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A NOTE ON CHEBISHEV'S INEQUALITY VIA k-GENERALIZED FRACTIONAL INTEGRALS

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ABSTRACT. We establish certain integral inequalities for the Chebyshev functional in the case of synchronous functions, using the k-generalized fractional integrals. In this new framework, we prove several known integral inequalities.

1. INTRODUCTION

One of the most developed mathematical areas in recent years is that of integral inequalities, in particular using various fractional and generalized integral operators (see, for example, [4, 6, 11, 13, 14, 19, 24]). In earlier work, generalized k-proportional fractional integral operators with general kernel were defined, containing many of the known fractional operators.

To facilitate the understanding of this work, we need to present some preliminary results. We will use throughout the functions Γ (see [26, 28, 36, 37]) and Γ_k (defined in [9]):

$$\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau} \mathrm{d}\tau, \quad \mathrm{Re}(z) > 0,$$
$$\Gamma_k(z) = \int_0^\infty \tau^{z-1} e^{-\tau^k/k} \mathrm{d}\tau, \quad k > 0.$$

It is clear that if $k \to 1$ we have $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$, and $\Gamma_k(z+k) = z\Gamma_k(z)$. We also define the k-beta function as follows:

$$B_k(u,v) = \frac{1}{k}\int_0^1 \tau^{\frac{u}{k}-1}(1-\tau)^{\frac{v}{k}-1}\,d\tau.$$

Notice that $B_k(u,v) = \frac{1}{k}B(\frac{u}{k},\frac{v}{k})$ and $B_k(u,v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}$.

Although there is a "basic" fractional integral operator, that of Riemann–Liouville, this has been the origin of various extensions and generalizations, one of which we present in this work, from the point of view of differential operators; by manipulating simple algebraic identities, we can follow the idea of fractional differential operators of the Riemann–Liouville or Caputo type. From the simple facts $\alpha = 1+\alpha-1$ or $\alpha = \alpha-1+1$ we have, respectively,

$${}^{RL}D^\alpha f(t) = \frac{d}{dt}\left\{J_{a_1}^{1-\alpha}(f)(t)\right\},$$
$${}^{C}D^\alpha f(t) = J_{a_1}^{1-\alpha}\left(\frac{df}{dt}\right)(t).$$

For the reader's convenience, we present several definitions of fractional integrals, some of them very recent (with $0 \le a_1 < \tau < a_2 \le \infty$). One of the first operators that can be called fractional is the Riemann–Liouville fractional derivative of order $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$, defined as follows (see [12]).

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Definition 1.1. Let $a_1 < a_2$ and $f \in L^1((a_1,a_2);\mathbb{R})$. The right- and left-sided Riemann–Liouville fractional integrals of order α, with $\mathrm{Re}(\alpha) > 0$, are defined, respectively, by

$$ {}^{RL}J^{\alpha}_{a_1+} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{t} (t-s)^{\alpha-1} f(s)\,ds $$

and

$$ {}^{RL}J^{\alpha}_{a_2-} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{a_2} (s-t)^{\alpha-1} f(s)\,ds, $$

with $t \in (a_1,a_2)$.

The following are other definitions of fractional integral operators.

Definition 1.2 ([15])**.** Let $a_1 < a_2$ and $f \in L^1((a_1,a_2);\mathbb{R})$. The right- and left-sided Hadamard fractional integrals of order α, with $\mathrm{Re}(\alpha) > 0$, are defined, respectively, by

$$ H^{\alpha}_{a_1+} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{t} \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s}\,ds $$

and

$$ H^{\alpha}_{a_2-} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{a_2} \left(\log\frac{s}{t}\right)^{\alpha-1} \frac{f(s)}{s}\,ds, $$

with $t \in (a_1,a_2)$.

New fractional integral operators, called the Katugampola fractional integrals, were introduced in [17]:

Definition 1.3. Let $0 < a_1 < a_2$. Let $f : [a_1,a_2] \to \mathbb{R}$ be an integrable function. Let $\alpha \in (0,1)$ and $\rho > 0$ be two fixed real numbers. The right- and left-sided Katugampola fractional integrals of order α are defined, respectively, by

$$ k^{\alpha,\rho}_{a_1+} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{a_1}^{t} \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}} f(s)\,ds, $$

and

$$ k^{\alpha,\rho}_{a_2-} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t}^{a_2} \frac{t^{\rho-1}}{(s^{\rho}-t^{\rho})^{1-\alpha}} f(s)\,ds, $$

with $t \in (a_1,a_2)$.

The left- and right-sided Riemann–Liouville k-fractional integrals are given in [22].

Definition 1.4. Let $f \in L_1[a_1,a_2]$. Then the Riemann–Liouville k-fractional integrals of order $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$ and $k > 0$, are given by the expressions

$$ {}^{\alpha}I^{k}_{a_1+} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{u} (u-\tau)^{\frac{\alpha}{k}-1} f(\tau)\,d\tau, \quad u > a_1, $$

$$ {}^{\alpha}I^{k}_{a_2-} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{u}^{a_2} (\tau-u)^{\frac{\alpha}{k}-1} f(\tau)\,d\tau, \quad u < a_2. $$

A more general definition of the Riemann–Liouville fractional integrals is given in [18].

Definition 1.5. Let $f:[a_1,a_2]\to\mathbb{R}$ be an integrable function. Let g be an increasing and positive function on $(a_1,a_2]$ with a continuous derivative g' on (a_1,a_2). The left- and right-sided fractional integrals of f with respect to g on $[a_1,a_2]$ of order $\alpha\in\mathbb{C}$, with $\mathrm{Re}(\alpha)>0$, are expressed by

$$
{}^{\alpha}_{g}I_{a_1^+}f(u)=\frac{1}{\Gamma(\alpha)}\int_{a_1}^{u}\big(g(u)-g(\tau)\big)^{\alpha-1}g'(\tau)f(\tau)\,d\tau,\quad u>a_1,
$$
$$
{}^{\alpha}_{g}I_{a_2^-}f(u)=\frac{1}{\Gamma(\alpha)}\int_{u}^{a_2}\big(g(\tau)-g(u)\big)^{\alpha-1}g'(\tau)f(\tau)\,d\tau,\quad u<a_2.
$$

The following is a k-fractional analogue of Definition 1.5 (see [3, 20, 29]):

Definition 1.6. Let $f:[a_1,a_2]\to\mathbb{R}$ be an integrable function. Let g be an increasing and positive function on $(a_1,a_2]$ with a continuous derivative g' on (a_1,a_2). The left- and right-sided k-fractional integrals of f with respect to g on $[a_1,a_2]$ of order $\alpha\in\mathbb{C}$, with $\mathrm{Re}(\alpha)>0$ and $k>0$, are expressed by

$$
{}^{\alpha}I^{k}_{a_1^+}f(u)=\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{u}\big(g(u)-g(\tau)\big)^{\frac{\alpha}{k}-1}g'(\tau)f(\tau)\,d\tau,\quad u>a_1,
$$
$$
{}^{\alpha}I^{k}_{a_2^-}f(u)=\frac{1}{k\Gamma_k(\alpha)}\int_{u}^{a_2}\big(g(\tau)-g(u)\big)^{\frac{\alpha}{k}-1}g'(\tau)f(\tau)\,d\tau,\quad u<a_2.
$$

An interesting definition is that of the generalized proportional fractional (GPF) integral (see [30]).

Definition 1.7. Let $U\in X^q_\Psi(0,+\infty)$, $0<a_1<a_2$, there is an increasing, positive monotone function Ψ defined on $[0,+\infty)$ having continuous derivative Ψ' on $[0,+\infty)$ with $\Psi(0)=0$. Then the left- and right-sided GPF integral operator of a function U in the sense of another function Ψ of order η are given by

$$
{}^{\Psi}T^{\eta,\varsigma}_{\upsilon_1+}U(\varsigma)=\frac{1}{\varsigma^{\eta}\Gamma(\eta)}\int_{\upsilon_1}^{\varsigma}\frac{\exp\left[\frac{\varsigma-1}{\varsigma}\left(\Psi(\varsigma)-\Psi(\xi)\right)\right]}{\left(\Psi(\varsigma)-\Psi(\xi)\right)^{1-\eta}}U(\xi)\Psi'(\xi)\,d\xi,\quad \upsilon_1<\varsigma,
$$

and

$$
{}^{\Psi}T^{\eta,\varsigma}_{\upsilon_2-}U(\varsigma)=\frac{1}{\varsigma^{\eta}\Gamma(\eta)}\int_{\upsilon_1}^{\varsigma}\frac{\exp\left[\frac{\varsigma-1}{\varsigma}\left(\Psi(\xi)-\Psi(\varsigma)\right)\right]}{\left(\Psi(\xi)-\Psi(\varsigma)\right)^{1-\eta}}U(\xi)\Psi'(\xi)\,d\xi,\quad \varsigma<\upsilon_2,
$$

where the proportionality index $\varsigma\in(0,1]$, $\eta\in\mathbb{C}$, $\mathrm{Re}(\eta)>0$, and Γ is the gamma function.

The functional space on which we develop our work is the following.

Definition 1.8. Let $h\in L_1[0,+\infty)$ and let F be a continuous and positive function on $[0,+\infty)$ with $F(0)=0$. The space $X^q_F(0,+\infty)$ $(1\le q<+\infty)$ consists of those real-valued Lebesgue measurable functions h on $[0,+\infty)$ for which

$$
\|h\|_{X^q_F}=\left(\int_{a_1}^{a_2}|h(s)|^qF(s)\,ds\right)^{\frac{1}{q}}<+\infty,\quad 1\le q<+\infty,
$$

and, for the case $q=+\infty$,

$$
\|h\|_{X^\infty_F}=\operatorname*{ess\,sup}_{0\le s<\infty}\left[F(s)h(s)\right].
$$

We are now in a position to define the generalized integral operators that we will use in our work (see [23]).

Definition 1.9. Let $h \in X_F^q(0,+\infty)$ and let F be a continuous, positive function on $[0,+\infty)$ with $F(0)=0$. The right- and left-sided generalized k-proportional fractional integral operators with general kernel of order γ of h are defined, respectively, by

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}h(\chi) = \frac{1}{\lambda^\gamma k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,s),\lambda)F(s)h(s)}{(\mathbb{F}_+(\chi,s))^{1-\frac{\gamma}{k}}}\,ds \tag{1}$$

and

$$J_{F,a_2-}^{\frac{\gamma}{k},\lambda}h(\chi) = \frac{1}{\lambda^\gamma k\Gamma_k(\gamma)}\int_{\chi}^{a_2}\frac{G(\mathbb{F}_-(s,\chi),\lambda)F(s)h(s)}{(\mathbb{F}_-(s,\chi))^{1-\frac{\gamma}{k}}}\,ds, \tag{2}$$

where the proportionality index $\lambda \in (0,1)$, $\gamma \in \mathbb{C}$, $\mathrm{Re}(\gamma) > 0$, $\chi \in (a_1,a_2)$, $\mathbb{F}_+(\chi,s) = \int_s^\chi F(r)\,dr$, $\mathbb{F}_-(s,\chi) = \int_\chi^s F(r)\,dr$, and $G(\mathbb{F}_+(\chi,s),1) = G(\mathbb{F}_-(\chi,s),1) = 1$.

Of course there are other integral fractional operators; variations of the previous ones can be considered, but we will omit them.

Remark 1.10. Next, we will show that many integral operators are particular cases of (1) and (2).

1. (1) If in Definition 1.9 we make $k=1$, $F=1$, and $\lambda=1$, we obtain the Riemann–Liouville operators of Definition 1.1.
2. (2) Under the above conditions, if $k \neq 1$ then, from Definition 1.9, the k-fractional operators of [22] are obtained.
3. (3) If $F(s)=\frac{1}{s}$, $\lambda=1$, and $k=1$, then the Hadamard fractional operator is reproduced; see Definition 1.2 and [15, 31].
4. (4) If $F(s)=\frac{1}{s^\rho}$, $\lambda=1$, and $k=1$, then we obtain the Katugampola fractional operator of Definition 1.3; see [17].
5. (5) Choosing $\lambda=1$, $F(s)=g'(s)$, and $k=1$, we get the integral operator of [18].
6. (6) Taking $F(s)=\frac{1}{s}$, $k\neq 1$, and $G(\mathbb{F}_+(\chi,s),\lambda) = \exp\left[\frac{\lambda-1}{\lambda}\left(\ln\frac{\chi}{s}\right)\right]$, we obtain the integral operator of [27].
7. (7) We can obtain an integral operator with a non-singular nucleus, of the Riemann–Liouville type, by putting $\gamma=k=1$, $F(t)=1$, and $G(\mathbb{F}_+(x,s),\alpha) = \exp\left[-\frac{1-\alpha}{\alpha}(x-s)\right]$; this is a slight modification of the operator of [2].
8. (8) Choosing $\lambda\neq 1$, $F(s)=g'(s)$, $k=1$, and $G(\mathbb{F}_+(\chi,s),\lambda) = \exp\left[\frac{\lambda-1}{\lambda}(g(\chi)-g(s))\right]$, we obtain the integral operator of [30], called GPF, presented in Definition 1.7.

Theorem 1.11. *Under the conditions $a_1 < \varsigma \leq a_2$, $\theta > 0$, $\sigma > 0$, and $\gamma > 0$, suppose that ω is a continuous positive non-decreasing function on $[a_1,a_2]$ and $h : [a_1,a_2] \to \mathbb{R}^+$ is also a continuous positive function. Then the following inequality is valid:*

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[(\varsigma-a_1)^\sigma h(\varsigma)\omega^\theta(\varsigma)]J_{F,a_1+}^{\frac{\gamma}{k},\lambda}h(\varsigma) \geq J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[(\varsigma-a_1)^\sigma h(\varsigma)]J_{F,a_1+}^{\frac{\gamma}{k},\lambda}[h(\varsigma)\omega^\theta(\varsigma)].$$

2. Results

One of the best known integral inequalities is Chebishev's inequality (see [7]), which establishes a relationship between the integral of the product of two functions and the product of their integrals. This inequality was stated in the framework of the classical Riemann integral:

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \geq \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{3}$$

where f and g are two integrable and synchronous functions on $[a,b]$, $a<b$, $a,b\in\mathbb{R}$. Inequality (3) has many applications in diverse research subjects such as numerical quadrature, transform theory, probability, existence of solutions of differential equations and statistical problems. Many authors have investigated generalizations of the Chebyshev inequality (3); these are called *Chebyshev-type inequalities* (see e.g. [1, 5, 8, 10, 16, 21, 25, 33, 34, 35]). This inequality will be generalized in the present work, using the generalized operator of Definition 1.9; many of the inequalities reported in the literature will be obtained as particular cases.

Definition 2.1. Two functions f and g are said to be synchronous (resp., asynchronous) on $[a,b]$ if

$$((f(u)-f(v))(g(u)-g(v)))\geq 0 \quad \text{(resp., } \leq 0)$$

for all $u,v\in[0,+\infty)$.

In Chebyshev's work cited above, the following functional is presented which has been the subject of attention in many investigations:

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right),$$

where f and g are two integrable functions which are synchronous on $[a,b]$.

We will consider the following generalization of the functional $T(f,g)$:

$$C_\lambda^{\frac{\gamma}{k}}(f,g)=J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)-\left(\frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}f(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}g(\chi)}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)}\right),$$

with

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)=\frac{1}{\lambda^\gamma k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)(1)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}.$$

Our first result is the following.

Theorem 2.2. *Let f and g be two synchronous functions on $[0,\infty)$. Then for all $\chi>a_1\geq 0$, $\alpha>0$, $\lambda\in(0,1)$, and $\gamma\in\mathbb{C}$ with $\mathrm{Re}(\gamma)>0$, we have $C_\lambda^{\frac{\gamma}{k}}(f,g)\geq 0$.*

Proof. As f and g are synchronous on $[0,\infty)$, it follows that, for all $u,v\geq 0$, we have

$$(f(u)-f(v))(g(u)-g(v))\geq 0.$$

Therefore,

$$f(u)g(u)+f(v)g(v)\geq f(u)g(v)+f(v)g(u). \tag{4}$$

Multiplying both sides of (4) by $\frac{1}{\lambda^\gamma k\Gamma_k(\gamma)}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}$ and then integrating the result with respect to u over (a_1,χ), we obtain

$$\frac{1}{\lambda^\gamma k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)f(u)g(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}+\frac{f(v)g(v)}{\lambda^\gamma k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}$$

$$\geq\frac{g(v)}{\lambda^\gamma k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)f(u)F(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}+\frac{f(v)}{\lambda^\gamma k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)g(u)F(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}},$$

expression that we can write as

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)+f(v)g(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)\geq g(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)+f(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi). \tag{5}$$

Multiplying both sides of (5) by $\frac{1}{\lambda^{\gamma}k\Gamma_k(\gamma)}\frac{G(\mathbb{F}_+(\chi,v),\lambda)F(v)}{(\mathbb{F}_+(\chi,v))^{1-\frac{\gamma}{k}}}$ and then integrating the resulting inequality with respect to v over (a_1,χ), we obtain

$$\begin{aligned} & J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)+J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \\ & \quad \geq J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)+J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi), \end{aligned} \tag{6}$$

that is,

$$J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)\geq J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi).$$

Reordering and taking into account the definition of $C_{\lambda}^{\frac{\gamma}{k}}(f,g)$, the desired inequality is obtained. This completes the proof. □

Remark 2.3. If in Theorem 2.2 we consider $k=1$, $a_1=1$, and $F(s)=\frac{1}{s}$, we have the Hadamard integral and the above result covers Theorem 3.1 of [8].

Theorem 2.4. *Let f and g be two synchronous functions on $[0,\infty)$. Then for all $\chi>a_1\geq 0$, $\lambda\in(0,1)$, and $\gamma,\delta\in\mathbb{C}$, with $\mathrm{Re}(\gamma)>0$ and $\mathrm{Re}(\delta)>0$, we have $C_{\lambda}^{\frac{\gamma}{k},\frac{\delta}{k}}(f,g)\geq 0$, where*

$$\begin{aligned} C_{\lambda}^{\frac{\gamma}{k},\frac{\delta}{k}}(f,g) = & \frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)}+\frac{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fg)(\chi)}{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)} \\ & -\frac{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi)}{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)} \end{aligned}$$

is the generalization of the functional $T(f,g)$ for γ and δ.

Proof. Multiplying both sides of (5) by $\frac{1}{\lambda^{\delta}k\Gamma_k(\delta)}\frac{G(\mathbb{F}_+(\chi,v),\lambda)F(v)}{(\mathbb{F}_+(\chi,v))^{1-\frac{\delta}{k}}}$ and then integrating the resulting inequality with respect to v over (a_1,χ), we obtain

$$\begin{aligned} & J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \\ & \quad \geq J_{F,a_1+}^{\frac{\delta}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi). \end{aligned}$$

Dividing both sides of this inequality by $J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)$, we get

$$\frac{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)}{J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)}+\frac{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fg)(\chi)}{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)}\geq\frac{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi)}{J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)}.$$

Reordering and taking into account the definition of $C_{\lambda}^{\frac{\gamma}{k},\frac{\delta}{k}}(f,g)$, the desired inequality is obtained. This completes the proof. □

Remark 2.5. Note that

(1) Applying Theorem 2.4 for $\gamma=\beta$, we obtain Theorem 2.2.

(2) If in Theorem 2.4 we consider $k=1$ and $\gamma=\delta=1$, then $C_{\lambda}^{\frac{\gamma}{k},\frac{\delta}{k}}(f,g)$ is $T(f,g)$.

Theorem 2.6. *Let $(f_i)_{i=1,2,\dots,n}$ be positive increasing functions on $[0,\infty)$. Then for all $\chi > a_1 \geq 0$, $\lambda \in (0,1)$, and $\gamma \in \mathbb{C}$ with $\operatorname{Re}(\gamma) > 0$, we have*

$$J^{\frac{\gamma}{k},\lambda}_{F,a_1+}\left(\prod_{i=1}^{n} f_i\right)(\chi) \geq \left[J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(1)(\chi)\right]^{1-n} \prod_{i=1}^{n} J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_i)(\chi).$$

Proof. We will use induction. Crearly, for $n = 1$,

$$J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_1)(\chi) \geq J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_1)(\chi)$$

for all $\chi > a_1 \geq 0$, $\alpha > 0$, $\lambda \in (0,1)$, and $\gamma \in \mathbb{C}$ with $\operatorname{Re}(\gamma) > 0$.

For $n = 2$, applying equation (6), we obtain

$$J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_1 f_2)(\chi) \geq \left[J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(1)(\chi)\right]^{-1} J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_1)(\chi) J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_2)(\chi).$$

Suppose that by induction hyphotesis

$$J^{\frac{\gamma}{k},\lambda}_{F,a_1+}\left(\prod_{i=1}^{n-1} f_i\right)(\chi) \geq \left[J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(1)(\chi)\right]^{2-n} \prod_{i=1}^{n-1} J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_i)(\chi)$$

for all $\chi > a_1 \geq 0$, $\alpha > 0$, $\lambda \in (0,1)$, and $\gamma \in \mathbb{C}$ with $\operatorname{Re}(\gamma) > 0$.

Now, since $(f_i)_{i=1,2,\dots,n}$ are positive and increasing on $[0,\infty)$, we have that $\left(\prod_{i=1}^{n-1} f_i\right)(\chi)$ is also positive and increasing on $[0,\infty)$. Therefore we can apply Theorem 2.2 to the functions $g = \prod_{i=1}^{n-1} f_i$ and $f = f_n$; we thus obtain

$$\begin{aligned}
J^{\frac{\gamma}{k},\lambda}_{F,a_1+}\left(\prod_{i=1}^{n} f_i\right)(\chi) &\geq J^{\frac{\gamma}{k},\lambda}_{F,a_1+}\left(\prod_{i=1}^{n-1} f_i f_n\right)(\chi) J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(gf)(\chi) \\
&\geq \left[J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(1)(\chi)\right]^{-1} J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(g)(\chi) J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f)(\chi) \\
&\geq \left[J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(1)(\chi)\right]^{-1} J^{\frac{\gamma}{k},\lambda}_{F,a_1+}\left(\prod_{i=1}^{n-a}\right)(\chi) J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_n)(\chi) \\
&\geq \left[J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(1)(\chi)\right]^{-1} \left[J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(1)(\chi)\right]^{2-n} \prod_{i=1}^{n-1} J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_i)(\chi) J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_n)(\chi) \\
&\geq \left[J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(1)(\chi)\right]^{1-n} \prod_{i=1}^{n} J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f_i)(\chi).
\end{aligned}$$

This completes the proof. □

The previous results can be extended if we consider a certain positive "weight" function h.

Theorem 2.7. *Let f and g be two synchronous functions on $[0,\infty)$, $h \geq 0$. Then for all $\chi > a_1 \geq 0$, $\lambda > 0$, and $\gamma \in \mathbb{C}$ with $\operatorname{Re}(\gamma) > 0$, we have the following inequality:*

$$\begin{aligned}
J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(1)(\chi) J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(fgh)(\chi) \geq J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(g)(\chi) J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(fh)(\chi) + J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(f)(\chi) J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(gh)(\chi) \\
- J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(fg)(\chi) J^{\frac{\gamma}{k},\lambda}_{F,a_1+}(h)(\chi).
\end{aligned}$$

Proof. Since $h \geq 0$ and the functions f and g are synchronous on $[0,\infty)$, we have that

$$(f(u) - f(v))(g(u) - g(v))(h(u) + h(v)) \geq 0$$

for all $u,v \geq 0$. Then,

$$\begin{aligned} f(u)g(u)h(u)+f(v)g(v)h(v) \geq{}& f(u)g(v)h(u)+f(v)g(u)h(u)-f(v)g(v)h(u) \\ &-f(u)g(u)h(v)+f(u)g(v)h(v)+f(v)g(u)h(v). \end{aligned} \tag{7}$$

Multiplying both sides of (7) by $\frac{1}{\lambda^{\gamma}k\Gamma_k(\gamma)} \frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}$ and then integrating the resulting inequality with respect to u over (a_1,χ), we obtain

$$\begin{aligned} &\frac{1}{\lambda^{\gamma}k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)f(u)g(u)h(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}+\frac{f(v)g(v)h(v)}{\lambda^{\gamma}k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}} \\ &\geq \frac{g(v)}{\lambda^{\gamma}k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)f(u)h(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}+\frac{f(v)}{\lambda^{\gamma}k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)g(u)h(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}} \\ &\quad -\frac{f(v)g(v)}{\lambda^{\gamma}k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)h(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}-\frac{h(v)}{\lambda^{\gamma}k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)f(u)g(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}} \\ &\quad +\frac{g(v)h(v)}{\lambda^{\gamma}k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)f(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}+\frac{f(v)h(v)}{\lambda^{\gamma}k\Gamma_k(\gamma)}\int_{a_1}^{\chi}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)g(u)\,du}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}, \end{aligned}$$

expression that we can write as

$$\begin{aligned} &J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi)+f(v)g(v)h(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \\ &\quad \geq g(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi)+f(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi) \\ &\qquad -f(v)g(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi)-h(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi) \\ &\qquad +g(v)h(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)+f(v)h(v)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi). \end{aligned} \tag{8}$$

Multiplying both sides of (8) by $\frac{1}{\lambda^{\gamma}k\Gamma_k(\gamma)} \frac{G(\mathbb{F}_+(\chi,v),\lambda)F(v)}{(\mathbb{F}_+(\chi,v))^{1-\frac{\gamma}{k}}}$ and then integrating the resulting inequality with respect to v over (a_1,χ), we obtain

$$\begin{aligned} &J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi)+J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi) \\ &\quad \geq J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi) \\ &\qquad +J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi)-J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi) \\ &\qquad -J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)+J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi) \\ &\qquad +J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi), \end{aligned}$$

that is,

$$\begin{aligned} J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi) \geq{}& J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi)+J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi) \\ &-J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi). \end{aligned}$$

□

Remark 2.8. Applying Theorem 2.7 for $h = 1$, we obtain Theorem 2.2.

Theorem 2.9. *Let f and g be two synchronous functions on $[0,\infty)$ and $h \geq 0$. Then for all $\chi > a_1 \geq 0$, $\lambda \in (0,1)$, $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$, and $\delta \in \mathbb{C}$ with $\mathrm{Re}(\delta) > 0$, we have the* *following inequality:*

$$
\begin{aligned}
&J_{F,a_1+}^{\frac{\delta}{k},\lambda}(h)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fgh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi)\\
&\qquad\geq J_{F,a_1+}^{\frac{\delta}{k},\lambda}(gh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi).
\end{aligned}
$$

Proof. Multiplying both sides of (8) by $\frac{1}{\lambda^{\delta}k\Gamma_k(\delta)}\frac{G(\mathbb{F}_+(\chi,v),\lambda)F(v)}{(\mathbb{F}_+(\chi,v))^{1-\frac{\delta}{k}}}$ and then integrating the resulting inequality with respect to v over (a_1,χ), we obtain

$$
\begin{aligned}
&J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fgh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)\\
&\qquad\geq J_{F,a_1+}^{\frac{\delta}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi)\\
&\qquad\quad -J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi)-J_{F,a_1+}^{\frac{\delta}{k},\lambda}(h)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)\\
&\qquad\quad +J_{F,a_1+}^{\frac{\delta}{k},\lambda}(gh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi).
\end{aligned}
$$

□

Remark 2.10. Applying Theorem 2.9 for $h=1$, we obtain Theorem 2.4.

More precise results can be obtained if we impose additional conditions on the function h in the previous Theorem.

Theorem 2.11. *Let f, g and h be three monotonic functions defined on $[0,\infty)$ satisfying the inequality*

$$(f(u)-f(v))(g(u)-g(v))(h(u)-h(v))\geq 0$$

for all $u,v\in[a_1,\chi]$. Then, for all $\chi>a_1\geq 0$, $\lambda\in(0,1)$, $\gamma\in\mathbb{C}$ with $\mathrm{Re}(\gamma)>0$, and $\delta\in\mathbb{C}$ with $\mathrm{Re}(\delta)>0$, we have

$$
\begin{aligned}
&J_{F,a_1+}^{\frac{\delta}{k},\lambda}(1)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fgh)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fgh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(1)(\chi)\\
&\qquad\geq J_{F,a_1+}^{\frac{\delta}{k},\lambda}(g)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fh)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(f)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(gh)(\chi)\\
&\qquad\quad -J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fg)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(h)(\chi)-J_{F,a_1+}^{\frac{\delta}{k},\lambda}(h)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)\\
&\qquad\quad +J_{F,a_1+}^{\frac{\delta}{k},\lambda}(gh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(f)(\chi)+J_{F,a_1+}^{\frac{\delta}{k},\lambda}(fh)(\chi)J_{F,a_1+}^{\frac{\gamma}{k},\lambda}(g)(\chi).
\end{aligned}
$$

Proof. As in the proof of Theorem 2.9, if we multiply both sides of (8) by $\frac{1}{\lambda^{\delta}k\Gamma_k(\delta)}\times\frac{G(\mathbb{F}_+(\chi,v),\lambda)F(v)}{(\mathbb{F}_+(\chi,v))^{1-\frac{\delta}{k}}}$ and then integrate the resulting inequality with respect to v over (a_1,χ), we obtain the desired inequality. □

An inequality involving the square of the functions f and g can be stated as follows.

Theorem 2.12. *Let f and g be defined on $[0,\infty)$. Then for all $\chi>a_1\geq 0$, $\lambda\in(0,1)$, and $\gamma\in\mathbb{C}$ with $\mathrm{Re}(\gamma)>0$, we have:*

$$J_{F,a_1^+}^{\frac{\gamma}{k},\lambda}(1)[J_{F,a_1^+}^{\frac{\gamma}{k},\lambda}f^2(\chi)+J_{F,a_1^+}^{\frac{\gamma}{k},\lambda}g^2(\chi)]\geq 2J_{F,a_1^+}^{\frac{\gamma}{k},\lambda}f(\chi)J_{F,a_1^+}^{\frac{\gamma}{k},\lambda}g(\chi) \tag{9}$$

and

$$J_{F,a_1^+}^{\frac{\gamma}{k},\lambda}f^2(\chi)J_{F,a_1^+}^{\frac{\gamma}{k},\lambda}g^2(\chi)\geq[J_{F,a_1^+}^{\frac{\gamma}{k},\lambda}(fg)(\chi)]^2. \tag{10}$$

Proof. Since $(f(u)-g(v))^2 \geq 0$, we have

$$f^2(u)+g^2(v) \geq 2f(u)g(v). \tag{11}$$

Multiplying both sides of (11) by $\frac{1}{\lambda^{\gamma}k\Gamma_k(\gamma)}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}$ and then integrating the resulting inequality with respect to u and v over (a_1,χ), we obtain (9).

On the other hand, since $(f(u)g(v)-f(v)g(u))^2 \geq 0$, the same arguments as before let us obtain (10). ☐

Theorem 2.13. *Let f and g be defined on $[0,\infty)$. Then for all $\chi > a_1 \geq 0$, $\lambda \in (0,1)$, $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$, and $\delta \in \mathbb{C}$ with $\mathrm{Re}(\delta) > 0$, we have the following integral inequalities:*

$$J_{F,a_1^+}^{\frac{\gamma}{k},\lambda} f^2(\chi) J_{F,a_1^+}^{\frac{\delta}{k},\lambda}(1) + J_{F,a_1^+}^{\frac{\gamma}{k},\lambda}(1) J_{F,a_1^+}^{\frac{\delta}{k},\lambda} g^2(\chi) \geq 2J_{F,a_1^+}^{\frac{\gamma}{k},\lambda} f(\chi) J_{F,a_1^+}^{\frac{\delta}{k},\lambda} g(\chi), \tag{12}$$

$$J_{F,a_1^+}^{\frac{\gamma}{k},\lambda} f^2(\chi) J_{F,a_1^+}^{\frac{\delta}{k},\lambda} g^2(\chi) + J_{F,a_1^+}^{\frac{\delta}{k}} f^2(\chi) J_{F,a_1^+}^{\frac{\gamma}{k},\lambda} g^2(\chi) \geq 2J_{F,a_1^+}^{\frac{\gamma}{k},\lambda}(fg)(\chi) J_{F,a_1^+}^{\frac{\delta}{k},\lambda}(fg)(\chi). \tag{13}$$

Proof. Like before, since $(f(u)-g(v))^2 \geq 0$, we have

$$f^2(u)+g^2(v) \geq 2f(u)g(v). \tag{14}$$

Multiplying both sides of (14) by $\frac{1}{\lambda^{\gamma}k\Gamma_k(\gamma)}\frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}$ and $\frac{1}{\lambda^{\delta}k\Gamma_k(\delta)}\frac{G(\mathbb{F}_+(\chi,v),\lambda)F(v)}{(\mathbb{F}_+(\chi,v))^{1-\frac{\delta}{k}}}$, and then integrating the resulting inequality with respect to u and v over (a_1,χ), respectively, we obtain (12).

On the other hand, since $(f(u)g(v)-f(v)g(u))^2 \geq 0$, using the same arguments as before we obtain (13). ☐

Remark 2.14. If we consider $\gamma = \delta$, we obtain Theorem 2.12.

The following is a result in a different direction.

Theorem 2.15. *Let $f:\mathbb{R}\to\mathbb{R}$ with $\bar{f}(u) = \int_{a_1}^{u} F(z,s)f(z)\,dz$, $u > a_1 \geq 0$, $\lambda \in (0,1)$, and $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$. Then, for $\gamma \geq k > 0$, we have*

$$J_{F,a_1^+}^{\frac{\gamma+k}{k},\lambda} f(\chi) = \frac{1}{k} J_{F,a_1}^{\frac{\gamma}{k},\lambda} \bar{f}(\chi).$$

Proof. Here

$$\begin{aligned} J_{F,a_1^+}^{\frac{\gamma}{k},\lambda} \bar{f}(\chi) &= \frac{1}{\lambda^{\gamma}k\Gamma_k(\gamma)} \int_{a_1}^{\chi} \frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)\bar{f}(u)}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}}\,du \\ &= \frac{1}{\lambda^{\gamma}k\Gamma_k(\gamma)} \int_{a_1}^{\chi} \frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma}{k}}} \int_{a_1}^{u} F(z,s)f(z)\,dz\,du. \end{aligned}$$

Then, by Dirichlet's formula, we see that the last expression becomes

$$\begin{aligned} &\frac{1}{\lambda^{\gamma}k\Gamma_k(\gamma)} \int_{a_1}^{\chi} F(z,s)f(z) \int_{z}^{\chi} G(\mathbb{F}_+(\chi,u),\lambda)F(u)(\mathbb{F}_+(\chi,u))^{\frac{\gamma}{k}-1}\,du\,dz \\ &= \frac{1}{\lambda^{\gamma}\Gamma_k(\gamma+k)} \int_{a_1}^{\chi} \frac{G(\mathbb{F}_+(\chi,u),\lambda)F(u)f(u)}{(\mathbb{F}_+(\chi,u))^{1-\frac{\gamma+k}{k}}} \\ &= kJ_{F,a_1^+}^{\frac{\gamma+k}{k}} f(u). \end{aligned}$$

This completes the proof. ☐

3. Conclusions

In this work we introduce a generalized formulation of the Riemann–Liouville fractional integral, which contains, as particular cases, many of the integral operators reported in the literature. In this context, we present a number of integral inequalities that generalize several known inequalities.

We want to highlight the strength of Definition 1.9: if we consider the kernel $F(\chi,s)=\chi^{1-s}$ and $G\equiv 1$, we obtain a variant of the (k,s)-Riemann–Liouville fractional integral of [32]:

$$ {}^{s}_{a_1}I^{\alpha}_{u}f(u)=\frac{(2-s)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_{a_1}^{u}(u^{2-s}-\chi^{2-s})^{\frac{\alpha}{k}-1}\chi^{1-s}f(\chi)\,d\chi. $$

This opens up wide possibilities of obtaining new integral inequalities.

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