)}$, $p \in P$, then $A^*(p) = p_*(e)A$, where p is here regarded as the injection of G into P given by $p(g) = pg$.

Let $K \in \Gamma(\tau P)$ we says that K is G-invariant if $R_{g*} K_p = K_{pg}$ for all $p \in P$ and $g \in G$. We denote by $\Gamma_G(\tau P)$ the space of G-invariant sections of τP.

Let M be a manifold, $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ a filtered probability space satisfying the usual conditions ([5],[10]), $L \in \Gamma(\tau M)$ and X a continuous semimartingale in M, one says that X is a diffusion for $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ with infinitesimal generator $L \in \Gamma(\tau M)$ if, for every $f \in \mathcal{C}_0^\infty(M)$

$$f \circ X_t - f \circ X_0 - \int_0^t Lf(X_s)ds \textit{ is a local martingale.}$$

Another form of this definition ([8]) is
For every 1-form θ with compact support

$$\int_0^t \langle \theta, \delta X\rangle - \int_0^t \langle D\theta, L\rangle (X_s)ds \textit{ is a local martingale.}$$

Where $\int_0^t \langle \theta, \delta X\rangle$ is the Stratonovich integral of θ along X and $D : TM \longrightarrow \tau M$ is the mapping defined by P.Meyer in [8].([4] pg. 92-93).

We remark that, if X is a diffusion for $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ with infinitesimal generators L and R, then $L(X_s) = R(X_s)$ a.e.

Proposition 1.2 *Let M and N be manifolds, $\phi : M \longrightarrow N$ a smooth map, and X a diffusion with infinitesimal generator L. Then $\phi \circ X$ is a diffusion if only if there is a $R \in \Gamma(N)$ such that $(\phi_* L)(X_s) = R(\phi(X_s))$. In this case $\phi \circ X$ is a diffusion with infinitesimal generator R.*

Proof: If $\phi \circ X$ is a diffusion then for every $f \in \mathcal{C}_0^\infty(N)$ we have that

$$f(\phi \circ X_t) - f(\phi \circ X_0) - \int_0^t Rf(\phi \circ X_s)ds \textit{ is a local martingale.}$$

and as

$$(f \circ \phi)X_t - (f \circ \phi)X_0 - \int_0^t L(f \circ \phi)(X_s)ds \textit{ is a local martingale.}$$

we obtain that

$$\phi_* L(X_s) = R(\phi(X_s))$$

And obviously, if $\phi_* L(X_s) = R(\phi(X_s))$ then $\phi \circ X$ is a diffusion with infinitesimal generator R. □

Proposition 1.3 *Let M and N be manifolds, $K \in \Gamma(M)$ and $L \in \Gamma(N)$. Let X and Y independent diffusions with infinitesimal generators K and L respectively. Then (X, Y) is a diffusion in $M \times N$ with infinitesimal generator $\overline{K} + \overline{L}$. Where $\overline{K}$ and $\overline{L}$ are the natural inclusions of K and L in $\Gamma(M \times N)$.*

Proof: Let (U, x^i) and (V, y^α) local charts of M and N respectively. In this charts $K = k^i D_i + k^{ij} D_{ij}$ and $L = l^\alpha D_\alpha + l^{\alpha\beta} D_{\alpha\beta}$. Let $f \in C_0^\infty(M \times N)$, then by an application of Itô formula

$$\begin{aligned} f(X_t^i, Y_t^\alpha) - f(X_0^i, Y_0^\alpha) &= \int_0^t D_i f(X_s^i, Y_s^\alpha) dX_s^i + \int_0^t D_\alpha f(X_s^i, Y_s^\alpha) dY_s^\alpha + \\ &\frac{1}{2}\int_0^t D_{ij} f(X_s^i, Y_s^\alpha) d[X^i, X^j]_s + \\ &\frac{1}{2}\int_0^t D_{\alpha\beta} f(X_s^i, Y_s^\alpha) d[Y^\alpha Y^\beta]_s + \\ &\frac{1}{2}\int_0^t D_{i\alpha} f(X_s^i, Y_s^\alpha) d[X^i, Y^\alpha]_s \end{aligned}$$

Now, as $dX_t^i = k^i(X_t)dt + d\,local\ mart._t$, $d[X^i, X^j]_t = k^{ij}(X_t)dt$, $dY_t^\alpha = l^\alpha(Y_t)dt + d\,local\ mart._t$, $d[Y^\alpha, Y^\beta]_t = l^{ij}(Y_t)dt$ because of that X and Y are diffusions with infinitesimal generators K and L respectively. And as X and Y are independent, $d[X^i, Y^\alpha]_t = 0$. Combining this facts we obtain

$$\begin{aligned} f(X_t^i, Y_t^\alpha) - f(X_0^i, Y_0^\alpha) &= \int_0^t (k^i(X_s)D_i + k^{ij}(X_s)D_{ij}) f(X_s, Y_s) ds + \\ &\int_0^t (l^i(Y_s)D_i + l^{ij}(Y_s)D_{ij}) f(X_s, Y_s) ds + \\ &local\ mart._t \end{aligned}$$

Hence we have

$$f(X_t, Y_t) - f(X_0, Y_0) = \int_0^t (\overline{K} + \overline{L}) f(X_s, Y_s) ds + local\ mart._t$$

This completes the proof. □

We have the following result about composition of independent diffusions in principal fiber bundles.

Corollary 1.4 *Let $P(M, G)$ be a principal fiber bundle, $K \in \Gamma_G(\tau P)$ and $A \in \mathcal{G}^{(2)}$. If Y and g are independent diffusions with infinitesimal generators*

K and A respectively, then $Z = Y \cdot g$ is a diffusion in P with infinitesimal generator $K + A^*$.

Proof: We know from above proposition that (Y, g) is a diffusion in $M \times G$ with infinitesimal generator $\overline{K} + \overline{A}$. We remember that $\overline{K}_{(p,g)}$ and $\overline{A}_{(p,g)}$ are given by $i_{g*}(p)K$ and $i_{p*}(g)A$ respectively, where $i_g(p) = (p, g)$ and $i_p(g) = (p, g)$. Let $\phi : P \times G \longrightarrow P$ be defined by $\phi(p, h) = ph$. Then by the proposition 1.2, $Z = \phi(Y, g)$ is a diffusion if there is a $R \in \Gamma(P)$ such that $\phi_*(Y, g)(\overline{K} + \overline{A}) = R(Z)$. Now, by invariance to left of A and G-invariance of K, we have

$$\begin{aligned} \phi_*(p, g)(\overline{K} + \overline{A}) &= \phi_*(p, g)(i_{g*}(p)K) + \phi_*(p, g)(i_{p*}(g)A) \\ &= (\phi \circ i_g)_*(p)K + (\phi \circ i_p)_*(g)A \\ &= R_{g*}(p)K + p_*(g)A \\ &= K_{pg} + A^*_{pg} \end{aligned}$$

Then Z is a diffusion with infinitesimal generator $K + A^*$. □

We remember the definition of 2-connection ([2])

Definition 1.5 *Let $P(M, G)$ be a principal fiber bundle. A family of Schwartz morphism $\mathbf{H} = \{H_p : p \in P\}$ is called a 2-connection if*

1) $H_p : \tau_{\pi p}M \longrightarrow \tau_p P$.

2) $\pi_* \circ H_p = id_{\tau_{\pi p}M}$

3) $H_{pg} = R_{g*}H_p$ for all $p \in P$ and $g \in G$ where R_g stands for the right action of G in P.

4) The mapping $p \longrightarrow H_p L$ belongs to $\Gamma(\tau P)$ if $L \in \Gamma(\tau M)$.

Let be $L \in \Gamma(\tau M)$, then we denote by $\mathbf{H}L$ the differential operator defined by $\mathbf{H}L_p = H_p L$ for all $p \in P$. We have that $\mathbf{H}L \in \Gamma_G(\tau P)$.

We have the following Corollary of Proposition 1.2.

Corollary 1.6 *Let $P(M, G)$ be a principal fiber bundle, $\mathbf{H}$ a 2-connection and Y a diffusion in P with infinitesimal generator $\mathbf{H}L$. Then $\pi \circ X$ is a diffusion with infinitesimal generator L.*

Let $P(M, G)$ be a principal fiber bundle, $\mathbf{H} = \{H_p : p \in P\}$ a 2-connection, X an M-valued semimartingale and Z a P-valued $\mathcal{F}_0$-random variable such that $\pi \circ Z = X_0$. We says that a P-valued semimartingale Y is a stochastic horizontal lift of X initialized in Z if satisfies

i) $Y_0 = Z$,

ii) $\pi \circ Y = X$,

iii) $\int_0^t \langle \Theta, d_2 Y \rangle = 0$ for all $t \geq 0$ a.e. Here Θ is an arbitrary element of $\mathbf{H}^{\perp}$, which is the subspace of 2-forms on P defined by

$$\mathbf{H}^{\perp} = \{\Theta : \Theta \circ H_p \; for \; all \; p \in P\}$$

Let X and Z be as above, is known ([2]) that there is an unique stochastic horizontal lift of X initialized in Z, it is given as solution of the following stochastic differential equation

$$\begin{aligned} d_2 Y &= H_Y d_2 X \\ Y_0 &= Z \end{aligned}$$

Now, we prove that the stochastic horizontal lift preserve diffusions.

Proposition 1.7 *Let $P(M, G)$ be a principal fiber bundle, $\mathbf{H}$ a 2-connection and X a diffusion in M with infinitesimal generator L. Then the stochastic horizontal lifts of X are diffusions with infinitesimal generator $\mathbf{H}L$.*

Proof: Let HX a stochastic horizontal lift of X by $\mathbf{H}$, then HX satisfies the following equation:

$$\begin{aligned} d_2 H X &= \mathbf{H}(HX_t) d_2 X_t \\ HX_0 &= Y \end{aligned}$$

where Y is a random variable such that $\pi \circ Y = X_0$.
As

$$\begin{aligned} f(HX_t) - f(HX_0) &= \int_0^t \langle d_2 f, d_2 HX \rangle \\ &= \int_0^t \langle \mathbf{H}(HX)^* d_2 f, d_2 X \rangle \\ &= \int_0^t \langle \mathbf{H}(HX_s)^* d_2 f, L(X_s) ds \rangle + local \; mart. \\ &= \int_0^t ((HL)f)(HX_s) ds + \; local \; mart. \end{aligned}$$

Then HX is a diffusion with infinitesimal generator HL. □

From above Proposition and Corollary 1.4, we have the following Corollary.

Corollary 1.8 *Let $P(M,G)$ be a principal fiber bundle, $\mathbf{H}$ a 2-connection $K \in \Gamma(\tau M)$ and $A \in \mathcal{G}^{(2)}$. If X and g are diffusions in M and G with infinitesimal generators K and A respectively, then $Z = HX \cdot g$ is a diffusion in P with infinitesimal generator $\mathbf{H}K + A^*$.*

2 Factorization of L-Diffusions.

Lemma 2.1 *Let $P(M,G)$ be a principal fiber bundle, $\mathbf{H}$ a 2-connection and $L \in \Gamma(\tau M)$. Then if Y and Z are diffusions in P with infinitesimal generator $\mathbf{H}L$ such that:*

i) $Y_0 = Z_0$,
ii) $\pi \circ Y = \pi \circ Z$

then $Y = Z$.

Proof: Let $\mathbf{H}_1$ be a connection induced by $\mathbf{H}$ and $\mathbf{H}^S = \mathbf{H}_1^S$ their Stratonovich prolongation ([2]), then $\mathbf{H}L = \mathbf{H}^S L + U$ where U is a vertical right invariant field.

Let ω be the connection form associated with $\mathbf{H}_1$, and Y a diffusion with infinitesimal generator $\mathbf{H}L$, then

$$\int_0^t \langle \omega, \delta Y \rangle = \int_0^t \langle \omega, U \rangle (Y_s) ds$$

In fact, let $\theta \in \mathbf{H}_1^\perp$. We have

$$\begin{aligned} \int_0^t \langle D\theta, \mathbf{H}L \rangle (Y_s) ds &= \int_0^t \left\langle D\theta, \mathbf{H}^S L + U \right\rangle (Y_s) ds \\ &= \int_0^t \langle D\theta, U \rangle (Y_s) ds \\ &= \int_0^t \langle \theta, U \rangle (Y_s) ds \end{aligned}$$

As $\int_0^t \langle \theta, \delta Y\rangle - \int_0^t \langle D\theta, \mathbf{H}L\rangle (Y_s)ds$ is a local martingale, we have that

$$\int_0^t \langle \theta, \delta Y\rangle - \int_0^t \langle \theta, U\rangle (Y_s)ds \textit{ is a local martingale} \qquad (1)$$

Let $f \in C_0^\infty(P)$ as $f\theta \in \mathbf{H}_1^\perp$, we have

$$\int_0^t \langle f\theta, \delta Y\rangle - \int_0^t \langle f\theta, U\rangle (Y_s)ds \textit{ is a local martingale}$$

As

$$\int_0^t \langle f\theta, \delta Y\rangle = \int_0^t f(Y_s)d(\int_0^s \langle \theta, \delta Y\rangle) + \tfrac{1}{2}[f\circ Y, \int \langle \theta, \delta Y\rangle]_t$$

We have that

$$\int_0^t (f\circ Y)d(\textit{local martingale}_s + \int_0^s \langle \theta, U\rangle (Y_r)dr) + \tfrac{1}{2}[f\circ Y, \int \langle \theta, \delta Y\rangle]_t - \int_0^t \langle f\theta, U\rangle (Y_s)ds$$

is a local martingale. We conclude that $[f\circ Y, \int \langle \theta, \delta Y\rangle] = 0$ for every $f \in C_0^\infty(P)$, then $\int \langle \theta, \delta Y\rangle$ is a bounded variation processes. Now, by (1) we have that

$$\int_0^t \langle \theta, \delta Y\rangle = \int_0^t \langle \theta, U\rangle (Y_s)ds \quad \textit{for every } \theta \in \mathbf{H}_1^\perp$$

In particular

$$\int_0^t \langle \omega, \delta Y\rangle = \int_0^t \langle \omega, U\rangle (Y_s)ds$$

Now, as $\pi\circ Y = \pi\circ Z$ there exists $g_t \in G$ such that $Z_t = Y_t g_t$, is not difficult to prove that g_t is a semimartingale. As

$$\begin{aligned}\int_0^t \langle \omega, U\rangle (Z_s)ds &= \int_0^t \langle \omega, \delta Z\rangle \\ &= \int_0^t \langle \omega, \delta Y g\rangle \\ &\quad \int_0^t \langle \Theta, \delta g\rangle + \int_0^t \langle R_{g*}\omega, \delta Y\rangle \quad ([12, \text{lemma } 3.4])\end{aligned}$$

where Θ is the canonical form of G. And as

$$\begin{aligned}\int_0^t \langle R_{g*}\omega, \delta Y\rangle &= \int_0^t \langle Adg^{-1}, \delta Y\rangle \\ &= \int_0^t Adg^{-1}\delta(\int \langle \omega, \delta Y\rangle) \\ &= \int_0^t Adg_s^{-1} \langle \omega, U\rangle (Y_s)ds \\ &= \int_0^t \langle \omega, U\rangle (Z_s)ds\end{aligned}$$

Then $\int_0^t \langle \Theta, \delta g\rangle = 0$,and $g_t = e$ because of $g_0 = e$ and ([12, lemma 3.3]). This is $Y = Z$. □

Corollary 2.2 *Let $P(M, G)$ be a principal fiber bundle,* $\mathbf{H}$ *a 2-connection and $L \in \Gamma(\tau M)$. Then every diffusion with infinitesimal generator $\mathbf{H}K$ is a horizontal lift by* $\mathbf{H}$ *of a diffusion with infinitesimal generator K.*

Proof: Let Y be a diffusion with infinitesimal generator $\mathbf{H}L$, then $\pi Y = X$ is a diffusion. Let Z the horizontal lift by $\mathbf{H}$ of X initialized in Y_0 , we know that Z is a diffusion with infinitesimal generator $\mathbf{H}K$ and by the above lemma $Y = Z$. □

Let M be a manifold and $L \in \Gamma(\tau M)$. We recall [5] that a stochastic process X is called a L-diffusion process if the probability law of X coincides with $\mathbf{P}_v(\cdot) = \int_M \mathbf{P}_x(\cdot)v(dx)$ where $\{\mathbf{P}_x : x \in M\}$ is a diffusion measure generated by L and v is the probability law of X_0.

We have the following result about factorization of L-diffusions.

Proposition 2.3 *Let $P(M, G)$ be a principal fiber bundle,* $\mathbf{H}$ *a 2-connection $K \in \Gamma(\tau M)$ and $A \in \mathcal{G}^{(2)}$. Let Z be a $\mathbf{H}K + A^*$-diffusion. Then there is an unique pair (Y, g) such that:*

i) Y is a $\mathbf{H}K$-diffusion and g is a A-diffusion,

ii) Y and g are independent, and

iii) $Z = Y \cdot g$.

Proof: By to prove the existence is the argument of [7, Proposition 6]. By the uniqueness, we consider Y' and g' another pair of diffusions such that satisfies the enunciate. As $\pi \circ Y = \pi \circ Y'$ and $Y_0 = Y'_0$, by the Corollary 2.2 we have that $Y = Y'$. Then $g = g'$, this completes the proof. □

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