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A TOPOLOGICAL CHARACTERIZATION OF THE DEDEKIND–MACNEILLE COMPLETION FOR HEYTING ALGEBRAS

CARLOS ERNESTO SCIRICA AND ALEJANDRO GUSTAVO PETROVICH

ABSTRACT. An important problem in the theory of Heyting algebras is to find completions with different properties. In this paper we introduce the notion of a pseudoregular open set in topological spaces. By means of this notion and based on Stone's and Priestley's topologies we construct a topological completion of Heyting algebras and we prove that this completion coincides, up to isomorphism, with the well-known completion of Dedekind–MacNeille. Finally we prove an interesting minimality condition.

1. INTRODUCTION

Heyting algebras are the algebraic counterpart of the intuitionistic propositional calculus. An important problem in algebraic logic is to find completions of different ordered algebraic structures. In the case of Heyting algebras there are several known completions. Among them, we can mention the completion of Dedekind–MacNeille [6, pp. 187–188], the cones completion of Maksimova [7] and the profinite completion [2]. Of all the previous completions, the Dedekind–MacNeille completion is the only one which is regular. The regularity condition in a completion is used by Bezhanishvili and Harding in [4] to prove the functional representation of Heyting monadic algebras, which were introduced by Monteiro and Varsavsky in [8].

In the first part of this paper we construct a complete Heyting algebra by means of a set endowed with two topologies. We use this tool to give a topological characterization of the Dedekind–MacNeille completion by means of Stone's and Priestley's topologies. Finally we prove an interesting minimality condition related to this completion.

2. COMPLETE HEYTING ALGEBRAS

Definition 1. Let T_1 and T_2 be two topologies on a set X such that $T_1 \subseteq T_2$. We say that A is a *pseudoregular open set* if $A = \left(\overline{A}^2\right)^{0_1}$, where $\overline{A}^2$ denotes the closure of A under the topology T_2, and $\left(\overline{A}^2\right)^{0_1}$ is the interior of $\overline{A}^2$ under the topology T_1.

Note that if $T_1 = T_2$ then the notion of pseudoregular open set coincides with the usual notion of regular open set.

Lemma 1. *Let T_1 and T_2 be two topologies on a set X such that $T_1 \subseteq T_2$. Then $\left(\overline{A}^2\right)^{0_1}$ is a pseudoregular open set for every $A \subseteq X$.*

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Proof. Let $B = \left(\overline{A}^2\right)^{0_1}$. We must prove that $\left(\overline{B}^2\right)^{0_1} = B$. Since $B \in T_1$ and $B \subseteq \overline{B}^2$, then $B \subseteq \left(\overline{B}^2\right)^{0_1}$. On the other hand we have that $\left(\overline{B}^2\right)^{0_1} \subseteq \left(\overline{\overline{A}^2}^2\right)^{0_1} \subseteq \left(\overline{A}^2\right)^{0_1}$, which proves the reverse inclusion $\left(\overline{B}^2\right)^{0_1} \subseteq B$. □

In the literature a set X endowed with two topologies is called a *bitopological space* ([5]). In this paper we shall only consider bitopological spaces where one of the topologies is finer than the other one. In this case we shall denote by $\mathrm{PReg}(X)$ the set determined by the pseudoregular open subsets of X.

Theorem 2. *If X is a bitopological space then $\mathrm{PReg}(X)$ is a complete Heyting algebra where the order is given by inclusion, the lattice operations are given by $A_1 \wedge A_2 = A_1 \cap A_2$; $A_1 \vee A_2 = \left(\overline{A_1 \cup A_2}^2\right)^{0_1}$, and implication is given by the formula $A_1 \to A_2 = \left(\overline{A_1^c \cup A_2}^2\right)^{0_1}$. Moreover, if $(A_i)_{i\in I}$ is a family of pseudoregular open subsets of X, then $\bigvee_{i\in I} A_i = \left(\overline{\bigcup_{i\in I} A_i}^2\right)^{0_1}$ and $\bigwedge_{i\in I} A_i = \left(\overline{\bigcap_{i\in I} A_i}^2\right)^{0_1}$.*

Proof. Let A_1, A_2 be in $\mathrm{PReg}(X)$. In order to verify that $A_1 \wedge A_2 = A_1 \cap A_2$ it is enough to prove that $A_1 \cap A_2$ is also a pseudoregular open set. Since $A_1 \cap A_2$ is an open set in the topology T_1 it follows that $A_1 \cap A_2 \subseteq \left(\overline{A_1 \cap A_2}^2\right)^{0_1}$. On the other hand $\left(\overline{A_1 \cap A_2}^2\right)^{0_1} \subseteq \left(\overline{A_1}^2 \cap \overline{A_2}^2\right)^{0_1} = \left(\overline{A_1}^2\right)^{0_1} \cap \left(\overline{A_2}^2\right)^{0_1} = A_1 \cap A_2$, proving that $A_1 \cap A_2 = \left(\overline{A_1 \cap A_2}^2\right)^{0_1}$.

It follows from Lemma 1 that $\left(\overline{A_1 \cup A_2}^2\right)^{0_1}$ is a pseudoregular open set and it is clear that it contains both A_1 and A_2. Let $B \in \mathrm{PReg}(X)$ such that $A_1 \subseteq B$ and $A_2 \subseteq B$. Then $A_1 \cup A_2 \subseteq B$, so $\left(\overline{A_1 \cup A_2}^2\right)^{0_1} \subseteq \left(\overline{B}^2\right)^{0_1} = B$, thus $A_1 \vee A_2 = \left(\overline{A_1 \cup A_2}^2\right)^{0_1}$. Since $\emptyset$ and X are pseudoregular open sets, then $\mathrm{PReg}(X)$ is a bounded lattice. Our next task will be to prove that it is distributive. To this end we shall use the following well known fact: a lattice L is distributive if and only if given elements $a, b, c \in L$ such that $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ then $b = c$. Let A, B and C be pseudoregular open subsets of X such that $A \cap B = A \cap C$ and $\left(\overline{A \cup B}^2\right)^{0_1} = \left(\overline{A \cup C}^2\right)^{0_1}$. We claim that $\overline{B}^2 = \overline{C}^2$. Let $x \in \overline{B}^2$ and assume that $x \notin \overline{C}^2$. Then there exists an open set U in the topology T_2 such that $x \in U$ and $U \cap C = \emptyset$. Since $x \in \overline{B}^2$ it follows that $U \cap B \neq \emptyset$. Take $y \in U \cap B$. Since $U \cap B \subseteq B \subseteq \left(\overline{A \cup B}^2\right)^{0_1}$ and $\left(\overline{A \cup B}^2\right)^{0_1} = \left(\overline{A \cup C}^2\right)^{0_1}$ it follows that $y \in \left(\overline{A \cup C}^2\right)^{0_1}$. In particular we have that $y \in \overline{A \cup C}^2$. Since $U \cap B$ is an open set in the topology T_2 we would have that $U \cap B \cap (A \cup C) \neq \emptyset$ and since $U \cap B \cap C = \emptyset$ we obtain that $U \cap B \cap A \neq \emptyset$. But since $A \cap B = A \cap C$ we would arrive to $U \cap C \cap A \neq \emptyset$, which is in contradiction with the fact that $U \cap C = \emptyset$. Hence $\overline{B}^2 \subseteq \overline{C}^2$ and in a similar way the other inclusion is proved. Therefore $\overline{B}^2 = \overline{C}^2$. It follows that $B = \left(\overline{B}^2\right)^{0_1} = \left(\overline{C}^2\right)^{0_1} = C$ and hence $\mathrm{PReg}(X)$ is a bounded distributive lattice.

We know from Lemma 1 that $\left(\overline{A_1^c \cup A_2}^2\right)^{0_1}$ is in $\mathrm{PReg}(X)$. On the other hand, since A_1^c is a closed set under the topology T_2 we have that $A_1 \cap \left(\overline{A_1^c \cup A_2}^2\right)^{0_1} = A_1 \cap \left(A_1^c \cup \overline{A_2}^2\right)^{0_1} \subseteq$

$\overline{A_2}^2$ and then $A_1 \cap \left(\overline{A_1^c \cup A_2}^2\right)^{0_1} \subseteq \left(\overline{A_2}^2\right)^{0_1} = A_2$, because $A_1 \cap \left(\overline{A_1^c \cup A_2}^2\right)^{0_1}$ is an open set under the topology T_1.

Let W be a pseudoregular open set such that $A_1 \cap W \subseteq A_2$. Hence $W \subseteq A_1^c \cup A_2$ and then $W = \left(\overline{W}^2\right)^{0_1} \subseteq \left(\overline{A_1^c \cup A_2}^2\right)^{0_1}$.

Therefore $A_1 \to A_2 = \left(\overline{A_1^c \cup A_2}^2\right)^{0_1}$, which proves that $\mathrm{PReg}(X)$ is a Heyting algebra.

Let $(A_i)_{i\in I}$ be a family of pseudoregular open sets. It follows from Lemma 1 that $\left(\overline{\bigcup_{i\in I} A_i}^2\right)^{0_1}$ is a pseudoregular open set and it is clear that it is an upper bound of the family $(A_i)_{i\in I}$. Let B be another pseudoregular open set which is also an upper bound of the given family. Since $\bigcup_{i\in I} A_i \subseteq B$ then $\left(\overline{\bigcup_{i\in I} A_i}^2\right)^{0_1} \subseteq \left(\overline{B}^2\right)^{0_1} = B$. Therefore $\bigvee_{i\in I} A_i = \left(\overline{\bigcup_{i\in I} A_i}^2\right)^{0_1}$. In a similar way it is proved that $\bigwedge_{i\in I} A_i = \left(\overline{\bigcap_{i\in I} A_i}^2\right)^{0_1}$. □

Let H be a Heyting algebra and let $a \in H$. Let $\chi(H)$ be the set of lattice prime filters of H and let $\sigma(a) = \{F \in \chi(H) \mid a \in F\}$. We shall consider the following topologies defined on $\chi(H)$: the Stone topology, denoted by T_S, having as a basis the sets of the form $\{\sigma(a)\}_{a\in H}$, and the Priestley topology, denoted by T_P, having as a basis the sets of the form $\{\sigma(a) \cap \sigma^c(b)\}_{a,b\in H}$. We shall denote by $\mathrm{PReg}_{SP}(\chi(H))$ the Heyting algebra determined by the pseudoregular open sets according to these topologies.

Theorem 3. *Let H be a Heyting algebra. The following conditions are equivalent:*

(1) *H is a complete Heyting algebra.*
(2) *If $A \in T_S$, then $\overline{A}^P \in T_S$.*

Proof. (1) ⇒ (2). Let $A \in T_S$. Then there exists a family $\{a_i\}_{i\in I}$ of elements in H such that $A = \bigcup_{i\in I} \sigma(a_i)$. Since H is complete, there exists $(\bigvee_{i\in I} a_i) \in H$. In order to prove that $\overline{A}^P \in T_S$ it is enough to see that $\overline{A}^P = \sigma(\bigvee_{i\in I} a_i)$. Since σ is an order preserving map it follows that $\sigma(\bigvee_{i\in I} a_i) \supseteq \sigma(a_i)$ $(\forall i \in I)$. Thus $\sigma(\bigvee_{i\in I} a_i) \supseteq \bigcup_{i\in I} \sigma(a_i)$ and then $\overline{\sigma(\bigvee_{i\in I} a_i)}^P \supseteq \overline{\bigcup_{i\in I} \sigma(a_i)}^P = \overline{A}^P$. Since $\sigma(\bigvee_{i\in I} a_i)$ is closed in T_P we infer that $\overline{\sigma(\bigvee_{i\in I} a_i)}^P = \sigma(\bigvee_{i\in I} a_i)$. Therefore $\overline{A}^P \subseteq \sigma(\bigvee_{i\in I} a_i)$. To prove the reverse inclusion take $Q \in \sigma(\bigvee_{i\in I} a_i)$. Suppose that $Q \notin \overline{A}^P$. Hence there are elements $b, c \in H$ such that $Q \in \sigma(b) \cap \sigma^c(c)$ and $\sigma(b) \cap \sigma^c(c) \cap A = \emptyset$. It follows that $\sigma(b) \cap \sigma^c(c) \cap \sigma(a_i) = \emptyset$ $\forall i \in I$ or, equivalently, $\sigma(b) \cap \sigma(a_i) \subseteq \sigma(c)$ for every index $i \in I$. Thus $b \wedge a_i \leq c$ $(\forall i \in I)$, which implies $a_i \leq b \to c$ $(\forall i \in I)$. Therefore $(\bigvee_{i\in I} a_i) \leq b \to c$. Since $(\bigvee_{i\in I} a_i) \in Q$ we obtain that $b \to c \in Q$ and since $b \in Q$ we would infer that $c \in Q$, which contradicts the fact that $Q \in \sigma(b) \cap \sigma^c(c)$. Hence $\sigma(\bigvee_{i\in I} a_i) \subseteq \overline{A}^P$ and then $\overline{A}^P = \sigma(\bigvee_{i\in I} a_i)$.

(2) ⇒ (1). Let $T = \{a_i : i \in I\} \subseteq H$ be any subset and let $A = \bigcup_{a_i\in T} \sigma(a_i)$. It is clear that $A \in T_S$. By hypothesis $\overline{A}^P \in T_S$. Hence $\overline{A}^P = \bigcup_{j\in J} \sigma(b_j)$ for some family $\{b_j\}_{j\in J}$ of elements of H. Since T_P is compact and $\overline{A}^P$ is closed in T_P there is a finite subset $\{j_1, \ldots, j_n\}$ of J such that $\overline{A}^P = \bigcup_{k=1}^n \sigma(b_{j_k}) = \sigma(\bigvee_{k=1}^n b_{j_k})$. Let $b = \bigvee_{k=1}^n b_{j_k}$. Thus $\overline{A}^P = \sigma(b)$ and hence $\sigma(b) \supseteq \bigcup_{i\in I} \sigma(a_i)$, which means that b is an upper bound of the family $\{a_i\}_{i\in I}$. Let c be another upper bound of $\{a_i\}_{i\in I}$. It follows that $\sigma(c) \supseteq \bigcup_{i\in I} \sigma(a_i)$ and then $\sigma(c) = \overline{\sigma(c)}^P \supseteq \overline{\bigcup_{i\in I} \sigma(a_i)}^P = \overline{A}^P = \sigma(b)$, because $\sigma(c)$ is closed in T_P. Therefore $c \geq b$ and hence $\bigvee_{i\in I} a_i = b$. □

According to the proof of the previous theorem we have that if H is a complete Heyting algebra, then the correspondence $a \mapsto \sigma(a)$ is an isomorphism of Heyting algebras between H and $\mathrm{PReg}_{SP}(\chi(H))$. Indeed, in this case we have that $\mathrm{PReg}_{SP}(\chi(H))$ coincides with the lattice $\mathrm{Clopen}(\chi(H))$ determined by the increasing closed and open sets of $\chi(H)$. By Priestley duality we have that σ is a lattice isomorphism between H and $\mathrm{Clopen}(\chi(H))$ and hence automatically preserves implication. Note also that if H is an arbitrary Heyting algebra then $\sigma(a)$ is always a pseudoregular open set for every $a \in H$.

3. A TOPOLOGICAL CHARACTERIZATION OF THE DEDEKIND–MACNEILLE COMPLETION OF HEYTING ALGEBRAS

Definition 4. Let H be a Heyting algebra and let S be a subalgebra of H. Then S is said to be $\bigvee$-*dense* (respectively $\bigwedge$-*dense*) in H, if every element of H is supremum (respectively infimum) of elements of S, i.e., if $\forall\ h \in H$, there exists a family $\{s_i\}_{i\in I}$ of elements of S such that $h = \bigvee_{i\in I} s_i$ (respectively $h = \bigwedge_{i\in I} s_i$).

Note that if S is either $\bigvee$-dense or $\bigwedge$-dense in H, then S is dense in H.

Lemma 2. *Let H be a Heyting algebra and $A \in \mathrm{PReg}_{SP}(\chi(H))$. Then there exists $(a_i)_{i\in I} \subseteq H$ such that $\bigvee_{i\in I} \sigma(a_i) = \bigcup_{i\in I} \sigma(a_i) = A$. Furthermore, if $A = \bigcup_{i\in I} \sigma(a_i)$, then $\bigvee_{i\in I} \sigma(a_i) = \bigcup_{i\in I} \sigma(a_i) = A$.*

Proof. Since $A \in \mathrm{PReg}_{SP}(\chi(H))$, $A \in T_S$, thus there is $(a_i)_{i\in I} \subseteq H$ such that $A = \bigcup_{i\in I} \sigma(a_i)$. Thus $\bigvee_{i\in I} \sigma(a_i) = \left(\overline{\bigcup_{i\in I} \sigma(a_i)}^{P}\right)^{0_S} = \left(\overline{A}^{P}\right)^{0_S} = A$. □

Recall that an injective homomorphism of Heyting algebras is said to be an *embedding of Heyting algebras*.

Definition 5. A morphism of Heyting algebras $j : H_1 \to H_2$ is said to be *regular* if given a family $\{a_i\}_{i\in I} \subseteq H_1$ the following two conditions are satisfied:

I) If there exists $\bigvee_{i\in I} a_i \in H_1$ then there exists $\bigvee_{i\in I} j(a_i) \in H_2$ and $j(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} j(a_i)$.

II) If there exists $\bigwedge_{i\in I} a_i \in H_1$ then there exists $\bigwedge_{i\in I} j(a_i) \in H_2$ and $j(\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} j(a_i)$.

In the case that $\mathrm{Im}(j)$ is a $\bigvee$-dense (respectively $\bigwedge$-dense) subalgebra of H_2 then j is said to be $\bigvee$-dense (respectively $\bigwedge$-dense).

Theorem 6. *Let H be a Heyting algebra. Then the correspondence $a \mapsto \sigma(a)$ is a regular, $\bigvee$-dense and $\bigwedge$-dense embedding between H and $\mathrm{PReg}_{SP}(\chi(H))$.*

Proof. We know from Theorem 2 that $\mathrm{PReg}_{SP}(\chi(H))$ is a complete Heyting algebra and we know that σ is an order embedding. To prove that σ is a Heyting morphism it only remains to prove the following facts:

I) $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b)$

II) $\sigma(a \vee b) = \left(\overline{\sigma(a) \cup \sigma(b)}^{P}\right)^{0_S}$

III) $\sigma(a \to b) = \left(\overline{\sigma^c(a) \cup \sigma(b)}^{P}\right)^{0_S}$.

The proof of items I) and II) are immediate. To prove III) note first that $\left(\overline{\sigma^c(a) \cup \sigma(b)}^{P}\right)^{0_S} = (\sigma^c(a) \cup \sigma(b))^{0_S}$. Since $\sigma(a \to b) \subseteq \sigma^c(a) \cup \sigma(b)$ it follows that $\sigma(a \to b) \subseteq (\sigma^c(a) \cup \sigma(b))^{0_S}$. To prove the other inclusion take $P \in (\sigma^c(a) \cup \sigma(b))^{0_S}$ and let $c \in H$ be such that $P \in \sigma(c)$ and $\sigma(c) \subseteq \sigma^c(a) \cup \sigma(b)$. Hence $\sigma(c) \cap \sigma(a) = \sigma(a \wedge c) \subseteq \sigma(b)$ and then $a \wedge c \leq b$, which implies $c \leq a \to b$. Therefore $P \in \sigma(a \to b)$, thus proving III). It follows from Lemma 2 that σ is $\bigvee$-dense.

Let us prove that σ is $\bigwedge$-dense. Let $A \in \mathrm{PReg}_{SP}(\chi(H))$ and let $Z = \{x \in H : \sigma(x) \supseteq A\}$. We claim that $A = \bigwedge_{x\in Z} \sigma(x) = (\bigcap_{x\in Z} \sigma(x))^{0_S}$. It is clear that $A \subseteq (\bigcap_{x\in Z} \sigma(x))^{0_S}$. Let $P \in (\bigcap_{x\in Z} \sigma(x))^{0_S}$ and take $b \in H$ be such that $P \in \sigma(b)$ and $\sigma(b) \subseteq \bigcap_{x\in Z} \sigma(x)$. Since A is pseudoregular it is enough to prove that $\sigma(b) \subseteq \overline{A}^P$. Suppose it is not the case. Then we could find elements $c, d \in H$ and a prime filter Q of H such that $b \in Q, c \in Q, d \notin Q$ and $\sigma(b) \cap \sigma(c) \cap \sigma(d)^c \cap A = \emptyset$. By Lemma 2 we know that A is a union of sets of the form $\sigma(z)$ and each one of these elements z satisfies $\sigma(z) \subseteq \sigma((b \wedge c) \to d)$. It follows that $A \subseteq \sigma((b \wedge c) \to d)$. Therefore $(b \wedge c) \to d \in Z$ and then $b \leq ((b \wedge c) \to d)$, which implies $((b \wedge c) \to d) \in Q$. Since Q is implicative we infer that $d \in Q$, a contradiction.

Finally it remains to prove that σ is regular. To this end let $\{a_i\}_{i\in I}$ be a family of elements of H such that $a = \bigvee_{i\in I} a_i \in H$ exists. To prove that $\sigma(a) = \bigvee_{i\in I} \sigma(a_i)$ we need to prove that $\sigma(a) = \left(\overline{\bigcup_{i\in I} \sigma(a_i)}^P\right)^{0_S}$. Since $a_i \leq a$ then $\sigma(a_i) \subseteq \sigma(a)$. Hence $\bigcup_{i\in I} \sigma(a_i) \subseteq \sigma(a)$, which implies that $\left(\overline{\bigcup_{i\in I} \sigma(a_i)}^P\right)^{0_S} \subseteq \left(\overline{\sigma(a)}^P\right)^{0_S} = \sigma(a)$. In order to prove the reverse inclusion note first that since $\sigma(a) \in T_S$ then it is enough to prove that $\sigma(a) \subseteq \overline{\bigcup_{i\in I} \sigma(a_i)}^P$. Let $F \in \sigma(a)$. Suppose that $F \notin \overline{\bigcup_{i\in I} \sigma(a_i)}^P$. Therefore there are $(b, c \in H)$ such that $F \in \sigma(b) \cap \sigma^c(c)$ and $(\sigma(b) \cap \sigma^c(c)) \cap (\bigcup_{i\in I} \sigma(a_i)) = \emptyset$. Hence $\sigma(b \wedge a_i) \cap \sigma^c(c) = \emptyset$ for every index i or, equivalently, $\sigma(b \wedge a_i) \subseteq \sigma(c)$ for all $i \in I$. Since σ is an order morphism it follows that $b \wedge a_i \leq c$ for all $i \in I$ and then $a_i \leq (b \to c)$ for all $i \in I$. Thus $a = \bigvee_{i\in I} a_i \leq (b \to c)$. Since $F \in \sigma(a)$ then $a \in F$, thus $(b \to c) \in F$. Since $b \in F$ then $c \in F$, which is impossible, because $F \in \sigma(b) \cap \sigma^c(c)$. Thus, $F \in \overline{\bigcup_{i\in I} \sigma(a_i)}^P$. Finally let $\{a_i\}_{i\in I}$ be a family of elements of H such that $a = \bigwedge_{i\in I} a_i$ exists in H. To prove that $\sigma(a) = \bigwedge_{i\in I} \sigma(a_i)$ we need to prove that $\sigma(a) = \left(\overline{\bigcap_{i\in I} \sigma(a_i)}^P\right)^{0_S} = (\bigcap_{i\in I} \sigma(a_i))^{0_S}$. It is plain that $\sigma(a) \subseteq (\bigcap_{i\in I} \sigma(a_i))^{0_S}$. Suppose that $(\bigcap_{i\in I} \sigma(a_i))^{0_S} \nsubseteq \sigma(a)$. Thus there is $P \in (\bigcap_{i\in I} \sigma(a_i))^{0_S}$ such that $P \notin \sigma(a)$. Therefore we could find $c \in H$ such that $P \in \sigma(c) \subseteq (\bigcap_{i\in I} \sigma(a_i))$ and $P \notin \sigma(a)$. Then $\sigma(c) \subseteq \sigma(a_i)$ for every index i, which implies that $c \leq a_i \ \forall i \in I$. Hence $c \leq \bigwedge_{i\in I} a_i = a$ and then $\sigma(c) \subseteq \sigma(a)$, which contradicts the fact that $P \in \sigma(c) \backslash \sigma(a)$. Therefore σ is regular. □

Let H be a Heyting algebra. It is known that, up to isomorphism, the Dedekind–MacNeille completion of H, denoted by $DM(H)$, is the only complete Heyting algebra which is regular, $\bigvee$-dense and $\bigwedge$-dense (see for instance [1], [3]). As a corollary of Theorem 6 we obtain the following result:

Theorem 7. *Let H be a Heyting algebra. Then* $\mathrm{PReg}_{SP}(\chi(H))$ *is isomorphic to* $DM(H)$.

Our next task will be to show an explicit isomorphism between $\mathrm{PReg}_{SP}(\chi(H))$ and $DM(H)$ as follows. Given $S \subseteq H$, let us denote by S^l the set of lower bounds of S and by S^u the set of upper bounds of S. We define

$$C(H) = \{S \subseteq H : S = S^{lu}\}$$

with the order given by $S \leq T$ if and only if $S \supseteq T$ (i.e. the reverse inclusion). It is easy to check that $C(H)$ is a Heyting algebra isomorphic to $DM(H)$.

For each $S \in C(H)$ we define $U_S = \bigcup_{x\in S^l} \sigma(x)$. Then the map $S \mapsto U_S$ is well defined and is an isomorphism between $C(H)$ and $\mathrm{PReg}_{SP}(\chi(H))$. We leave the proof of this fact to the reader.

Remark 8. If B is a Boolean algebra then Stone's topology and Priestley's topology coincide on $\chi(B)$. Hence $\mathrm{PReg}_{SP}(\chi(B)) = \mathrm{Reg}(\chi(B))$. Thus, the Stone morphism

$$\sigma : B \to \mathrm{Reg}(\chi(B))$$
$$a \to \sigma(a)$$

is a regular, $\bigvee$-dense and $\bigwedge$-dense embedding in a complete Boolean algebra.

4. Relationship between $\mathrm{PReg}_{SP}(\chi(H))$ and different completions of Heyting algebras

Definition 9. Let H be a Heyting algebra. We shall say that H satisfies the S property, if for every family $\{x_l\}_{l\in L}$ of elements of H and every $x \in H$ such that $x \le s$ for every upper bound s of $\{x_l\}_{l\in L}$, there exists $\bigvee_{l\in L}\{x_l\}$, or there is a finite subset of $F \subseteq \{x_l\}_{l\in L}$ such that $x \le \bigvee F$.

There are Heyting algebras which do not satisfy the S property. Take, for instance, the following example.

Let B be the Boolean algebra determined by those subsets A of the real line such that A is countable or its complement is countable. For each real number i in the interval $[0,1]$ let $x_i = \{i\}$ and let $x = Q \cap [0,1]$. If s is an upper bound of $\{i\}_{i\in[0,1]}$ then $[0,1] \subseteq s$, thus

$$\text{if } s \text{ is an upper bound of } \{x_i\}_{i\in[0,1]} \text{ then } x \le s. \tag{1}$$

Let s be an upper bound of $\{x_i\}$. Then $[0,1] \subseteq s$, but $[0,1] \notin B$, thus $[0,1] \subsetneq s$, so that there exists $P \in (s\backslash[0,1])$. Let $\overline{s} = s\backslash\{P\}$. Since s is not countable it follows that $\#(s^c) \le \aleph_0$, then $\#(\overline{s}^c = s^c \bigcup \{P\}) \le \aleph_0$. Therefore $\overline{s} \in B$. Since $[0,1] \subseteq \overline{s}$, $\overline{s}$ is an upper bound of $\{x_i\}_{i\in[0,1]}$ and $\overline{s} \le s$. So we have proved that if s is an upper bound of $\{x_i\}_{i\in[0,1]}$, then there exists $\overline{s} < s$ such that $\overline{s}$ is an upper bound of the family $\{x_i\}_{i\in[0,1]}$.

$$\text{Therefore this family does not have a supremum in } B. \tag{2}$$

Moreover, if $F \subseteq \{x_i\}_{i\in[0,1]}$ is finite, then $\bigvee_{x_i\in F}\{x_i\}$ is finite, thus

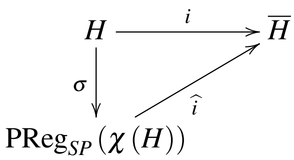
$$x \nleq \bigvee_{x_i\in F}\{x_i\}. \tag{3}$$

As a consequence of (1), (2) and (3), B does not satisfy the S property.

Definition 10. Let H and $\overline{H}$ be lattices. A map $i : H \to \overline{H}$ is a *lower morphism of lattices* if $i(a \wedge b) = i(a) \wedge i(b)$. Furthermore, if i is inyective, i is said to be an *embedding of lower morphism*.

Theorem 11. *Let H and $\overline{H}$ be Heyting algebras such that $\overline{H}$ is complete, and let $i : H \to \overline{H}$ be an embedding of Heyting algebras. Then*

(1) *There exists an embedding lower morphism of lattices $\widehat{i} : \mathrm{PReg}_{SP}(\chi(H)) \to \overline{H}$ such that $\widehat{i} \circ \sigma = i$*



(2) *If $i : H \to \overline{H}$ is regular and H has the S property, then $\widehat{i} : \mathrm{PReg}_{SP}(\chi(H)) \to \overline{H}$ is an embedding of lattices.*

(3) *If $i : H \to \overline{H}$ is $\bigvee$-dense and regular and H has the S property, then $\widehat{i} : \mathrm{PReg}_{SP}(\chi(H)) \to \overline{H}$ is an embedding of Heyting algebras.*

Proof. (1) We define

$$\widehat{i}: \mathrm{PReg}_{SP}(\chi(H)) \to \overline{H}$$
$$\bigcup_{l\in L}\sigma(a_l) \to \bigvee_{l\in L} i(a_l).$$

We claim that $\widehat{i}$ is well defined. Let $A \in \mathrm{PReg}_{SP}(\chi(H))$ and let $(a_l)_{l\in L}$ and $(b_t)_{t\in T} \subseteq H$ be such that $A = \bigcup_{l\in L}\sigma(a_l) = \bigcup_{t\in T}\sigma(b_t)$. We must show that $\bigvee_{l\in L} i(a_l) = \bigvee_{t\in T} i(b_t)$. Let $l \in L$. Suppose that $i(a_l) \nleq \bigvee_{t\in T} i(b_t)$, then

$$\text{there is } P \in \chi\left(\overline{H}\right) \text{ such that } i(a_l) \in P \text{ and } \bigvee_{t\in T} i(b_t) \notin P. \tag{4}$$

$i^{-1}(P) \in \chi(H)$ and $a_l \in i^{-1}(P)$ or, equivalently,

$$i^{-1}(P) \in \sigma(a_l). \tag{5}$$

Let $t \in T$. If $b_t \in i^{-1}(P)$, then $i(b_t) \in P$, thus $\bigvee_{t\in T} i(b_t) \in P$, but that contradicts (4). Thus $(\forall t \in T)$, $i(b_t) \notin P$ or, equivalently, $b_t \notin i^{-1}(P)$. Hence $(\forall t \in T)\ i^{-1}(P) \notin \sigma(b_t)$. Therefore $i^{-1}(P) \notin \bigcup_{t\in T}\sigma(b_t) = \bigcup_{l\in L}\sigma(a_l)$, but that contradicts (5). Thus, $i(a_l) \leq \bigvee_{t\in T} i(b_t)$ and $l \in L$ is arbitrary, and hence $\bigvee_{l\in L} i(a_l) \leq \bigvee_{t\in T} i(b_t)$. The other inequality is proved analogously. We have proved that $\widehat{i}$ is well defined as function. Let $A = \bigcup_{l\in L}\sigma(a_l)$ and $B = \bigcup_{t\in T}\sigma(b_t) \in \mathrm{PReg}_{SP}(\chi(H))$.

$$\begin{aligned} A \wedge B &= \left(\bigcup_{l\in L}\sigma(a_l)\right) \wedge \left(\bigcup_{t\in T}\sigma(b_t)\right) = \bigcup_{l\in L}\bigcup_{t\in T}(\sigma(a_l)\cap\sigma(b_t)) \\ &= \bigcup_{l\in L}\bigcup_{t\in T}(\sigma(a_l \wedge b_t)). \end{aligned}$$

Thus $\widehat{i}(A \wedge B) = \bigvee_{l\in L}\bigvee_{t\in T} i(a_l \wedge b_t)$. On the other hand,

$$\widehat{i}(A) \wedge \widehat{i}(B) = \left(\bigvee_{l\in L} i(a_l)\right) \wedge \left(\bigvee_{t\in T} i(b_t)\right) = \bigvee_{l\in L}\bigvee_{t\in T} i(a_l \wedge b_t).$$

We have proved that $\widehat{i}$ is a lower morphism of lattices. Let's see that $\widehat{i}$ is an embedding. Let $A = \bigcup_{l\in L}\sigma(a_l)$ and $B = \bigcup_{t\in T}\sigma(b_t) \in \mathrm{PReg}_{SP}(\chi(H))$ be such that $\widehat{i}(A) \leq \widehat{i}(B)$. Then $(\bigvee_{l\in L} i(a_l)) \leq (\bigvee_{t\in T} i(b_t))$ In order to prove that $A \leq B$ it is enough to prove that $\overline{A}^P \leq \overline{B}^P$. Suppose it is not the case. Then there is $P \in \left(\overline{A}^P \backslash \overline{B}^P\right)$. Since $P \notin \overline{B}^P$, there are $x, z \in H$ such that $P \in \sigma(x) \cap \sigma(z)^c$ and $\sigma(x) \cap \sigma(z)^c \cap B = \emptyset$. Since $P \in \overline{A}^P$, $\emptyset \neq \sigma(x) \cap \sigma(z)^c \cap A = \sigma(x) \cap \sigma(z)^c \cap (\bigcup_{l\in L}\sigma(a_l))$, thus

$$(\exists l_0 \in L) \quad \text{such that} \quad \sigma(x) \cap \sigma(z)^c \cap (\sigma(a_{l_0})) \neq \emptyset. \tag{6}$$

Thus,

$$\exists Q \in \sigma(x) \cap \sigma(z)^c \cap \sigma(a_{l_0}). \tag{7}$$

Since $\sigma(x) \cap \sigma(z)^c \cap (\bigcup_{t\in T}\sigma(b_t)) = \emptyset$, then $(\forall t \in T)\ \sigma(x) \cap \sigma(z)^c \cap \sigma(b_t) = \emptyset$. Thus $\sigma(x) \cap \sigma(b_t) \subseteq \sigma(z)$. Since σ is an order embedding, $x \wedge b_t \leq z$ or, equivalently, $b_t \leq (x \to z)$. Thus $i(b_t) \leq i(x \to z) = i(x) \to i(z)$. Because this inequality is valid for all $t \in T$, $\widehat{i}(B) = \bigvee_{t\in T} i(b_t) \leq i(x) \to i(z)$. Since $(\bigvee_{l\in L} i(a_l)) \leq (\bigvee_{t\in T} i(b_t))$, $(\bigvee_{l\in L} i(a_l)) \leq i(x) \to i(z)$. Therefore $i(a_l) \leq i(x) \to i(z)$ for every $l \in L$. In particular $i(a_{l_0}) \leq i(x) \to i(z)$. Thus $i(x) \wedge i(a_{l_0}) \leq i(x) \wedge (i(x) \to i(z)) = i(x \wedge z) \leq i(z)$. Thus $x \wedge a_{l_0} \leq z$. Because of (7) $x \wedge a_{l_0} \in Q$ and $z \notin Q$, but this is impossible. The contradiction arises because we assumed that $\overline{A}^P \nleq \overline{B}^P$.

(2) Assume now that H satisfies the S property and let $i : H \to \overline{H}$ be regular. We know that $\widehat{i} : \mathrm{PReg}_{SP}(\chi(H)) \to \overline{H}$ is an embedding lower morphism of lattices. Due to this fact $\widehat{i}(A) \vee \widehat{i}(B) \leq \widehat{i}(A \vee B)$ $\forall A, B \in \mathrm{PReg}_{SP}(\chi(H))$. Let's suppose that $\widehat{i}(A \vee B) \nleq \widehat{i}(A) \vee \widehat{i}(B)$. $A = \bigcup_{l \in L} \sigma(a_l)$, $B = \bigcup_{t \in T} \sigma(b_t)$ and $A \vee B = \bigcup_{j \in J} \sigma(c_j) = (\bigcup_{l \in L} \sigma(a_l)) \vee (\bigcup_{t \in T} \sigma(b_t))$. Therefore $\widehat{i}(A) = (\bigvee_{l \in L} i(a_l))$, $\widehat{i}(B) = (\bigvee_{t \in T} i(b_t))$ and $\widehat{i}(A \vee B) = \left(\bigvee_{j \in J} i(c_j)\right)$. Since $\widehat{i}(A \vee B) \nleq \widehat{i}(A) \vee \widehat{i}(B)$,

$$\text{there exists } j_0 \in J \text{ such that } i(c_{j_0}) \nleq \left(\bigvee_{l \in L} i(a_l)\right) \vee \left(\bigvee_{t \in T} i(b_t)\right). \tag{8}$$

Then

$$\text{there exists } P \in \chi\left(\overline{H}\right) \text{ such that } i(c_{j_0}) \in P \text{ and } \left(\bigvee_{l \in L} i(a_l)\right) \vee \left(\bigvee_{t \in T} i(b_t)\right) \notin P. \tag{9}$$

Hence,

$$i^{-1}(P) \in \chi(H) \quad \text{and} \quad i^{-1}(P) \in \sigma(c_{j_0}). \tag{10}$$

Let $l \in L$. If $a_l \in i^{-1}(P)$, then $i(a_l) \in P$, but this contradicts (9). Thus, $a_l \notin i^{-1}(P)$ or, equivalently, $i^{-1}(P) \notin \sigma(a_l)$. Hence $i^{-1}(P) \notin \bigcup_{l \in L} \sigma(a_l) = A$. In the same way, $i^{-1}(P) \notin B$. and consequently

$$i^{-1}(P) \notin (A \cup B). \tag{11}$$

Let $F \subseteq \{a_l; b_t\}$ be finite. Let $M = \bigvee F$, and let's suppose that $i^{-1}(P) \in \sigma(M)$. Since P is prime, there is $x \in F$ such that $i^{-1}(P) \in \sigma(x)$, but this contradicts (9). Thus, $i^{-1}(P) \notin \sigma(M)$. Because of (10), $\sigma(c_{j_0}) \nsubseteq \sigma(M)$. Thus $c_{j_0} \nleq M$. Thus,

$$\text{if } F \subseteq \{a_l; b_t\} \text{ is finite and } M = \bigvee F, \text{ then } c_{j_0} \nleq M. \tag{12}$$

Let x be an upper bound of $\{a_l; b_t\}_{l \in L, t \in T}$. Since $\sigma(a_l) \subseteq \sigma(x)$ and $\sigma(b_t) \subseteq \sigma(x)$, $(\bigcup_{l \in L} \sigma(a_l)) \cup (\bigcup_{t \in T} \sigma(b_t)) \subseteq \sigma(x)$. Thus $A \vee B \subseteq \sigma(x)$. Since $\sigma(c_{j_0}) \subseteq A \vee B$, $\sigma(c_{j_0}) \subseteq \sigma(x)$. Thus

$$c_{j_0} \leq x \quad \forall x \text{ upper bound of } \{a_l; b_t\}_{l \in L, t \in T}. \tag{13}$$

If c_{j_0} is an upper bound of $\{a_l; b_t\}_{l \in L, t \in T}$, $c_{j_0} = \sup\{a_l; b_t\}_{l \in L, t \in T}$. Since i is regular, $i(c_{j_0}) = (\bigvee_{l \in L} i(a_l)) \vee (\bigvee_{t \in T} i(b_t))$, but this contradicts (8). Thus c_{j_0} isn't an upper bound of $\{a_l; b_t\}_{l \in L, t \in T}$. If there exists $z = \sup\{a_l; b_t\}_{l \in L, t \in T}$, then $i(z) = (\bigvee_{l \in L} i(a_l)) \vee (\bigvee_{t \in T} i(b_t))$. Thus, because of (9) $i(z) \notin P$. Thus $z \notin i^{-1}(P)$. Because of (13), $c_{j_0} \leq z$, but $i(c_{j_0}) \in P$ or, equivalently, $c_{j_0} \in i^{-1}(P)$, but this is impossible. Thus

$$\nexists \sup\{a_l; b_t\}_{l \in L, t \in T}. \tag{14}$$

But (12), (13) and (14) imply H does not have the S property, which is against the hypothesis. The contradiction arises because we supposed that $\widehat{i}(A \vee B) \nleq \widehat{i}(A) \vee \widehat{i}(B)$. Thus $\widehat{i}(A \vee B) = \widehat{i}(A) \vee \widehat{i}(B)$. We have proved that $\widehat{i}$ is an embedding of lattices.

(3) Assume now that H satisfies the S property and let $i : H \to \overline{H}$ be $\bigvee$-dense and regular. Let $A = \bigcup_{l \in L} \sigma(a_l)$ and $B = \bigcup_{t \in T} \sigma(b_t)$. Since $\widehat{i}$ is a lower morphism of lattices, $\widehat{i}(A) \wedge$

$\widehat{i}(A \to B) = \widehat{i}(A \wedge B) \leq \widehat{i}(B)$, thus $\widehat{i}(A \to B) \leq \widehat{i}(A) \to \widehat{i}(B)$. On the other hand,

$$A \to B = \left(\overline{A^c \cup B}^P\right)^{0_S} = \left(\overline{\left(\bigcap_{l\in L} \sigma^c(a_l)\right) \cup \left(\bigcup_{t\in T} \sigma(b_t)\right)}^P\right)^{0_S}$$
$$= \left(\bigcup_{k\in K} \sigma(c_k)\right).$$

Thus, $\widehat{i}(A \to B) = \bigvee_{k\in K} i(c_k)$. Let $E \in \overline{H}$ such that

$$\widehat{i}(A) \wedge E \leq \widehat{i}(B). \tag{15}$$

Since $\mathrm{Im}(i)$ is $\bigvee$-dense in $\overline{H}$, $\left(\exists (x_j)_{j\in J} \subseteq H\right) : E = \bigvee_{j\in J} i(x_j)$. Thus $\widehat{i}(A) \wedge \left(\bigvee_{j\in J} i(x_j)\right) \leq \widehat{i}(B)$. Let $j \in J$. $\widehat{i}(A) \wedge i(x_j) \leq \widehat{i}(B)$ or, equivalently, $\widehat{i}(A) \wedge \widehat{i}(\sigma(x_j)) \leq \widehat{i}(B)$. Since $\widehat{i}$ is an embedding of lattices, $A \wedge \sigma(x_j) \leq B$. Thus, $\sigma(x_j) \leq A \to B$, so $\widehat{i}(\sigma(x_j)) \leq \widehat{i}(A \to B)$ or, equivalently, $i(x_j) \leq \widehat{i}(A \to B)$, and as a consequence of being $j \in J$ arbitrary, $E = \bigvee_{j\in J} i(x_j) \leq \widehat{i}(A \to B)$. Since E, which verifies (15), is arbitrary, $\widehat{i}(A) \to \widehat{i}(B) \leq \widehat{i}(A \to B)$. □

REFERENCES

[1] R. Balbes and P. Dwinger, Distributive lattices. University of Missouri Press, 1974. MR 0373985.

[2] G. Bezhanishvili, M. Gehrke, R. Mines, P. J. Morandi, Profinite completions and canonical extensions of Heyting algebras. Order **23** (2006), 143–161. MR 2308901.

[3] G. Bezhanishvili, J. Vosmaer, Comparison of MacNeille, canonical, and profinite completions. Order **25** (2008), 299–320. MR 2470200.

[4] G. Bezhanishvili, J. Harding, Functional monadic Heyting algebras. Algebra Universalis **48** (2002), 1–10. MR 1930030.

[5] G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, A. Kurz, Bitopological duality for distributive lattices and Heyting algebras, Math. Structures Comput. Sci. **20** (2010), 359–393. MR 2646630.

[6] B. A. Davey, H. A. Priestley, Introduction to lattices and order. Cambridge University Press, 1990. MR 1058437.

[7] L. Maksimova, Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras, Algebra and Logic **16** (1977), 427–455. MR 0516426.

[8] A. Monteiro, O. Varsavsky, Álgebras de Heyting monádicas. Actas de las X Jornadas de la Unión Matemática Argentina, 52–62, Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca, 1957.

ESCUELA DE CIENCIA Y TECNOLOGÍA, UNIVERSIDAD NACIONAL DE SAN MARTÍN, ARGENTINA
E-mail: `cscirica@dm.uba.ar, cscirica@unsam.edu.ar`

FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES, ARGENTINA,
E-mail: `apetrov@dm.uba.ar`