ACTAS DEL XIII CONGRESO
DR. ANTONIO A. R. MONTEIRO (2015)
2016, Páginas 57–62

ON THE SPECTRUM OF THE JOHNSON GRAPHS $J(n,k,r)$

JOSÉ ARAUJO AND TIM BRATTEN

ABSTRACT. It is known that the spectrum of the Johnson graph $J(n,k,r)$ can be obtained using Eberlein polynomials, as set forth in an article from 1973 by P. Delsarte and in a more recent work by M. Krebs and A. Shaheen. In this article we give a formula that describes the spectrum of the Johnson graph, which we obtain independently of the Eberlein polynomials, using instead the realization of the irreducible representations of the symmetric group $\mathfrak{S}_n$ in the polynomial ring.

1. INTRODUCTION

It is known that the spectrum of the Johnson graph $J(n,k,r)$ can be obtained using Eberlein polynomials, as set forth in [2] and [4]. In this article we give a formula that describes the spectrum, obtained independently of the Eberlein polynomials, using instead the realization of the irreducible representations of the symmetric group $\mathfrak{S}_n$ in the polynomial ring such as is discussed in [1].

In all that follows we denote by $\mathbb{N}_0$ the set of nonnegative integer numbers, by I_n the set $\{1,2,\dots,n\}$, by $\mathbb{C}$ the set complex numbers, and by $\mathscr{P}$ the polynomial ring $\mathbb{C}[x_1,\dots,x_n]$.

Let Γ be a graph and G a subgroup of the automorphisms of Γ. Let M be the incidence matrix of Γ and let V be the complex vector space generated by the set of vertices of Γ. The group G operates on V in a natural way, induced from the permutation action of G on the vertices of Γ. The matrix M induces a morphism in V given by $\Upsilon(v) = \sum_{w\sim v} w$, where $w \sim v$ means that (w,v) is an edge of Γ.

It is not difficult to see that Υ belongs to the centralizer of G in $End_{\mathbb{C}}(V)$, so that Υ stabilizes the isotypic components of V with respect to G.

A special case occurs when V is multiplicity-free with respect to G. In this case, the isotypic components of V are part of the eigenspaces of Υ, so that the spectrum of Γ can be known by evaluating Υ at a nonzero element of each isotypic component.

This is the situation that corresponds to a Johnson graph, where also V can be identified with a subspace of $\mathbb{C}[x_1,\dots,x_n]$ and the symmetric group $\mathfrak{S}_n$ takes the role of G.

The spectrum of Johnson graphs is known and has been treated by several authors. Delsarte in [2] realizes that this spectrum can be established using Eberlein polynomials introduced in [3]. From the perspective of the theory of representations of the symmetric group, the result of Delsarte is given in [4]. In this article we give a closed formula for the spectrum of $J(n,k,r)$, independent of Eberlein polynomials; this formula is obtained from the realization given in [1], of the irreducible representations of the symmetric group $\mathfrak{S}_n$ in the ring of polynomials $\mathscr{P}$.

2. JOHNSON GRAPHS

The vertices of a Johnson graph $J(n,k,r)$ are the subsets of $\{1,2,\dots,n\}$ whose cardinality is k with $2k \le n$. In this graph, two vertices P and Q are joined by an edge if $|P\cap Q| = k-r$.

2010 *Mathematics Subject Classification.* Primary: 20B30; Secondary: 05D25.

The vertices of $J(n,k,r)$ can be naturally identified with a family of monomials as follows. If $P=\{i_1,i_2,\ldots,i_k\}$ is a vertex of the graph, P is associated with the monomial $x_{i_1}x_{i_2}\cdots x_{i_k}$. So, the space of vertices V is identified as $\mathfrak{S}_n$-module with the space $\mathscr{S}_k$ generated by the family of monomials $x_{i_1}x_{i_2}\cdots x_{i_k}$, where $\mathfrak{S}_n$ acts by permutation of the variables, that is, with the natural action of $\mathfrak{S}_n$ in $\mathbb{C}[x_1,\ldots,x_n]$.

To simplify the notation, we will use:

$$x_{i_1}x_{i_2}\cdots x_{i_k}=x^{\alpha}=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n},$$

where $\alpha:\{1,2,\ldots,n\}\to\mathbb{N}_0$ is the characteristic function of the set P.

As we shall see, S_k is a multiplicity-free $\mathfrak{S}_n$-module and we will select a non-zero element in each isotypic component, to obtain the spectrum of $J(n,k,r)$. Moreover, the operators, Υ_r, associated with the incidence matrix of the graph $J(n,k,r)$ will be expressed in terms of certain symmetric operators to facilitate their evaluation at elements of the isotypic components.

It is appropriate to note that

$$\Upsilon_r\left(x^{\alpha}\right)=\sum_{\|\beta-\alpha\|^2=2r}x^{\beta}, \tag{1}$$

where β runs the characteristic functions of subsets with k elements of $\{1,2,\ldots,n\}$ and

$$\|\beta-\alpha\|^2=\sum_{i=1}^{n}\left(\alpha_i-\beta_i\right)^2.$$

3. IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{S}_n$

Representations of $\mathfrak{S}_n$ induced by the trivial representation of certain subgroups, called parabolic subgroups, can be realized naturally in subspaces of $\mathscr{P}$. In these representations, an irreducible subrepresentation is distinguished. The symmetric operator given in (2) was introduced in [1] in order to characterize the space associated with this subrepresentation.

Let $\lambda=(\lambda_0,\lambda_1,\ldots,\lambda_m)$ be a partition of n. Associated with λ we consider the set of multi-indices $\mathscr{O}_\lambda$ given by

$$\mathscr{O}_\lambda=\left\{\alpha:I_n\to\mathbb{N}_0:\left|\alpha^{-1}\left(i\right)\right|=\lambda_i,\ 0\le i\le m\right\}.$$

Clearly $\mathscr{O}_\lambda$ is a $\mathfrak{S}_n$-orbit in the set $\mathscr{M}=\{\alpha:I_n\to\mathbb{N}_0\}$ under the action

$$\pi\cdot\alpha=\alpha\pi^{-1},\quad \pi\in\mathfrak{S}_n,\ \alpha\in\mathscr{M}.$$

Denote by $\mathscr{S}_\lambda$ the subspace of $\mathscr{P}$ given by

$$\mathscr{S}_\lambda=\left\{\sum_{\alpha\in\mathscr{O}_\lambda}c_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n}:c_\alpha\in K\right\}.$$

Note that $\mathscr{S}_\lambda$ is a $\mathfrak{S}_n$-module under the natural action of $\mathfrak{S}_n$ on $\mathscr{P}$.

Let ∂ be the symmetric differential operator defined as

$$\partial=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}. \tag{2}$$

We have $\pi\partial\pi^{-1}=\partial$, $\forall\pi\in\mathfrak{S}_n$, where π is considered as an automorphism of $\mathscr{P}$.

Let $\mathscr{S}_\lambda^0$ be the $\mathfrak{S}_n$-submodule of $\mathscr{S}_\lambda$ given by

$$\mathscr{S}_\lambda^0=\left\{P\in\mathscr{S}_\lambda:\partial\left(P\right)=0\right\}.$$

Theorem 1. *i) S_λ^0 is a simple S_n-module.*

ii) If $\lambda\neq\mu$ are partitions of n, then $\mathscr{S}_\lambda^0\not\simeq\mathscr{S}_\mu^0$.

Proof. See [1]. □

Note that for the partitions λ of the form (l,k) with $l+k=n$, $l\geq k$, $\mathscr{S}_\lambda$ coincides with $\mathscr{S}_k$, the space associated with the vertices of a Johnson graph.

For $0\leq i\leq k$ we define the polynomial ξ_i as

$$\xi_i=\prod_{j=1}^{i}\left(x_{2j-1}-x_{2j}\right)\times\phi_{k-i}\left(x_{2i+1},\ldots,x_n\right), \tag{3}$$

where $\phi_m\left(x_1,x_2,\ldots,x_h\right)$ is the symmetric polynomial

$$\phi_m\left(x_1,x_2,\ldots,x_h\right)=\sum_{1\leq i_1<i_2<\cdots<i_m\leq h}x_{i_1}x_{i_2}\cdots x_{i_m},$$

where $\phi_0=1$.

The following proposition shows that $\mathscr{S}_k$ is multiplicity-free and also that the family ξ_i $(0\leq i\leq k)$ runs the isotypic components of $\mathscr{S}_k$.

Proposition 2. *i)* $S_k=\oplus_{i=0}^{k}S_k^i$, *where* $S_k^i\simeq S_i^0$ *as* S_n*-modules.*

ii) Let ξ_i *be as in* (3)*, then* $\xi_i\in S_k^i$*.*

Proof. i) First we show that $\partial:\mathscr{S}_k\to\mathscr{S}_{k-1}$ is a surjective morphism. We put $I=\{1,2,\ldots,k-1\}$ and $J=\{k,k+1,\ldots,2k-1\}$. Denote by ϕ_m^I and ϕ_m^J the symmetric polynomials in variables $x_1,\ldots,x_{k-1}$ and $x_k,\ldots,x_{2k-1}$ respectively. Let α_m and β_m the sequences given by

$$\partial\phi_m^I=\alpha_m\phi_{m-1}^I,\quad\text{with }1\leq m\leq k-1$$

$$\partial\phi_m^J=\beta_m\phi_{m-1}^J,\quad\text{with }1\leq m\leq k$$

and γ_i are the scalars defined by

$$\gamma_i=(-1)^{i+1}\,\frac{\alpha_{k-1}\alpha_{k-2}\cdots\alpha_{k-i+1}}{\beta_1\beta_2\cdots\beta_i}.$$

Let P be the polynomial

$$P=\sum_{i=1}^{k-1}\gamma_i\phi_{k-i}^I\phi_i^J,$$

where $\phi_0^I=1$. It is clear that $P\in\mathscr{S}_k$ since the number of variables n is greater than or equal to $2k$.

From the identity $\alpha_{k-i}\gamma_i+\beta_{i+1}\gamma_{i+1}=0$, it follows that

$$\begin{aligned}\partial\left(P\right)&=\sum_{i=1}^{k-1}\gamma_i\alpha_{k-i}\phi_{k-i-1}^I\phi_i^J+\sum_{i=1}^{k-1}\gamma_i\beta_i\phi_{k-i}^I\phi_{i-1}^J\\&=\gamma_1\beta_1\phi_{k-1}^I+\sum_{i=2}^{k-1}\left(\alpha_{k-i}\gamma_i+\beta_{i+1}\gamma_{i+1}\right)\phi_{k-i-1}^I\phi_i^J\\&=\phi_{k-1}^I=x_1x_2\cdots x_{k-1}.\end{aligned}$$

Thus $x_1x_2\cdots x_{k-1}$ belongs to the image of $\partial:\mathscr{S}_k\to\mathscr{S}_{k-1}$, but this is a $\mathfrak{S}_n$-module, so the image is $\mathscr{S}_{k-1}$. Moreover, the kernel of this map is precisely the simple $\mathfrak{S}_n$-module S_k^0, thus

$$\mathscr{S}_k\simeq\mathscr{S}_k^0\oplus\mathscr{S}_{k-1}.$$

Proceeding recursively, the proof of *i)* concludes.

ii) Let $\mathscr{P}_k$ be the homogeneous component of degree k of $\mathscr{P}$ and $\pi_k:\mathscr{P}_k\to\mathscr{S}_k$ the orthogonal projection associated with the inner product $\langle\;,\;\rangle$ defined by bilinearity from identities

$$\langle x^\alpha,x^\beta\rangle=\delta_{\alpha,\beta}.$$

We define $\mathscr{F} : \mathscr{S}_i^0 \to \mathscr{S}_k$ by

$$\mathscr{F}(P) = \pi_k \left(P \times \phi_{k-i}(x_1, x_2, \ldots, x_n)\right).$$

$\mathscr{F}$ is a $\mathfrak{S}_n$-morphism, since π_k is a $\mathfrak{S}_n$-morphism and $\phi_{k-i}(x_1, x_2, \ldots, x_n)$ is a symmetric polynomial. Denote by $\zeta^i \in \mathscr{S}_i^0$ the polynomial $\prod_{j=1}^{i} (x_{2j-1} - x_{2j})$ and by ζ_l^i the polynomial $\prod_{j=1, j\neq l}^{i} (x_{2j-1} - x_{2j})$. Putting

$$\begin{aligned}\phi_{k-i}(x_1, x_2, \ldots, x_n) &= (x_1 + x_2)\, \phi_{k-i-1}(x_3, \ldots, x_n) \\ &\quad + x_1 x_2 \phi_{k-i-2}(x_3, \ldots, x_n) + \phi_{k-i}(x_3, \ldots, x_n)\end{aligned}$$

we have

$$\begin{aligned}\zeta^i \times \phi_{k-i}(x_1, x_2, \ldots, x_n) &= \zeta_1^i \left(x_1^2 - x_2^2\right) \phi_{k-i-1}(x_3, \ldots, x_n) \\ &\quad + \zeta_1^i \left(x_1^2 x_2 - x_1 x_2^2\right) \phi_{k-i-2}(x_3, \ldots, x_n) \\ &\quad + \zeta^i \phi_{k-i}(x_3, \ldots, x_n);\end{aligned}$$

thus

$$\begin{aligned}\mathscr{F}\left(\zeta^i\right) &= \pi_k \left(\zeta^i \times \phi_{k-i}(x_1, x_2, \ldots, x_n)\right) \\ &= \pi_k \left(\zeta^i \phi_{k-i}(x_3, \ldots, x_n)\right)\end{aligned}$$

and recursively $\mathscr{F}\left(\zeta^i\right) = \xi_i$ is obtained. We conclude that $\xi_i \in \mathscr{S}_k^i$. □

Remark 1. *The proof of i) in the previous proposition could be simpler if it were shown that* $\dim\left(S_k^0\right) = \binom{n}{k} - \binom{n}{k-1}$*, but in our opinion the demonstration given is more constructive.*

4. The spectrum

We consider the operator $\mathscr{D}_r : \mathscr{S}_k \to \mathscr{S}_k$ given by

$$\mathscr{D}_r = \pi_k \circ \mu_r \circ \frac{1}{r!} \partial^r, \tag{4}$$

where π_k is as before and $\mu_r(P) = P \times \phi_r(x_1, \ldots, x_n)$, i.e., multiplication by the elementary symmetric polynomial of degree r. Clearly $\mathscr{D}_r$ is a $\mathfrak{S}_n$-morphism of $\mathscr{S}_k$, because each one of its factors is a $\mathfrak{S}_n$-morphism.

Proposition 3. *Let $\mathscr{D}_r$ and Υ_i be defined as in* (4) *and* (1)*, respectively. Then we have:*
i)

$$\mathscr{D}_r = \sum_{i=0}^{r} \binom{k-i}{r-i} \Upsilon_i.$$

ii) $\mathscr{D}_r(\xi_i) = \mu_r^i \xi_i$, *where*

$$\mu_r^i = \begin{cases} \binom{k-i}{r}\binom{n-k+r-i}{r} & \text{if } 0 \le i \le k-r, \\ 0 & \text{if } i > k-r. \end{cases}$$

Proof. Evaluate $\mathscr{D}_r(x^\alpha)$ with $\operatorname{Im}(\alpha) = \{i_1, \ldots, i_k\}$. Assuming $r \le k$, we have

$$\frac{1}{r!} \partial^r (x^\alpha) = \phi_{k-r}(x_{i_1}, \ldots, x_{i_k}).$$

In the product

$$\phi_{k-r}(x_{i_1}, \ldots, x_{i_k})\, \phi_r(x_1, \ldots, x_n)$$

the terms that π_k does not annihilate are those formed by multiplying a monomial of $\phi_{k-r}(x_{i_1},\ldots,x_{i_k})$ with a monomial of $\phi_r(x_1,\ldots,x_n)$ without common factors, so that the monomials in $\mathscr{S}_k$ that have between k and $k-r$ variables in the set $\{x_{i_1},\ldots,x_{i_k}\}$ are reconstructed. Indeed, let $x^\beta \in S_k$ such that $k-r \leq |\mathrm{Im}(\alpha)\cap\mathrm{Im}(\beta)| \leq k$, put $J=\mathrm{Im}(\alpha)\cap\mathrm{Im}(\beta)$ and $m=|J|$. For each subset $K\subseteq J$ such that $|K|=k-r$, we have

$$x^\beta = \prod_{k\in K} x_k \times \prod_{l\in \mathrm{Im}(\beta)-K} x_l. \tag{5}$$

The first factor is a term of $\phi_{k-r}(x_{i_1},\ldots,x_{i_k})$ and the second factor is a term of $\phi_r(x_1,\ldots,x_n)$. The decomposition in (5) can be obtained by $\binom{m}{k-r}=\binom{m}{m-k+r}$ ways, and putting $m=k-i$, $(0\leq i\leq r)$, this number is $\binom{k-i}{r-i}$.

In conclusion, in the product $\phi_{k-r}(x_{i_1},\ldots,x_{i_k})\,\phi_r(x_1,\ldots,x_n)$, the terms that are not canceled who share exactly $k-i$ variables x^α appear with multiplicity $\binom{k-i}{r-i}$. It follows that

$$\pi_k\left(\phi_{k-r}(x_{i_1},\ldots,x_{i_k})\,\phi_r(x_1,\ldots,x_n)\right)=\sum_{i=0}^{r}\binom{k-i}{r-i}\Upsilon_i(x^\alpha);$$

thus

$$\mathscr{D}_r=\sum_{i=0}^{r}\binom{k-i}{r-i}\Upsilon_i. \tag{6}$$

ii) We evaluate $\mathscr{D}(\xi_i)$. As ∂ is a derivation, it holds that $\partial(PQ)=P\partial(Q)$ if $\partial(P)=0$. Moreover $\partial(x_i-x_j)=0$ and from the expression of ξ_i in (3) yields

$$\begin{aligned}\frac{1}{r!}\partial^r(\xi_i)&=\prod_{j=1}^{i}(x_{2j-1}-x_{2j})\times\frac{1}{r!}\partial^r\phi_{k-i}(x_{2i+1},\ldots,x_n)\\&=\binom{n-k+r-i}{r}\prod_{j=1}^{i}(x_{2j-1}-x_{2j})\,\phi_{k-i-r}(x_{2i+1},\ldots,x_n);\end{aligned} \tag{7}$$

here, the following identity was used:

$$\partial^m\phi_h(x_1,x_2,\ldots,x_l)=\frac{(l-h+m)!}{(l-h)!}\phi_{h-m}(x_1,x_2,\ldots,x_l).$$

The remainder is similar to the proof of *ii)* Proposition 2. Again, we put $\zeta^i=\prod_{j=1}^{i}(x_{2j-1}-x_{2j})$, $\zeta_l^i=\prod_{j=1,j\neq l}^{i}(x_{2j-1}-x_{2j})$ and $\psi=\phi_{k-i-r}(x_{2i+1},\ldots,x_n)$. Writing

$$\begin{aligned}\phi_r(x_1,x_2,\ldots,x_n)&=(x_1+x_2)\,\phi_{r-1}(x_3,\ldots,x_n)\\&\quad+x_1x_2\phi_{r-2}(x_3,\ldots,x_n)+\phi_r(x_3,\ldots,x_n)\end{aligned}$$

we have

$$\begin{aligned}\zeta^i\psi\phi_r(x_1,x_2,\ldots,x_n)&=\psi\zeta_1^i\left(x_1^2-x_2^2\right)\phi_{r-1}(x_3,\ldots,x_n)\\&\quad+\psi\left(x_1^2x_2-x_1x_2^2\right)\phi_{r-2}(x_3,\ldots,x_n)\\&\quad+\psi\zeta^i\phi_r(x_3,\ldots,x_n),\end{aligned}$$

so that

$$\pi_k\left(\zeta^i\psi\phi_r(x_1,x_2,\ldots,x_n)\right)=\pi_k\left(\psi\zeta^i\phi_r(x_3,\ldots,x_n)\right)$$

and reiterating the process

$$\pi_k\left(\zeta^i\psi\phi_r(x_1,x_2,\ldots,x_n)\right)=\pi_k\left(\psi\zeta^i\phi_r(x_{2i+1},\ldots,x_n)\right).$$

To evaluate the second member of the previous identity, we use the same argument as in the proof of *i)* of this proposition; that is,

$$\pi_k\left(\zeta^i\phi_{k-i-r}\left(x_{2i+1},\ldots,x_n\right)\phi_r\left(x_{2i+1},\ldots,x_n\right)\right)=\binom{k-i}{r}\zeta^i\phi_{k-i}\left(x_{2i+1},\ldots,x_n\right); \qquad (8)$$

then *ii)* follows from identities (7) and (8). □

Remark 2. *The matrix* (6) *which links the families of operators $\mathscr{D}_r$ and Υ_i is lower triangular and unipotent, that is, with ones on the diagonal. The same relationship holds for the eigenvalues of the operators*

$$\mu_r^j=\sum_{i=0}^{r}\binom{k-i}{r-i}\nu_i^j \qquad (9)$$

where $\Upsilon_i\left(\xi_j\right)=\nu_i^j\xi_j$. The eigenvalues of the family $\mathscr{D}_r$ were established in Proposition 3. Using Cramer's rule in (9) *the following result is obtained.*

Theorem 4. *The spectrum of $J(n,k,r)$ is given by*

$$\det\begin{bmatrix} 1 & 0 & \cdots & 0 & \mu_0^i \\ \binom{k}{1} & 1 & \ddots & \vdots & \mu_1^i \\ \binom{k}{2} & \binom{k-1}{1} & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \binom{k}{r} & \cdots & \binom{k-r+2}{2} & \binom{k-r+1}{1} & \mu_r^i \end{bmatrix} \qquad (10)$$

with $0\leq i\leq k$ and where

$$\mu_m^i=\begin{cases}\binom{k-i}{m}\binom{n-k+m-i}{m} & \text{if } 0\leq i\leq k-m,\\ 0 & \text{if } i>k-m.\end{cases}$$

We should clarify that the values given in (10) covering the spectrum of $J(n,k,r)$ do not take into account the multiplicities. Moreover, each value given in (10) is in the spectrum of $J(n,k,r)$; however, in principle it is not clear that the values in (10) are all different.

REFERENCES

[1] Aguado, J.L. and Araujo, J.O.: *A Gel'fand model for the symmetric group* Comm. Algebra 29 (2001), 1841–1851. MR 1853129.
[2] Delsarte, P.: *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl. No. 10, 1973. MR 0384310.
[3] Eberlein, P. J.: *A two parameter test matrix*. Math. Comp. 18 (1964), 296–298. MR 0170462.
[4] Krebs, M. and Shaheen, A.: *On the spectra of Johnson graphs*. Electron. J. Linear Algebra. 17 (2008), 154–167. MR 2390366.

UNICEN, FACULTAD DE CIENCIAS EXACTAS, TANDIL, ARGENTINA
E-mail: araujo@exa.unicen.edu.ar

UNICEN, FACULTAD DE CIENCIAS EXACTAS, TANDIL, ARGENTINA
E-mail: bratten@exa.unicen.edu.ar