ON COMPLEX BASES FOR NUMBER SYSTEMS WITH THE DIGIT SET {0,1}

Agnes Benedek and Rafael Panzone

ABSTRACT. One of the purposes of this paper is to prove an essentially known result: the family of complex numbers entitled to be eligible as bases for **C** with 0 and 1 as the admissible digits has five elements: $i\sqrt{2}$, $\pm 1/2+(i/2)\sqrt{7}$, $\pm 1+i$, if we ignore conjugates (§4). However it is shown (Th. 10) that only four are truly eligible as bases for **C**: $i\sqrt{2}$, $\pm 1/2+(i/2)\sqrt{7}$, $-1+i$. We describe the fractionary sets of these bases among which the twindragon and the tame dragon are found. Other results of more general nature precede the fundamental theorem (§2 and §3). They are interesting in themselves and some generalize partial contents of the main proposition. A final note refers the different sections of the paper to the items of the Bibliography.

I. INTRODUCTION.

1. POSITIONAL REPRESENTATION. Let $b \in$ **R** (**C**), $|b|>1$, $D=\{0,d_1,d_2,\ldots,d_k\}\subset$ $\subset$**R** (**C**). α is said *representable* in base b with ciphers D if there exists $\{a_j \in D ; j=M, M-1,\ldots\}$ such that $\alpha=\sum_{-\infty}^{M} a_j b^j$. We write $\alpha=a_M\ldots a_0.a_{-1}a_{-2}\ldots=(e.f)_b$ and call (e) the integral part of α and (f) the fractional part of α. Denote G the set of all representable numbers and define the set F of *fractional numbers* as those numbers in G with a representation such that $(e)=0$ and the set W of *integers* of the system as those with a representation such that $(f)=0$. A number r will be called a *rational* of the numerical system (b,D) if it has a finite positional representation, i.e., with $a_j=0$ for $j<J(r)$. U will denote the set of rationals of the system.

If $b=10$, $D=\{0,1,\ldots,9\}$ then $F=[0,1]$, W=N, $G=[0,\infty)$. Observe that U$\subset$**Q** and U$\neq$**Q**.

Assume $b=m$, m a positive integer, $D=\{0,1,\ldots,m-1\}$. In this case we obtain: $F=[0,1]$, W=N, $G=[0,\infty)$. In both cases we need a prefix to represent the negative numbers.

However a negative base fits to represent **R** without prefixes.

THEOREM 1. Let b be a negative integer, $m=|b|>1$ and D={0,1,...,m-1}. Then, W=**Z** and if $t\in$W then

(1) $$t = a_r b^r + a_{r-1}b^{r-1} + ... + a_1 b + a_0, \quad a_j \in \mathrm{D}$$

where the a_j are the non negative remainders of the divisions by b. Besides, t is uniquely represented as

(2) $$t = (a_r a_{r-1} ... a_0)_b = (a).(\overline{0}) \text{ in base } b \bullet$$

($\overline{...}$ denotes a period). The proof is left to the reader (cf. Th. 3).

1.1. THE BASE $-m$. It is well known and generally true that if the base $b\in$**R** (**C**) and D=$\{0,d_1,d_2,...,d_k\}\subset$ **R** (**C**) and $|b|>1$ then not every number of **R** (**C**) belongs to G or there are numbers with more than one positional representation (cf. §2 and Th. 4). We want to characterize the multirepresentable numbers of **R** in case $b=-m$. Now D={0,1,...,$m-1$} and by Th. 1, W=Z. Since

(3) $$-m/(m+1) = \sum_0^\infty (m-1)/(-m)^{2j+1}, \quad 1/(m+1) = \sum_1^\infty (m-1)/(-m)^{2j}$$

we have F⊂F':=[$-m/(m+1)$,$1/(m+1)$]. If $z'\in$F' and z:=$m/(m+1)+z'$ then $z\in$I:=[0,1]. Developing z in base m and same D and using (3) we get

$$z' = -m/(m+1) + z = \sum_0^\infty (m-1)/(-m)^{2j+1} + \sum_1^\infty (-1)^j a_{-j}/(-m)^j \in \mathrm{F}.$$

Then F=F'. It follows that G=**R** since **R**=$\bigcup\{g$+F:$g\in$W}. Recalling that $\overline{...}$ denotes a period, we have:

(4) $$\frac{1}{m+1} = 0.\overline{0(m-1)} = 1.\overline{(m-1)0} \qquad \frac{-m}{m+1} = 0.\overline{(m-1)0} = 1(m-1).\overline{0(m-1)}$$

THEOREM 2. If x has two representations in base $-m$, $x=(n_1).a_{-1}a_{-2}...=(n_2).b_{-1}b_{-2}...$ with $n_1>n_2$, then n_1-n_2=1 and $x=(n).\overline{0(m-1)}=(n+1).\overline{(m-1)0}$, $n=n_2$ $\bullet$

PROOF. Observe that $n_1-n_2=0.b_{-1}b_{-2}...-0.a_{-1}a_{-2}...=f_2-f_1$, $f_j\in\mathrm{F}=\left[\frac{-m}{m+1},\frac{1}{m+1}\right]$, can only be a positive integer for $f_2=\frac{1}{m+1}$ and $f_1=\frac{-m}{m+1}$. These numbers have unique representations with zero integral part given in (4), QED.

COROLLARY 1. A number x can not have three representations •

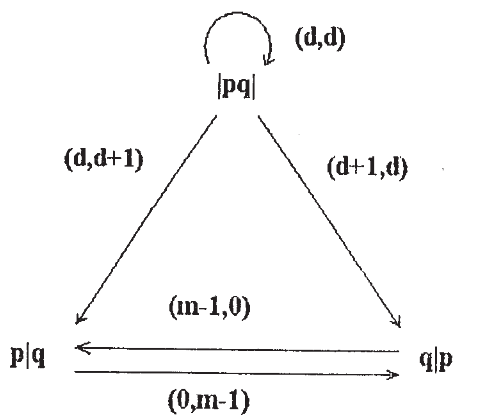
PROOF. Suppose there are three different representations of x. Then there is a j such that $b^j x$ has three representations with integral parts: $n_1 < n_2 < n_3$. But then $n_3 - n_1 > 1$, absurd, QED.

COROLLARY 2. The set B of numbers with two representations are those finally periodic of period 0(m-1) or (m-1)0. **R**\B consists of points uniquely representable •

1.2. THE GRAPH ASSOCIATED TO B. Let x be equal to $p = a_L ... a_0 . a_{-1} ...$ and to $q = c_L ... c_0 . c_{-1} ...$. The *state k* of the representations is the pair $p(k)$, $q(k)$ (cf.[G2]):

$p(k) = \sum_k^L a_j (-m)^{j-k}$, $q(k) = \sum_k^L c_j (-m)^{j-k}$. Therefore, $\sigma(k) := p(k) - q(k)$ is an integer

GRAPH Δ



equal to the sum of a series like that defining $n_1 - n_2$ in Th. 2. Thus, $\sigma(k)$ belongs to S:={-1,0,1}. We call $\sigma(k)$ the *type* of the state k and denote the types -1, 0, 1 symbolically and respectively as $p|q$, $| pq |$, $q|p$.

THEOREM 3. The succesive states of a number in B are obtained following an infinite string in the graph Δ •

PROOF. Observe that the following relation holds:

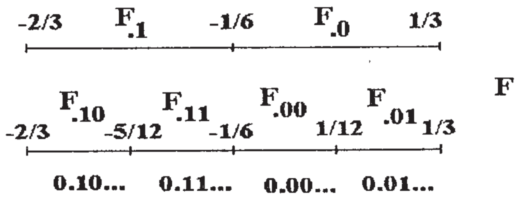
(5) $\sigma(k-1) = \sigma(k).b + (a_{k-1} - c_{k-1})$

The vectors (a_j, c_j) in the graph Δ near an arrow are the succesive ciphers of the representations p, q since $a_{k-1} - c_{k-1} \in \{-(m-1),...,0,...,m-1\}$ and $\sigma(k-1) \in \{-1,0,1\}$.

1.3. THE UNITARY SET F. The set F is the invariant set of the contractions

(6) $\Phi_j(z) = z/b + j/b$, $j = 0,...,(m-1)$.

In fact, $\Phi_j(F) = \left[-\frac{(m+1)j+1}{m(m+1)}, \frac{m-(m+1)j}{m(m+1)} \right]$ is the set of numbers that can be represented as $0.j$ ... .

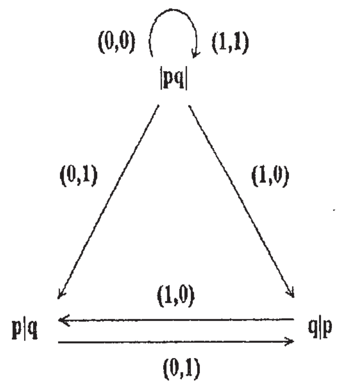


1.4. THE BASE $b=-2$.

The diagram shows the set F in case m=2. Here

$$F_{.a_{-1}a_{-2}...a_{-n}} := \{x; x=(0.a_{-1}...a_{-n}...)_{-2}\}$$

The graph Δ takes now the form:



1.5. THE BASE $b=-\sqrt{2}$. Let D={0,1}. In this case we have W=$\{m-n\sqrt{2}: m,n\in \mathrm{N}\}$. In fact, a number $w\in$W can be written as

$$(7)\ w=\sum_{j=0}^{L} a_j.2^j-\sqrt{2}\sum_{j=0}^{L} a'_j 2^j = m-\sqrt{2}.n,\ m,n\in \mathrm{N}.$$

Because of the irrationality of $\sqrt{2}$, m and n are uniquely determined, that is, w is uniquely representable as an integer of the system. Moreover, W is dense in **R**. In fact, it holds that

$$\left|\sqrt{2}-p_n/q_n\right|\le 1/q_n^2,\quad q_n\uparrow\infty,\quad p_{2k}/q_{2k}<\sqrt{2}<p_{2k+1}/q_{2k+1}$$

for the consecutive convergents to the continued fraction whose value is $\sqrt{2}$ and consequently, $\{k.(p_j-q_j\sqrt{2}): k\in N, j=1,2,...\}$ is dense in **R**. Furthermore, a number α belongs to F if and only if it can be written as

$$(8)\qquad \alpha=\sum_{-\infty}^{-1} a_j 2^j-\sqrt{2}\sum_{\infty}^{-1} a'_j 2^j = x-\sqrt{2}y,\quad x,y\in[0,1]$$

Then, F=$[-\sqrt{2},1]$. Moreover, each number in $(-\sqrt{2},1)$ has infinite representations of the form (8). Since **R**=$\bigcup\{w+F: w\in W\}$, the next proposition 1 follows.

PROPOSITION 1. A real number has infinite representations in the numerical system $(-\sqrt{2},\{0,1\})$ •

2. BASIC RESULTS. RESTRICTIVE ASSUMPTIONS. Given a complex number b, $|b|>1$, the *base*, and a set of numbers D=$\{d_0,...,d_k\}$, $k>0$, $d_0=0$, the *ciphers*, we write $x=(a_M...a_0.a_{-1}a_{-2}...)_b, a_j \in D$, instead of $x=\sum_{\infty}^{M} a_j b^j$. We have D⊂W. Besides, F is a compact set since D is finite. Let us consider the following hypothesis:

H0) 0 *has a unique positional representation in* (b,D).

THEOREM of equivalence. H0) is equivalent to

HB) *for any* M *there is a* $v=v(M)\in$ N *such that if* $w\in W$ *and* $|w|\leq M$ *then* $w=\sum_0^v a_i b^i, a_i \in D$ •

PROOF. If 0=$(w.f)_b$ then 0=$b^J(w.f)_b=(w_J.f_J)_b$. $\{w_J\}$ is a bounded set and this contradicts HB) if $w.f$ has a non null cipher. Assume H0). If $g_N=\left(a_N^{(N)}...a_0^{(N)}\right)_b$ is a bounded sequence with $a_N^{(N)}\neq 0$, $\{N\}$ an increasing sequence of integers, then a subsequence verifies: $g_{N_i}=\left(c_1...c_i a_{N_i-i}^{(N_i)}...a_0^{(N_i)}\right)$, $\quad N_i\uparrow\infty, \quad c_1\neq 0$.

Then, $0=lim \ b^{-N_i-1}g_{N_i}=0.c_1c_2...$, a contradiction.

THEOREM on the boundary. If H0) holds then any number in ∂F that is a limit of numbers in G\F has two representations •

PROOF. If $\left\{g_j=a_{M(j)}^{(j)}...a_0^{(j)}.a_{-1}^{(j)}...\right\}$ $\quad a_{M(j)}^{(j)}\neq 0$ converges then $\{M(j)\}$ must be a bounded set. Therefore $\{g_j\}$ has a convergent subsequence to a number $x=(w.f)_b$, $w\neq 0$ and the theorem follows.

THEOREM on the closure. Assume H0). If $\{g_j\}$⊂G and $g_j\rightarrow r$ then r∈G •

COROLLARY 3. Assume H0). Then $\overline{U}=G$ •

THEOREM on the ciphers. Assume H0). If $d_i,d_j\in D \quad d_i\neq d_j$ implies that $d_i-d_j\neq b.r$, $r\in W$, then the following proposition holds:

HD) *The ciphers have a unique representation as integers of the number system* (b,D) •

PROOF. Assume $d_i = (a_M...a_1d_j)_b$. Then, $d_i - d_j = b.(a_M...a_1)_b = 0$. Because of H0), $a_M = 0$, and the theorem follows.

THEOREM of uniqueness. (i) Assume H0). If W is a **Z**-module then it is uniquely representable in (b,D).

(ii) Assume that HD) and HW) hold:

HW) *If r and s belong to* W *then* $r-s$ *or* $s-r \in$W.

Then, every number of W has a unique representation as an integer of (b,D) •

PROOF. (i) If $w_1 = w_2.f$ then $(w_2 - w_1).f = 0$. Because of H0), $f = 0 = w_2 - w_1$.

(ii) Assume $w = (a_M...a_0) = (c_N...c_0)$. Then, $a_0 - c_0 = b[(c_N...c_1) - (a_M...a_1)] =$ $= b(r-s)$. If $0 \neq r-s \in W$ then $a_0 = c_0 + b(r-s)$ and a_0 has two representations, QED

COROLLARY 4. Let W be a **Z**-module. If the ciphers have a unique representation in (b,D), or they have a unique representation as integers of the number system, then the same holds for any number in W •

We introduce in §3 a new set of restrictive conditions on a number system. In what follows K will *denote* the field **C** or **R** and D⊂**K** will be assumed. The base b, $|b|>1$, will be real *if and only if* K=**R**. H° or *int*(H) will *denote* the open interior of the set H. The next theorem is of a very general nature

THEOREM 4. (i) $0 \in \text{int}(F) \Rightarrow$ K=G

(ii) K=G⇒ there are points with more than one representation •

PROOF. (i) If $z \in$K then $x = zb^{-j} \in \text{int}(F)$ for a certain j. Therefore $x = 0.a_{-1}...$ implies $z = a_{-1}...a_{-j}.a_{-j-1}...$. (ii) Assume that G=K. Because of the Baire category theorem, int(F)$\neq \varnothing$. Therefore, if W is discrete then every point in ∂F has two representations and if not, there are two tiles, w+F, w'+F, with interior points in common, QED.

3. POINT-LATTICES. Let us *define*: L=$\{m+nb : m,n \in Z\}$. We want L to be a point-lattice ([HW]). It surely is if $b\in$**C\R**. But if K=**R** we have to requiere that $b\in$**Q**. In fact, if b is an irrational number there exist pairs of integers such that the sequence $p_j - q_jb$ tends to zero (this can be seen as in the proof of proposition 1 using the convergents of

the continued fraction of b). Assume $b=p/q$, a reduced fraction. Call $r=1/q$. Then, $m+nb=(mq+np)r=kr$ and $L=\{kr:k\in Z\}$.

The following statements are the hypotheses mentioned in the preceding paragraph:

HC) $\mathbf{Z}\supset D\supset\{0,1\}$ *and for* $d_i,d_j\in D$, $|d_i-d_j|<|b|^\zeta$, $\zeta=dim$ K

HL) $W\subset L$ *and* L *is a point-lattice.*

THEOREM 5. Assume HL). Then,

(i) W=L $\Rightarrow$ K=G

(ii) If K=**R** then $b\in Z$ and $L\subset\mathbf{Z}$

(iii) If K=**C** then b satisfies a quadratic equation: $b^2-mb+K=0$ where $m=2\mathrm{Re}(b)$ and $K=|b|^2$ are integers with $4K>m^2$ •

PROOF. HL) implies $b,b^2\in L$. Then $b^2=mb-K$ with m,K integers and (iii) follows. If K=**R** and $b=p/q$, a reduced fraction, then $b^2=(mp-qK)/q$. This implies (ii), (cf. §1.4, §1.5 and also [HW], T.206).

(i). $U=\bigcup b^{-j}W=\bigcup_j\{mb^{-j}+nb^{-j+1}:m,n\in Z\}$ is dense in K. Thus, for $z\in K$ there is a sequence of rationals such that $u_n\to z$. Let $u_n=(w_n.f_n)_b$. Since F is compact, $\{|w_n|\}$ is bounded and $w_n=w$ for an infinite number of indices. Suppose then $u_n=(w.f_n)_b$. There is a subsequence such that, for any j, the ciphers of $\{f_n\}$ of index j are constant from some moment on. In consequence, $u_n\to z=(w.f)_b$, QED.

THEOREM 6. Assume HL). Then, K=G $\Rightarrow$ $0\in int$(F) •

PROOF. There is only a finite number of $w_j\in W$, $j=1,\ldots,n$, such that $|w_j|<2diam(F)$. Then, $V:=\bigcup\{w_j+F\}$ is a neighborhood of 0. In consequence, there is an m such that $b^{-m}V\subset F$, QED.

COROLLARY 5. If HL) holds then K=G $\Leftrightarrow$ $0\in int$(F) •

THEOREM 7. Assume HC) and HL). Then, any number in W is uniquely representable as an integer of the numerical system •

PROOF. Assume $w = bw_1 + d_1 = bw_2 + d_2$. In consequence

(9) $\qquad d_1 - d_2 = b(w_2 - w_1) = b(ub + v) \qquad u, v \in Z$.

Because of HC), if b is real, $w_1 = w_2$, and if not, $d_1 - d_2 = -uK + b(mu + v)$. Thus, $mu+v=0$ and $|d_1 - d_2| = |u|.|b|^2$ Then, $u=0$ and $v=0$. In both cases, $d_1 = d_2$ and the theorem follows.

THEOREM 8. HC) and HL) $\Rightarrow$ H0) •

PROOF. If 0 had more than one representation it would exist a $d \in D\setminus\{0\}$ such that $0 = (0.d\ldots)_b$. Multiplying by $b^k, k = 1,2,\ldots$, we would have $0 = (w_k . f_k)_b$ and necessarily some $w \in$W will have more than one representation as an integer, a contradiction, QED.

COROLLARY 6. HC) and HL) $\Rightarrow$ H0), HB) and HD) •

The theorem of uniqueness and theorem 8 imply the following result, (cf. Th. 5).

COROLLARY 7. Assume HC) and HL). Then, W = L $\Rightarrow$ F$\cap$(L\{0}) = $\varnothing$ •

3.1. EXAMPLES. (i) Let b=3, D={0,1}. Then **K**=**R**, F=T/2 where T is the ordinary Cantor set, N$\neq$W$\subset$N. H0) holds and any number of G is non negative and has a unique representation. Moreover, L=**Z** and 0 is not an interior point of F.

(ii) b=1+i, D={0,1}. b satisfies the equation $b^2 - 2b + 2 = 0$ and this implies that W$\subset$L=the set of Gaussian integers. Neither i=ib+1 (cf. Th. 7 and [G3]) nor $-1 = ib^2 + 1$ (see next Lemma 1) nor $-i = ib^3 + b + 1$ is representable as an integer of the system. However, $-i = (.\bar{1})_b \in$F. Since F is contained in the rectangle [-2/3,2/3]x[-4/3,1/3], (cf.[BP]), it follows that $i \notin$F. But $i = b^2 + (-i) = (100.\bar{1})_b$. Thus $i \in$(G$\cap$L)\(W$\cup$F).

Observe that if $p + iq \in$L then $b(p + iq) = (p - q) + i(p + q)$ has *its real and imaginary parts simultaneously odd or even*. This yields,

LEMMA 1. If $\alpha \in$ L\W then $b\alpha$ and $b\alpha + 1$ also belong to L\W •

Because of $-1, i \in$ L\W, the numbers in $V = \{-1-i, -i, 1-i, -1-2i, -2i, -2i+1\}$ belong to L\W. Applying three times the lemma and beginning with V, one obtains a set V' that contains a point β such that *dist* (β,W) ≥ $\sqrt{5}$ > 2. Since 2 > *diam* F (cf. [BP]), it follows that β is a *not representable number* : β∈L\G.

3.2. CIPHERS AND POINT-LATTICES. We say that L has D as a *complete set of residues*, or more precisely, D is a complete set of residues of bL with respect to L, if for any $l \in L$ there are an $l' \in L$ and a $d \in D$ such that $l = bl' + d$. The next theorem is a sort of converse of Corollary 7.

THEOREM 9. Assume that HL) holds and L has D as a complete set of residues. Then, $W \neq L \Rightarrow (L\backslash\{0\}) \cap F \neq \varnothing$ •

PROOF. Let $l_0 \in L \backslash W$ $l_0 = bl_1 + d_0$ $d_0 \in D$. Then $l_1 \in L \backslash W$. We define recursively $l_j = bl_{j+1} + d_j$. Let $C = max\{|d_j|\}$. We get $|l_{j+1}| \leq \frac{C + |l_j|}{|b|} \leq \frac{C}{|b|} + \frac{C}{|b|^2} + ... + \frac{C + |l_0|}{|b|^{j+1}} \leq$ $\leq \frac{C}{|b| - 1} + \frac{|l_0|}{|b|^{j+1}}$. Therefore, $\{l_j\} \subset L \backslash W$ is a bounded sequence. In consequence, there exist j and k, j<k, such that $l_j = l_k$. Without loss of generality we may assume that j=0. Then, $l_0 = d_0 + bl_1 = d_0 + bd_1 + b^2 l_2 = ... = d_0 + bd_1 + ... + b^{k-1} d_{k-1} + b^k l_0$ and $b^{Mk} l_0 + \sum_{s=0}^{M-1} b^{ks} (d_0 + bd_1 + ... + b^{k-1} d_{k-1}) = l_0$. This yields $-l_0 = 0.\overline{d_{k-1} ... d_0} \in F$, QED.

II. FOUNDATION.

4. COMPLEX BASES WITH DIGITS 0,1. Let $b \in \mathbf{C} \backslash \mathbf{R}$, $|b| > 1$, D={0,1}. We ask for conditions on b which ensure that for the unitary set F=F(b):= $= \{z = 0.a_{-1}a_{-2}... = \sum_1^\infty a_{-i} b^{-i} : a_{-i} \in D\}$ there exists a point-lattice $L := [u, v]$, $[u, v] := \{l = mu + nv : m, n \in Z\}$, u and v being linearly independent vectors, such that the sets $\{l + F : l \in L\}$ tessellate the plane in the sense of the following definition.

DEFINITION 1. *A tessellation of a set in the plane is a locally finite covering by bounded subsets which intersect pairwise in sets of Lebesgue measure zero.*

With $m(.)$ we denote the Lebesgue measure in $\mathbf{R}^2$ and $< u, v >$ will stand for the parallelogram with sides u and v. We state next two useful propositions.

PROPOSITION 2. Let L=[u,v] and F be a compact set. Assume that $\bigcup\{l+F^o:l\in L\}$ is a disjoint union. Then

(i) $m(F^o)\le m(<u,v>)$

(ii) if $m(F^o)=m(<u,v>)$ then $\bigcup\{l+F:l\in L\}=$ $\mathbf{R}^2$ and $m(F)=m(F^o)$ •

PROOF. Cf. [HW], §3.11.

PROPOSITION 2'. Let L=[u,v] and F be a compact set. Suppose that. $\bigcup\{l+F:l\in L\}=$ =R^2 Then

(a) F has nonvoid interior and $m(F)\ge m(<u,v>)$

(b) if $\{l+F:l\in L\}$ is a tessellation of the plane then $m(F)=m(<u,v>)$=$m(F^o)$

(c) conversely, if $m(F)=m(<u,v>)$ then $\{l+F:l\in L\}$ is a tessellation of the plane •

PROOF. It follows the lines of the preceding proposition and is left to the reader.

4.1. THE CONTEXT. Since

(10) $$bF=\{z=a_0.a_{-1}a_{-2}\ldots:a_j\in D\}=F\bigcup(F+1)$$

one gets $b^2F=F\bigcup(F+1)\bigcup(F+b)\bigcup(F+1+b)$ and also

(11) $$b^nF=\bigcup\{F+g:g=(a_{n-1}\ldots a_0)\in W\}\qquad a_j\in D.$$

Observe that from $m(bF)=|b|^2m(F)=m(F\cup(F+1))$ it follows that $|b|^2=2$ by simply assuming that $F^o\ne\varnothing$, $m(F\cap(F+1))=0$. In this situation the union of the sets $F+g,g\in W$, contains balls of radii as great as we please.

In many examples W is part of a point-lattice [1,v] which necessarily contains L:=[1,b]. We shall see in §4.4 that this implies L⊃W. Accordingly, we *assume that hypothesis* HL) *holds with the lattice* L=[1,b]. From (iii) of theorem 5 we have,

12) $$b^2=mb-K\qquad m,K\in Z$$

(13) $$|b|^2=K>m^2/4.$$

If $\{F+l:l\in L\}$ is a tessellation of $\mathbf{R}^2$ then $F^o\ne\varnothing$ and $m(F\cap(F+1))=0$. It follows that $\{bF+\lambda:\lambda\in\Lambda\}$ $\quad\Lambda:=bL$ is also a tessellation of $\mathbf{R}^2$. Using (10), we obtain that $\{F+l':l'\in L'\}$ $\quad L':=\Lambda\bigcup\{\Lambda+1\}$ is also a tessellation of the plane. Now, the fact that

$L' \subset L$ implies $L = L'$. That is,

(14) $$L = \Lambda \cup (\Lambda + 1) \qquad \Lambda \cap (\Lambda + 1) = \varnothing .$$

This means that $\{0,1\}$ is a complete set of residues of Λ with respect to L. Since K=2 from formula (13) we obtain $8 > m^2$. Therefore, $m \in \{0, \pm 1, \pm 2\}$. We have proved

PROPOSITION 3. Let $b \in \mathbf{C} \setminus \mathbf{R}$, $|b|>1$, $D=\{0,1\}$. If $W \subset L := [1,b]$ then the following assertion i) implies ii):

i) $\{F + l : l \in L\}$ tessellates R^2

ii) $|b|^2 = 2$ and b satisfies a quadratic equation: $b^2 - mb + 2 = 0 \qquad m \in \{0, \pm 1, \pm 2\}$ •

REMARK. Observe that it may be proved with a point-lattice argument that K=2: $\lambda = \alpha b + \beta \in L' \subset L$ is of the form $\lambda = pb^2 + qb + \delta = -Kp + \delta + (mp + q)b$ where $\delta \in \{0,1\}$, i.e., $\alpha = mp + q, \beta = \delta - Kp$. To ensure that $L = L'$, β must take any integer value. Therefore, we must have K=2 since $K = |b|^2 > 1$. (Then α, β may take any integer value).

4.2. DEFINITION 2. *For* $\varphi \in \mathbf{C} \setminus \mathbf{R}$, $L(\varphi) := [1, \varphi]$. *We call* $\mu = -1/2 + i\sqrt{7}/2$, $\Gamma = -1 + i$ *and*

(15) $$S := \left\{ \pm i\sqrt{2},\ 1/2 \pm i\sqrt{7}/2,\ -1/2 \pm i\sqrt{7}/2,\ -1 \pm i,\ 1 \pm i \right\}, \qquad B := S \setminus \{1 \pm i\}.$$

S is the set of numbers b that satisfy ii) of proposition 3. The point-lattices associated to each pair of numbers in S are $L(i\sqrt{2}), L(\mu), L(\mu), L(\Gamma) = L(i), L(\Gamma)$, respectively. In fact,

PROPOSITION 4. Let $b \in \mathbf{C} \setminus \mathbf{R}$, $|b|>1$, $D=\{0,1\}$. Then, $F(-b) = F(b) - b/(b^2 - 1)$

$$F(\bar{b}) = \overline{F(b)} \qquad L(-b) = L(b) \qquad L(\bar{b}) = \overline{L(b)}$$ •

PROOF. Notice that $b/(b^2 - 1) = \sum_1^{\infty} b^{-2j+1}$. Thus, if $z \in F(b)$ then $z = \sum_1^{\infty} a_j b^{-j}$ and $z - b/(b^2 - 1) = \sum_1^{\infty} c_j (-b)^{-j}$ where $c_{2k} = a_{2k}, c_{2k+1} = 1 - a_{2k+1}$. In consequence, $c_j \in D$ and $z - b/(b^2 - 1)$ belongs to $F(-b)$. The proposition follows.

III. FUNDAMENTAL RESULT.

4.3. We shall prove that i) and ii) of proposition 3 are in fact equivalent. We shall say that $b\in S$ *is a binary basis if* G=C, that is, if any $z\in$ **C** can be written in the form

$$(16)\qquad z=(a_M a_{M-1}\dots a_0.a_{-1}a_{-2}\dots)_b=\sum_{-\infty}^{M}a_j b^j \qquad a_j\in\{0,1\}$$

PROPOSITION 5. $F(i\sqrt{2})=[-2/3,1/3]\times[-\sqrt{8}/3,\sqrt{2}/3]$; $W(i\sqrt{2})=L(i\sqrt{2})$ •

PROOF. $F(i\sqrt{2})=\left\{z=\sum_{j<0}a_j(i\sqrt{2})^j : a_j\in\{0,1\}\right\}=$

$$=\left\{z=x+iy\sqrt{2}: x=\sum_{j<0}a_{2j}(-2)^j, y=\sum_{j<0}a_{2j+1}(-2)^j\right\}$$

It follows from §1.4 that $x,y\in F(-2)=[-2/3,1/3]$ and from theorem 1 the equality

$$W(i\sqrt{2})=\left\{\sum_{j=0}^{N}a_j(i\sqrt{2})^j : a_j\in\{0,1\}\right\}=\{x+iy\sqrt{2}: x,y\in Z\}=L(i\sqrt{2}),\ \text{QED.}$$

THEOREM 10. i) If $b\in S$ then $\{F+\xi:\xi\in L(b)\}$ tessellates $\mathbf{R}^2$.

ii) If $b\in B$ then $W(b)=L(b)$.

iii) $b\in S$ is a binary basis if and only if $b\in B$.

iv) If $b\in S$ then $m(F(b))=|\mathrm{Im}(b)|$, $m(\partial F(b))=0$.

v) If $b\in S$ then the unitary set $F=F(b)$ is disk-like •

PROOF. iii), ii)$\Rightarrow$i): Suppose $b\in B$. Then b is a binary basis and $\mathbf{R}^2=\bigcup\{F+w: w\in W\}$. Using (11), we get

$$(17)\qquad 2^n m(F)=m(b^n F)=m\left(\bigcup\{F+w_j: j=1,2,\dots,2^n\}\right)\le\sum m(F+w_j)=2^n m(F)$$

and the inequality in (17) must be an equality. Therefore, if $w,w'\in W$ are different then $m((F+w)\cap(F+w'))=0$. In consequence, $\{F+l: l\in L=W\}$ tessellates $\mathbf{R}^2$ and i) holds for $b\in B$. By proposition 4, $\{F(-b)+l: l\in L(-b)=L(b)\}$, $b\in B$, also tessellates the plane. Thus i) holds for any $b\in S=B\bigcup(-B)$.

ii) Because of proposition 4, it is enough to prove $W(b)=L(b)$ for $b\in B,\mathrm{Im}(b)>0$. The case $b=\Gamma$ is treated in [KS] and $b=i\sqrt{2}$ in proposition 5. However in any case one can use the following result.

PROPOSITION 6. If $b\in\mathbf{C}\setminus\mathbf{R}$ satisfies $b^2-mb+k=0$ where $m,k\in Z$, $k>1$, $D:=\{0,1,\ldots,k\text{-}1\}$, then $W(b)=L(b)$ iff *the critical set* $\mathcal{C}(b,D):=\{g\in L(b):|g|\leq 1+\sqrt{k}\}$ is contained in $W(b)$ •

PROOF. L is the disjoint union $L(b)=\bigcup\{\Lambda+\delta:\delta\in D\}$ where $\Lambda:=\{b\xi:\xi\in L\}$. Thus $\forall z\in L$ there is a (unique) $z_1\in L$ and a $\delta\in D$ such that $z=bz_1+\delta$. Now $|z_1|\leq(k-1+|z|)/\sqrt{k}$. Therefore, if $|z|>\sqrt{k}+1=(k-1)/(\sqrt{k}-1)$ then $|z_1|<|z|$. Then, any number in L may be written as $z=b^n z_n+b^{n-1}\delta_{n-1}+\ldots+b\delta_1+\delta_0$ with $\delta_j\in D$ and $z_n\in\mathcal{C}(b,D)$. In consequence, $z\in W$ if $z_n\in W$, QED.

To finish the proof of ii) we simply exhibit for each $b\in B, i\sqrt{2}\neq b\neq\Gamma,\mathrm{Im}(b)>0$, the positional representation of the numbers in $\mathcal{C}(b,D)\setminus\{0,1,b,b+1\}$. The expresions that follow can be verified by using in each case the equation $b^2=mb-2$ with the corresponding m: $m=-1$, $m=+1$.

$b=\mu$: −2=(110) −1=(111) 2=(1010) b−1=(111001) b+2=(11100)
1−b=(1111) −b=(1110) −b−1=(101) −b−2=(100)

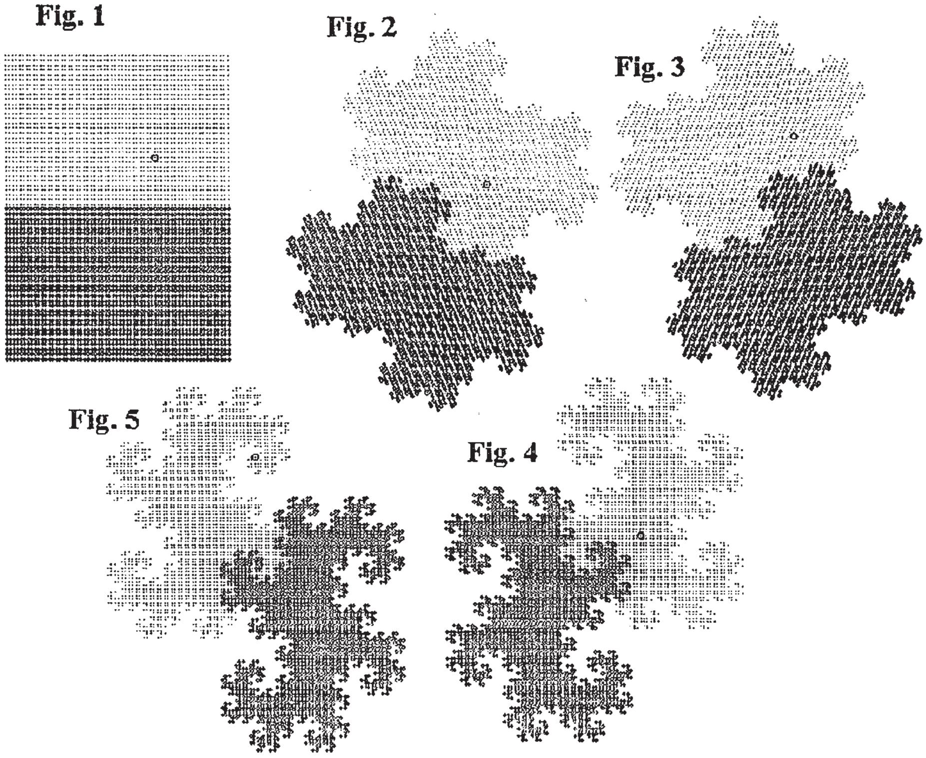
$b=-\bar{\mu}$: −1=(1011) −2=(1010) 2=(101110) b−1=(101) b−2=(100)
−b−1=(1001) −b=(10110) 1−b=(10111) 2−b=(101100)

iii) It is sufficient to prove iii) for $b\in S,\mathrm{Im}(b)>0$. $b=1+i$ is not a binary basis since in this case there is a $\beta\in L\setminus G$ as is shown in §3.1, Example (ii). Theorem 5 (i) and ii) of this theorem 10 yield **C**=G for $b=\Gamma, i\sqrt{2},\mu,-\bar{\mu}$.

iv) follows from proposition 2'.

v) It is enough to prove, again by proposition 4, that the three sets: $F(i\sqrt{2}), F(\Gamma)$, $F(\mu)$ are disk-like. The first is a rectangle (see proposition 5) and the second is shown to be disk-like in [BP]. We shall prove elsewhere the next result.

THEOREM 11. $T := F(\mu)$ is disk-like, i.e., T is the union of a Jordan curve $J = \partial T$ and its interior domain T^{o} •



° represents the origin. Fig.1 : F(i√2) Fig.2 : F(−½+i√7/2) Fig.3 : F(½+i√7/2) Fig.4 : F(−1+i) Fig.5 : F(1+i).

REMARK 2. Proposition 4 also implies that $dim(\partial F(-1+i)) = dim(\partial F(1+i))$ and $dim(\partial F(-\frac{1}{2}+i\sqrt{7}/2)) = dim(\partial F(\frac{1}{2}+i\sqrt{7}/2))$, improving iv) of theorem 10.

4.4. INTEGERS AND POINT-LATTICES. We assume that b and v are complex non-real numbers, b of modulus greater than one, D= $\{d_0 = 0, d_1 = 1, \ldots, d_k\} \subset \mathbf{Z}$ and $\mathcal{L} := [1, v]$. W denotes, as before, the set of integers of the numerical system (b,D) and L the lattice $[1,b]$; P:= $\{b^{1+j}; j=0,1,2,\ldots\}$. A greek letter denotes an ordinary integer.

THEOREM 12. *i) If* $\mathcal{L} \supset$ P *then* $b^2 \in$ L

ii) $\mathcal{L} \supset$ W *implies* L $\supset$ W •

PROOF. i) Let $b^{1+j} = m_j v + n_j$, $m_j, n_j \in \mathbf{Z}$. Then,

(18) $$b^{1+j} = b.b^j = m_0 m_{j-1} v^2 + (m_0 n_{j-1} + n_0 m_{j-1})v + n_0 n_{j-1} = m_j v + n_j$$

Taking j=1 in (18) we obtain,

(19) $$m_0^2 v^2 = (m_1 - 2m_0 n_0)v + (n_1 - n_0^2)$$

whence $(b-n_0)^2 = m_0 v.[(m_1/m_0) - 2n_0] + (n_1 - n_0^2)$. If m_1/m_0 is an ordinary integer then we shall have $(b-n_0)^2 = \lambda(b-n_0) + \mu$ and therefore $b^2 = \alpha b + \beta \in$ L. Thus, to finish the proof of i) it is enough to show that $m_0 \mid m_1$.

We obtain from (18), using (19), the next equality,

(20) $$m_j = (m_1/m_0 - n_0)m_{j-1} + m_0 n_{j-1}$$

Then,

(21) $$m_2 = m_1^2 / m_0 + \alpha m_1 + \beta m_0 \qquad m_0 \mid m_1^2$$

Assume, as an inductive hypothesis, that for k such that $2 \le k \le j-1$, it holds,

(22) $$m_0^{k-1} \mid m_1^k \qquad m_k = \sum_{h=0}^{k} \alpha_h m_1^h / m_0^{h-1} \qquad \alpha_k = 1$$

Because of (20) and the second set of relations in (22),

(23) $$m_j = \sum_{h=0}^{j} \beta_h m_1^h / m_0^{h-1} \qquad \beta_j = 1$$

and from (23) and the first set of relations in (22): $m_j = m_1^j / m_0^{j-1} + \varphi$. In consequence, $m_0^{j-1} \mid m_1^j$. Then, for any $j > 0$, $m_0^{j-1} \mid m_1^j$. From this, using the decomposition in prime factors, we obtain $m_0 \mid m_1$.

ii) Assume $\mathcal{L} \supset$W. Since W$\supset$P, i) implies that $b^2 \in$L. This is equivalent to L$\supset$W, QED.

NOTES. In relation to section 1 we refer the reader to [K], [G2], [E] and [KS]; for section 2 we refer to [G1] and [G2]; for §3, cf. [BG]; for §3.1, cf. [GD] and [G3]; for §4, cf. [HW] and [B]; for §4.1, cf. [BG]; for §4.3 and specifically for Th. 10, cf. [IKR] and [BG] and also [K], [BP].

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Dep. and Inst. of Mathematics, Universidad Nacional del Sur
(8000) Bahía Blanca, ARGENTINA