ACTAS DEL XII CONGRESO
DR. ANTONIO A. R. MONTEIRO (2013)
2014, Páginas 149–161

INEQUALITIES FOR ONE-SIDED OPERATORS IN ORLICZ SPACES

SERGIO J. FAVIER AND SONIA E. ACINAS

ABSTRACT. In this paper, we get strong type inequalities for one-sided maximal best approximation operators $\mathscr{M}^{\pm}$ which are very related to one-sided Hardy-Littlewood maximal functions $M^{\pm}$. In order to obtain our results, strong and weak type inequalities for $M^{\pm}$ are considered.

1. INTRODUCTION

We denote by $\mathscr{I}$ the set of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are nonnegative, even, nondecreasing on $[0,\infty)$, such that $\varphi(t) > 0$ for all $t > 0$, $\varphi(0+) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

We say that a nondecreasing function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies the Δ_2 condition, symbolically $\varphi \in \Delta_2$, if there exists a constant $\Lambda_\varphi > 0$ such that $\varphi(2a) \le \Lambda_\varphi \varphi(a)$ for all $a \ge 0$.

An even and convex function $\Phi : \mathbb{R} \to \mathbb{R}_0^+$ such that $\Phi(a) = 0$ iff $a = 0$ is said to be a Young function. Unless stated otherwise, the Young function Φ is the one given by $\Phi(x) = \int_0^x \varphi(t)\,dt$, where $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the right-continuous derivative of Φ.

If $\varphi \in \mathscr{I}$, we define $L^\varphi(\mathbb{R}^n)$ as the class of all Lebesgue measurable functions f defined on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \varphi(t|f|)\,dx < \infty$ for some $t > 0$ and where dx denotes the Lebesgue measure on $\mathbb{R}^n$. If φ is a Young function, then $L^\varphi(\mathbb{R}^n)$ is an Orlicz space (see [12]).

In the case of Φ being a Young function such that $\Phi \in \Delta_2$, then $L^\Phi(\mathbb{R}^n)$ is the space of all measurable functions f defined on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \Phi(|f|)\,dx < \infty$.

Also note that if $\Phi \in C^1 \cap \Delta_2$ such that $\Phi(2a) \le \Lambda_\Phi \Phi(a)$ for all $a > 0$, then its derivative function φ satisfies the Δ_2 condition and

$$\frac{1}{2}(\varphi(a) + \varphi(b)) \le \varphi(a+b) \le \frac{\Lambda_\Phi^2}{2}(\varphi(a) + \varphi(b)), \tag{1}$$

for every $a, b \ge 0$.

A nondecreasing function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies the ∇_2 condition, denoted $\varphi \in \nabla_2$, if there exists a constant $\lambda_\varphi > 2$ such that $\varphi(2a) \ge \lambda_\varphi \varphi(a)$ for all $a \ge 0$.

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the classical Hardy-Littlewood maximal function M defined over cubes $Q \subset \mathbb{R}^n$ is given by the formula

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(t)|\,dt.$$

For $f \in L^1_{\text{loc}}(\mathbb{R})$, the one-sided Hardy-Littlewood maximal functions M^+ and M^- are introduced in [5] as follows:

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|\,dy, \quad \text{with } x \in \mathbb{R},$$

and

$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|\,dy, \quad \text{with } x \in \mathbb{R}.$$

2010 *Mathematics Subject Classification.* Primary: 42B25. Secondary: 41A44, 46E30.
This research was partially supported by CONICET and UNSL grants.

For the sake of simplicity, in the sequel we write $M^{\pm}$ to refer to M^{+} or M^{-}.

It is well known that M is homogeneous, subadditive, weak type $(1,1)$ and it also satisfies $\|Mf\|_{\infty} \leq \|f\|_{\infty}$. The one-sided maximal functions $M^{\pm}$ are also homogeneous, subadditive, weak type $(1,1)$ (see [5]) and strong type (∞,∞). In addition, M may be defined from the one-sided maximal functions as follows

$$Mf(x) = \max\{M^{+}f(x), M^{-}f(x)\}. \tag{2}$$

In fact,

$$\frac{1}{s+t}\int_{x-s}^{x+t}|f(y)|\,dy \leq \frac{s}{s+t}M^{-}f(x) + \frac{t}{t+s}M^{+}f(x) \leq \max\{M^{-}f(x), M^{+}f(x)\}.$$

Now, taking supremum over all $s,t > 0$, we have

$$Mf(x) \leq \max\{M^{-}f(x), M^{+}f(x)\}.$$

On the other hand,

$$Mf(x) = \sup_{s,t>0}\frac{1}{s+t}\int_{x-s}^{x+t}|f(u)|\,du \geq \sup_{s,t>0}\frac{1}{s+t}\int_{x-s}^{x}|f(u)|\,du = M^{-}f(x).$$

Similarly, we have $Mf(x) \geq M^{+}f(x)$. Therefore

$$Mf(x) \geq \max\{M^{+}f(x), M^{-}f(x)\}.$$

In [1] and [6], weak and strong type inequalities for M in Orlicz spaces were obtained. The one-sided weighted maximal operator on $\mathbb{R}$ in L^{p} spaces was studied by Sawyer [13], Martín-Reyes, Ortega Salvador and de la Torre [8], and Martín-Reyes [7]. The weighted Orlicz space case was treated in Ortega Salvador [10] assuming the reflexivity of the space. Kokilashvili and Krbec in [6], based on Ortega Salvador [10] and Ortega Salvador and Pick [11], removed the restriction to reflexive spaces and weakened some hypothesis.

In this paper, we follow the idea of Kokilashvili and Krbec in [6] for one-sided maximal functions on $\mathbb{R}$ without dealing with weight functions. Namely, we specify conditions on $\varphi \in \mathscr{I}$ under which the weak type inequalities

$$|\{x \in \mathbb{R} : M^{\pm}(f)(x) > \lambda\}| \leq \frac{c_1}{\varphi(\lambda)}\int_{\mathbb{R}}\varphi(c_1 f(x))\,dx, \tag{3}$$

and

$$|\{x \in \mathbb{R} : M^{\pm}f(x) > \lambda\}| \leq c_2\int_{\mathbb{R}}\varphi\left(\frac{c_2 f(x)}{\lambda}\right)dx, \tag{4}$$

hold for all $\lambda > 0$ and where $f \in L^{1}_{\mathrm{loc}}(\mathbb{R})$. We also characterize the strong type inequality

$$\int_{\mathbb{R}}\varphi(M^{\pm}f(x))\,dx \leq c\int_{\mathbb{R}}\varphi(cf(x))\,dx, \tag{5}$$

for all $f \in L^{1}_{\mathrm{loc}}(\mathbb{R})$.

It is worth mentioning that inequalities (3), (4) and (5) are particular cases of results given in [6]; however, as we do not deal with weight functions, we include easier proofs.

Then, we get conditions to assure the validity of strong type inequalities like (5) for one-sided maximal operators $\mathscr{M}^{\pm}$, related to one-sided best φ-approximation by constants to a function $f \in L^{\varphi}_{\mathrm{loc}}(\mathbb{R})$.

Last, we get strong type inequalities for lateral maximal operators $M_p^{\pm}$ related to p-averages.

2. WEAK TYPE INEQUALITIES FOR $M^{\pm}$

The next concept is introduced in [6] and we will employ it to set conditions under which (3) and (4) are valid.

Definition 1. *A function $\varphi : [0,\infty) \to \mathbb{R}$ is quasiconvex on $[0,\infty)$ if there exist a convex function ω and a constant $c > 0$ such that*

$$\omega(t) \leq \varphi(t) \leq c\omega(ct),$$

for all $t \in [0,\infty)$.

2.1. **Necessary and sufficient condition.** Lemma 1.2.4 in [6] establishes the equivalence between the validity of a weak type inequality like (3) for M over $\mathbb{R}^n$ and the quasiconvexity of φ. Theorem 2.4.1 in [6] states an analogous equivalence for $M^{\pm}$ over $\mathbb{R}$ employing weight functions. The next result is a particular case of this theorem; nevertheless, as we deal without using weights, we include an easier proof.

Theorem 2. *Let $\varphi \in \mathscr{I}$. φ is quasiconvex if and only if there exists $c_1 > 0$ such that*

$$|\{x \in \mathbb{R} : M^{\pm} f(x) > \lambda\}| \leq \frac{c_1}{\varphi(\lambda)} \int_{\mathbb{R}} \varphi(c_1 f(x))\, dx, \tag{6}$$

for all $\lambda > 0$ and for all $f \in L^1_{\text{loc}}(\mathbb{R})$.

Proof. $\Rightarrow$) Let $\varphi \in \mathscr{I}$ be a quasiconvex function. By Lemma 1.2.4 in [6], there exists $c_1 > 0$ such that

$$|\{x \in \mathbb{R} : Mf(x) > \lambda\}| \leq \frac{c_1}{\varphi(\lambda)} \int_{\mathbb{R}} \varphi(c_1 f(x))\, dx, \tag{7}$$

for all $\lambda > 0$ and for all $f \in L^1_{\text{loc}}(\mathbb{R})$. From (2) and the monotonicity of Lebesgue measure, we have

$$|\{x \in \mathbb{R} : M^{\pm} f(x) > \lambda\}| \leq |\{x \in \mathbb{R} : Mf(x) > \lambda\}|. \tag{8}$$

Now, by (7) and (8), we get (6).

$\Leftarrow$) We will prove the statement for M^+ reasoning as in the proof of Theorem 2.4.1 in [6]. The same argument with a slight modification is also valid for the case of M^-.

Let $a < b < c$ and assume $\int_b^c |f(u)|\, du \neq 0$. If $x \in (a,b)$ there exists $h > 0$ such that $(x, x+h) \supset (b,c)$ and $x+h = c$, then $h < c-a$ and

$$\chi_{(a,b)}(x) \frac{1}{c-a} \int_b^c |f(u)|\, du < \frac{1}{h} \int_b^c |f(u)|\, du \leq \frac{1}{h} \int_x^{x+h} |f(u)|\, du.$$

Therefore, if $x \in (a,b)$ then

$$\frac{1}{c-a} \int_b^c |f(u)|\, du < M^+ f(x). \tag{9}$$

On the other hand, if $x \notin (a,b)$ then

$$\chi_{(a,b)}(x) \frac{1}{c-a} \int_b^c |f(u)|\, du \leq M^+ f(x), \tag{10}$$

as $M^{\pm} f(x) \geq 0$. Eventually, from (9) and (10),

$$M^+ f(x) \geq \chi_{(a,b)}(x) \frac{1}{c-a} \int_b^c |f(u)|\, du \quad \text{for all } x \in \mathbb{R}.$$

Let $\lambda = \frac{1}{c-a} \int_b^c |f(u)|\, du > 0$. By (6), there exists $c_1 > 0$ such that

$$\left| \left\{ x \in \mathbb{R} : M^+ f(x) > \frac{1}{c-a} \int_b^c |f(u)|\, du \right\} \right| \varphi\left(\frac{1}{c-a} \int_b^c |f(u)|\, du \right) \leq c_1 \int_{\mathbb{R}} \varphi(c_1 f(x))\, dx.$$

From (9) we have

$$(b-a) \leq \left| \left\{ x \in \mathbb{R} : M^+ f(x) > \frac{1}{c-a} \int_b^c |f(u)|\, du \right\} \right|,$$

then there exists $c_1 > 0$ such that

$$(b-a)\varphi\left(\frac{1}{c-a}\int_b^c |f(u)|\, du\right) \leq c_1 \int_b^c \varphi(c_1 f(x))\, dx + c_1 \int_{\mathbb{R}-(b,c)} \varphi(c_1 f(x))\, dx,$$

for all $f \in L^1_{\text{loc}}(\mathbb{R})$ provided that $\int_b^c |f(u)|\, du \neq 0$.

Now, let $f(x) = f(x)\chi_{(b,c)}(x)$, then there exists $c_1 > 0$ such that

$$(b-a)\varphi\left(\frac{1}{c-a}\int_b^c |f(u)|\, du\right) \leq c_1 \int_b^c \varphi(c_1 f(x))\, dx, \tag{11}$$

with $f \in L^1_{\text{loc}}(\mathbb{R})$ such that $\int_b^c |f(u)|\, du \neq 0$.

In case of $\int_b^c |f(u)|\, du = 0$, (11) holds trivially.

Let $c > 1$ and $a < b < c$ such that $b - a = c - b$. Let $t_1, t_2 > 0$ and $\theta \in (0,1)$. We decompose (b,c) into two disjoint sets F and F' such that $(b,c) = F \cup F'$, $|F| = \theta(c-b)$ and $|F'| = (1-\theta)(c-b)$. Let $h(x) = t_1\chi_F(x) + t_2\chi_{F'}(x)$ for $x \in (b,c)$, then

$$\frac{1}{c-a}\int_b^c |h(x)|\, dx = \frac{1}{2}[\theta t_1 + (1-\theta)t_2].$$

Replacing in the left hand side of (11), there exists $c_2 > 0$ such that

$$(b-a)\varphi\left[\frac{\theta t_1 + (1-\theta)t_2}{2}\right] \leq c_1 \int_b^c \varphi(c_1 h(x))\, dx = c_1(b-a)[\varphi(c_1 t_1)\theta + (1-\theta)\varphi(c_1 t_2)].$$

Let $0 < T_1 = \frac{t_1}{2}$, $0 < T_2 = \frac{t_2}{2}$, then there exists $K_2 = 2c_2 > 0$ independent of T_1, T_2 and h such that

$$\varphi[\theta T_1 + (1-\theta)T_2] \leq K_2[\theta\varphi(K_2 T_1) + (1-\theta)\varphi(K_2 T_2)]. \tag{12}$$

Finally, by Lemma 1.1.1 in [6], (12) is equivalent to the fact that φ is quasiconvex. □

2.2. **Sufficient conditions.** Next, we set sufficient conditions for (4) to be verified. The next result is a particular case of Theorem 2.4.2 in [6].

Theorem 3. *Let $\varphi \in \mathscr{I}$. If φ is quasiconvex, then there exists $c_2 > 0$ such that*

$$|\{x \in \mathbb{R} : M^{\pm} f(x) > \lambda\}| \leq c_2 \int_{\mathbb{R}} \varphi\left(\frac{c_2 f(x)}{\lambda}\right) dx,$$

for all $\lambda > 0$ and for all $f \in L^1_{\text{loc}}(\mathbb{R})$.

Proof. It follows straightforwardly taking $\rho = \sigma = g = 1$ in the proof of Theorem 2.4.2 in [6]. □

However, the quasiconvexity of $\varphi \in \mathscr{I}$ is not a necessary condition for the validity of (4). Let $\varphi(x) = |x|^p$ for $p \geq 1$, then $\varphi \in \mathscr{I}$ and φ is a quasiconvex function on $[0,\infty)$. By Theorem 3 there exists $c_2 > 0$ such that

$$|\{x \in \mathbb{R} : M^{\pm} f(x) > \lambda\}| \leq c_2 \int_{\mathbb{R}} \varphi\left(\frac{c_2 f(x)}{\lambda}\right) dx,$$

for all $\lambda > 0$ and for all $f \in L^1_{\text{loc}}(\mathbb{R})$.

Now, let

$$\tilde{\varphi}(x) = \begin{cases} |x|^p & \text{if } |x| \geq 1 \\ |x|^{\frac{1}{p}} & \text{if } |x| < 1 \end{cases} \quad \text{for } p > 1;$$

then $\tilde{\varphi} \in \mathscr{I}$ and $\tilde{\varphi}(x) \geq \varphi(x) \geq 0$ for all $x \in \mathbb{R}$. Therefore, there exists $c_2 > 0$ such that

$$|\{x \in \mathbb{R} : M^{\pm} f(x) > \lambda\}| \leq c_2 \int_{\mathbb{R}} \tilde{\varphi}\left(\frac{c_2 f(x)}{\lambda}\right) dx,$$

for all $\lambda > 0$ and for all $f \in L^1_{\text{loc}}(\mathbb{R})$, although $\tilde{\varphi}$ is not a quasiconvex function. Hence, the converse of Theorem 3 is not true.

Remark 4. Let $\varphi, \tilde{\varphi} \in \mathscr{I}$ such that $\varphi(x) \leq \tilde{\varphi}(x)$ for all $x \in \mathbb{R}$. If φ is quasiconvex on $[0,\infty)$, then there exists $c > 0$ such that

$$|\{x \in \mathbb{R} : M^{\pm}(f)(x) > \lambda\}| \leq c \int_{\mathbb{R}} \tilde{\varphi}\left(\frac{cf(x)}{\lambda}\right) dx,$$

for all $f \in L^1_{\text{loc}}(\mathbb{R})$ and for all $\lambda > 0$.

Moreover, we determine some characteristics of the class of functions that satisfy (4).

Theorem 5. *Let $\psi \in \mathscr{I}$. Assume there exist constants $c_1 > 0$ and $x_0 \geq 0$ such that $\psi(x) \geq c_1 x$ for all $x \geq x_0$ and there exists a subinterval $(0,x_v) \subseteq (0,x_0)$ where ψ is either a convex function or a concave one. Then there exists a constant $c > 0$ such that*

$$|\{x \in \mathbb{R} : M^{\pm}(f)(x) > \lambda\}| \leq c \int_{\mathbb{R}} \psi\left(\frac{cf(x)}{\lambda}\right) dx,$$

for all $f \in L^1_{\text{loc}}(\mathbb{R})$ and for all $\lambda > 0$.

Proof. From (2), the monotonicity of Lebesgue measure and Theorem 5.8 in [1]. □

Therefore, (4) is valid for all $f \in L^1_{\text{loc}}(\mathbb{R})$ and for all $\lambda > 0$, when $\psi \in \mathscr{I}$ belongs to a bigger subset than that of quasiconvex functions.

2.3. **Necessary condition.** We also find a necessary condition for (4) to be satisfied.

Theorem 6. *Let $\varphi \in \mathscr{I}$. If there exists $c > 0$ such that*

$$|\{x \in \mathbb{R} : M^{\pm} f(x) > \lambda\}| \leq c \int_{\mathbb{R}} \varphi\left(\frac{cf(x)}{\lambda}\right) dx, \tag{13}$$

for all $\lambda > 0$ and for all $f \in L^1_{\text{loc}}(\mathbb{R})$, then $\frac{y}{c^2} \leq \varphi(y)$ for all $y > c$.

Proof. First, we consider the case of M^+.

Let $0 < t_1 < t_2$, $I = \left(1 - \frac{t_1}{t_2}, 1\right)$ and $f(x) = t_2 \chi_I(x)$. For any $x \in (0,1)$ we have $M^+ f(x) > t_1 > 0$ and then

$$|\{x \in \mathbb{R} : M^+ f(x) > t_1\}| \geq 1.$$

Now, with $\lambda = t_1$ and $f(x) = t_2 \chi_I(x)$ in (13), there exists $c > 0$ such that

$$1 \leq c \int_{\mathbb{R}} \varphi\left(\frac{ct_2 \chi_I(u)}{t_1}\right) du = c\varphi\left(c\frac{t_2}{t_1}\right)\frac{t_1}{t_2}.$$

Finally, we set $y = c\frac{t_2}{t_1} > c$, then $y \leq c^2 \varphi(y)$ for all $y > c$.

With the aim of obtaining the result for M^-, we set $I = (0, \frac{t_1}{t_2})$ where $0 < t_1 < t_2$ and we reason as in the case of M^+. □

3. STRONG TYPE INEQUALITY FOR $M^{\pm}$

Theorem 1.2.1 in [6] establishes that the validity of a strong type inequality for M is equivalent to the fact that the function involved satisfies the ∇_2 condition. We obtain an analogous result for $M^{\pm}$.

Theorem 7. *Let $\varphi \in \mathscr{I}$. The next statements are equivalent:*

i) there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}} \varphi(M^{\pm} f(x))\, dx \leq c_1 \int_{\mathbb{R}} \varphi(c_1 f(x))\, dx \quad \text{for all } f \in L^1_{\text{loc}}(\mathbb{R}), \tag{14}$$

ii) the function φ^{α} is quasiconvex for some $\alpha \in (0,1)$,

iii) there exists $c_2 > 0$ such that $\int_0^{\sigma} \frac{\varphi(x)}{s^2}\, ds \leq \frac{c_2 \varphi(c_2 \sigma)}{\sigma}$ for $0 < \sigma < \infty$,

iv) there exists $c_3 > 0$ such that for $t > 0$ $\int_0^t \frac{d\varphi(u)}{u} \leq \frac{c_3 \varphi(c_3 t)}{t}$,

v) there exists $a > 1$ such that

$$\varphi(t) < \frac{1}{2a} \varphi(at), \quad t \geq 0.$$

Proof. The proof of Theorem 1.2.1 in [6] follows this scheme: $i) \Rightarrow iii) \Rightarrow v) \Rightarrow ii) \Rightarrow i)$ and $iii) \Leftrightarrow iv)$. In the case of $M^{\pm}$ it is sufficient to obtain $i) \Rightarrow iii)$ and $ii) \Rightarrow i)$ because the remaining implications are not modified when M is changed by $M^{\pm}$, as only properties of quasiconvex functions are employed.

$i) \Rightarrow iii)$ Let $f(x) = \chi_{[a,b]}(x)$. After some calculations (see [4, p. 79]), we have

$$Mf(x) = \begin{cases} \frac{b-a}{b-x} & \text{if } x < a \\ 1 & \text{if } a \leq x \leq b \\ \frac{b-a}{x-a} & \text{if } x > b, \end{cases}$$

$$M^+ f(x) = \begin{cases} \frac{b-a}{b-x} & \text{if } x < a \\ 1 & \text{if } a \leq x \leq b \\ 0 & \text{if } x > b \end{cases} \quad \text{and} \quad M^- f(x) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } a \leq x \leq b \\ \frac{b-a}{x-a} & \text{if } x > b. \end{cases}$$

Consequently, we can write $M^+ f(x) = Mf\chi_{(-\infty,b]}(x)$ and $M^- f(x) = Mf\chi_{[a,\infty)}(x)$. Then, $iii)$ follows from $i) \Rightarrow iii)$ of Theorem 1.2.1 in [6].

$ii) \Rightarrow i)$ Due to Theorem 1.2.1 in [6], $v)$ implies that there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}^n} \varphi(Mf(x))\, dx \leq c_1 \int_{\mathbb{R}} \varphi(c_1 f(x))\, dx, \quad \text{for all } f \in L^1_{\text{loc}}(\mathbb{R}), \tag{15}$$

thus, by (2) and the monotonicity of φ we have

$$\int_{\mathbb{R}} \varphi(M^{\pm} f(x))\, dx \leq \int_{\mathbb{R}} \varphi(Mf(x))\, dx, \quad \text{for all } f \in L^1_{\text{loc}}(\mathbb{R}). \tag{16}$$

From (15) and (16), we get the desired inequality (14). □

Remark 8. Item $v)$ in Theorem 7 is equivalent to say that $\varphi \in \nabla_2$.

We point out that there exists an alternative way to get the strong type inequality (14) applying interpolation techniques.

Theorems 2, 3 and 5 guarantee the existence of classes of functions $\varphi \in \mathscr{I}$ that satisfy weak type inequalities like (3) and (4); in addition, the operators $M^{\pm}$ are subadditive and strong type (∞, ∞). Then, by application of Theorem 2.4 in [9] or Theorem 5.2 in [1], we obtain

$$\int_{\mathbb{R}} \Psi(|M^{\pm}(f)|)\, dx \leq K \int_{\mathbb{R}} \Psi(4f)\, dx, \quad \text{for all } f \in L^1_{\text{loc}}(\mathbb{R}),$$

and for a family of Young functions Ψ such that $\Psi' = \psi$ is related to $\varphi \in \mathscr{I}$ and provided that the function φ satisfies additional conditions.

4. One-sided maximal operators $\mathscr{M}^{\pm}$

By $\widehat{\mathscr{I}}$ we denote the class of all nondecreasing functions φ defined for all real number $t \geq 0$ such that $\varphi(t) > 0$ for all $t > 0$, $\varphi(0+) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Let $\Phi \in \widehat{\mathscr{I}} \cap \Delta_2$ be a convex function and let B be a bounded measurable set of $\mathbb{R}^n$. The next definition is introduced in [2].

Definition 9. *A real number c is a best Φ-approximation of $f \in L^{\Phi}(B)$ if and only if*

$$\int_B \Phi(|f(x)-c|)\,dx \leq \int_B \Phi(|f(x)-r|)\,dx, \quad \text{for all } r \in \mathbb{R}.$$

With the symbol $\mu_\Phi(f)(B)$ the authors refers to the multivalued operator of all best approximation constants of the function $f \in L^{\Phi}(B)$. It is well known that $\mu_\Phi(f)(B)$ is a non empty set; and, if Φ is strictly convex, then $\mu_\Phi(f)(B)$ has an only one element.

In [2] the definition of $\mu_\Phi(f)(B)$ is extended in a continuous way for functions $f \in L^{\varphi}(B)$ such that $\varphi = \Phi'$ with $\Phi \in C^1$ as follows.

Definition 10. *Let $\Phi \in \widehat{\mathscr{I}} \cap \Delta_2$ be a function in C^1 and assume that $\Phi' = \varphi$. If $f \in L^{\varphi}(B)$, then a constant c is a extended best approximation of f on B if c is a solution of the next inequalities:*

$$a)\ \int_{\{f>c\}\cap B} \varphi(|f(y)-c|)\,dy \leq \int_{\{f\leq c\}\cap B} \varphi(|f(y)-c|)\,dy,$$

and

$$b)\ \int_{\{f<c\}\cap B} \varphi(|f(y)-c|)\,dy \leq \int_{\{f\geq c\}\cap B} \varphi(|f(y)-c|)\,dy.$$

Let $\tilde{\mu}_\Phi(f)(B)$ be the set of all constants c.

In the particular case of $B = I_\varepsilon^{\pm}(x)$ where $I_\varepsilon^{\pm}(x)$ is a bounded one-sided interval of $x \in \mathbb{R}$ with positive Lebesgue measure ε, we write $\mu_\varepsilon^{\pm}(f)(x)$ for $\mu_\Phi(f)(I_\varepsilon^{\pm}(x))$ which is the one-sided best approximation by constants and we set $\tilde{\mu}_\varepsilon^{\pm} f(x)$ for the set $\tilde{\mu}_\Phi(f)(I_\varepsilon^{\pm}(x))$ which is the extended one-sided best approximation by constants.

We define the one-sided maximal operators $\mathscr{M}^{\pm}$, associated to one-sided best approximation by constants, in the following way:

$$\mathscr{M}^{\pm} f(x) = \sup_{\varepsilon>0}\{|f_\varepsilon^{\pm}(x)| : f_\varepsilon^{\pm}(x) \in \tilde{\mu}_\varepsilon^{\pm}(f)(x)\}.$$

Remark 11. If $f_\varepsilon^{\pm}(x) \in \tilde{\mu}_\varepsilon^{\pm}(f)(x)$, there exists $c_\varepsilon^{\pm} \in \tilde{\mu}_\varepsilon^{\pm}(|f|)(x)$ such that $|f_\varepsilon^{\pm}(x)| \leq c_\varepsilon^{\pm}$.

In fact, since $|f| \geq f \geq -|f|$ and the extended one-sided best approximation operator is a monotonous one (Lemma 12 in [3]), there exist $a_\varepsilon^{\pm}, b_\varepsilon^{\pm} \geq 0$ with $-a_\varepsilon^{\pm} \in \tilde{\mu}_\varepsilon^{\pm}(-|f|)(x)$ and $b_\varepsilon^{\pm} \in \tilde{\mu}_\varepsilon^{\pm}(|f|)(x)$ such that $-a_\varepsilon^{\pm} \leq f_\varepsilon^{\pm}(x) \leq b_\varepsilon^{\pm}$.

However, $a_\varepsilon^{\pm} \in \tilde{\mu}_\varepsilon^{\pm}(|f|)(x)$ and $c_\varepsilon^{\pm} = \max\{a_\varepsilon^{\pm}, b_\varepsilon^{\pm}\} \in \tilde{\mu}_\varepsilon^{\pm}(|f|)(x)$ because $\tilde{\mu}_\varepsilon^{\pm}(|f|)(x)$ is a closed set (Lemma 11 in [3]). As $c_\varepsilon^{\pm} \geq a_\varepsilon^{\pm}, b_\varepsilon^{\pm}$, we have $\mathscr{M}^{\pm} f(x) \leq \mathscr{M}^{\pm}|f|(x)$ and we may assume $f \geq 0$.

Now, we reason as in [2], working on $I_\varepsilon^{\pm}$ of $\mathbb{R}$ instead of balls centered at $x \in \mathbb{R}^n$ with radius ε, and we get the following result.

Theorem 12. *Let $\Phi \in \widehat{\mathscr{I}} \cap \Delta_2$ be a C^1 convex function and we assume $\Phi' = \varphi$. Let $f \in L^{\varphi}_{\mathrm{loc}}(\mathbb{R})$ and we select $f^{\pm}_{\varepsilon}(x) \in \tilde{\mu}^{\pm}_{\varepsilon}(f)(x)$ with $x \in \mathbb{R}$ and $\varepsilon > 0$. Then*

$$\frac{1}{C}\varphi(|f^{\pm}_{\varepsilon}(x)|) \leq \frac{1}{\varepsilon}\int_{I^{\pm}_{\varepsilon}} \varphi(|f(y)|)\,dy \leq C\varphi(|f|^{\pm}_{\varepsilon}(x)), \tag{17}$$

and

$$\frac{1}{C}\varphi(|f^{\pm}_{\varepsilon}(x) - f(x)|) \leq \frac{1}{\varepsilon}\int_{I^{\pm}_{\varepsilon}} \varphi(|f(y) - f(x)|)dy, \tag{18}$$

being ε the Lebesgue measure of the intervals $I^{\pm}_{\varepsilon}$ and $C = \frac{3\Lambda^2_{\Phi}}{2}$ where Λ_{Φ} is the constant given by the Δ_2 condition on Φ.

Proof. By Remark 11 we can assume $f \geq 0$ and then $f^{\pm}_{\varepsilon}(x) \geq 0$. In effect, by a) in Definition 10, if $c < 0$

$$\begin{aligned}\int_{I^{\pm}_{\varepsilon}} \varphi(|f(y) - c|)\,dy &= \int_{\{f>c\}\cap I^{\pm}_{\varepsilon}} \varphi(|f(y) - x|)\,dy \\ &\leq \int_{\{f\leq c<0\}\cap I^{\pm}_{\varepsilon}} \varphi(|f(y) - x|)\,dy = 0.\end{aligned} \tag{19}$$

As Φ is a C^1 convex function, then $\varphi(x) > 0$ for $x > 0$; if $c < 0$ then $f(y) - c > -c > 0$, consequently $\varphi(|f(y) - c|) > \varphi(-c) > 0$ and

$$\int_{I^{\pm}_{\varepsilon}} \varphi(|f(y) - c|)\,dy > |\varphi(-c)|\varepsilon > 0. \tag{20}$$

From (19) and (20) we obtain a contradiction.

Now, applying (1) and $|I^{\pm}_{\varepsilon} \cap \{f^{\pm}_{\varepsilon} < f\}| \leq \varepsilon$, we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_{I^{\pm}_{\varepsilon}} \varphi(f(y))\,dy \leq \frac{\Lambda^2_{\Phi}}{2\varepsilon}\int_{I^{\pm}_{\varepsilon}\cap\{f^{\pm}_{\varepsilon}<f\}} \varphi(f(y) - f^{\pm}_{\varepsilon}(x))\,dy + \frac{\Lambda^2_{\Phi}}{2}\varphi(f^{\pm}_{\varepsilon}(x)) \\ + \frac{1}{\varepsilon}\int_{I^{\pm}_{\varepsilon}\cap\{f^{\pm}_{\varepsilon}\geq f\}} \varphi(f(y))\,dy.\end{aligned} \tag{21}$$

Next, by b) of Definition 10 and if we suppose, without loss of generality, that $\Lambda_{\Phi} \geq \sqrt{2}$, we get

$$(21) \leq \frac{\Lambda^2_{\Phi}}{2\varepsilon}\int_{I^{\pm}_{\varepsilon}\cap\{f^{\pm}_{\varepsilon}\geq f\}} [\varphi(-f(y) + f^{\pm}_{\varepsilon}(x)) + \varphi(f(y))]\,dy + \frac{\Lambda^2_{\Phi}}{2}\varphi(f^{\pm}_{\varepsilon}(x)). \tag{22}$$

From (1) and as $f^{\pm}_{\varepsilon}(x) - f(y) \geq 0$ and $f(y) \geq 0$, then

$$\varphi(f^{\pm}_{\varepsilon}(x) - f(y)) + \varphi(f(y)) \leq 2\varphi(f^{\pm}_{\varepsilon}(x) - f(y) + f(y)) = 2\varphi(f^{\pm}_{\varepsilon}(x)),$$

and since $|I^{\pm}_{\varepsilon} \cap \{f^{\pm}_{\varepsilon} \geq f\}| \leq \varepsilon$, we obtain

$$(22) \leq \frac{\Lambda^2_{\Phi}}{2\varepsilon}\int_{I^{\pm}_{\varepsilon}\cap\{f^{\pm}_{\varepsilon}\geq f\}} 2\varphi(f^{\pm}_{\varepsilon}(x))\,dy + \frac{\Lambda^2_{\Phi}}{2}\varphi(f^{\pm}_{\varepsilon}(x)) \leq \frac{3\Lambda^2_{\Phi}}{2}\varphi(f^{\pm}_{\varepsilon}(x)).$$

Therefore, there exists $C = \frac{3\Lambda^2_{\Phi}}{2}$ such that

$$\frac{1}{\varepsilon}\int_{I^{\pm}_{\varepsilon}} \varphi(f(y))\,dy \leq C\varphi(f^{\pm}_{\varepsilon}(x)). \tag{23}$$

On the other hand, applying (1),

$$\begin{aligned}\varphi(f_\varepsilon^\pm(x)) &= \frac{1}{\varepsilon}\int_{I_\varepsilon^\pm}\varphi(f_\varepsilon^\pm(x))\,dy\\ &\leq \frac{\Lambda_\Phi^2}{2\varepsilon}\int_{I_\varepsilon^\pm\cap\{f_\varepsilon^\pm>f\}}[\varphi(f_\varepsilon^\pm(x)-f(y))+\varphi(f(y))]\,dy+\frac{1}{\varepsilon}\int_{I_\varepsilon^\pm\cap\{f_\varepsilon^\pm\leq f\}}\varphi(f_\varepsilon^\pm(x))\,dy. \quad (24)\end{aligned}$$

Now, we apply a) of Definition 10 and we have

$$\begin{aligned}(24) &\leq \frac{\Lambda_\Phi^2}{2\varepsilon}\int_{I_\varepsilon^\pm\cap\{f_\varepsilon^\pm\leq f\}}\varphi(f(y)-f_\varepsilon^\pm(x))\,dy+\frac{\Lambda_\Phi^2}{2\varepsilon}\int_{I_\varepsilon^\pm\cap\{f_\varepsilon^\pm>f\}}\varphi(f(y))\,dy\\ &\quad+\frac{1}{\varepsilon}\int_{I_\varepsilon^\pm\cap\{f_\varepsilon^\pm\leq f\}}\varphi(f_\varepsilon^\pm(x))\,dy\\ &\leq \frac{\Lambda_\Phi^2}{2\varepsilon}\int_{I_\varepsilon^\pm\cap\{f_\varepsilon^\pm\leq f\}}\left[\varphi(f(y)-f_\varepsilon^\pm(x))+\varphi(f_\varepsilon^\pm(x))\right]\,dy+\frac{\Lambda_\Phi^2}{2\varepsilon}\int_{I_\varepsilon^\pm\cap\{f_\varepsilon^\pm>f\}}\varphi(f(y))\,dy,\end{aligned} \quad (25)$$

provided that $1\leq\frac{\Lambda_\Phi^2}{2}$. Now, by (1) we get

$$(25)\leq\frac{\Lambda_\Phi^2}{2\varepsilon}\int_{I_\varepsilon^\pm\cap\{f_\varepsilon^\pm\leq f\}}2\varphi(f(y))\,dy+\frac{\Lambda_\Phi^2}{2\varepsilon}\int_{I_\varepsilon^\pm\cap\{f_\varepsilon^\pm>f\}}\varphi(f(y))\,dy\leq\frac{\Lambda_\Phi^2}{\varepsilon}\int_{I_\varepsilon^\pm}\varphi(f(y))\,dy,$$

because $\frac{\Lambda_\Phi^2}{2}\leq\Lambda_\Phi^2$ and $I_\varepsilon^\pm=I_\varepsilon^\pm\cap\{f_\varepsilon^\pm\leq f\}\cup I_\varepsilon^\pm\cap\{f_\varepsilon^\pm>f\}$.

Then

$$\frac{1}{C}\varphi(f_\varepsilon^\pm(x))\leq\frac{1}{\varepsilon}\int_{I_\varepsilon^\pm}\varphi(f(y))\,dy, \quad (26)$$

where $C=\frac{3\Lambda_\Phi^2}{2}$ and (17) follows from (23) and (26).

It remains to prove (18). Note that if $f_\varepsilon^\pm(x)\in\tilde{\mu}_\varepsilon^\pm(f)(x)$, then $f_\varepsilon^\pm(x)-f(x)\in\tilde{\mu}_\varepsilon^\pm(f-f(x))(x)$. We apply (17) to the function $f-f(x)$ and we obtain

$$\frac{1}{C}\varphi(|f_\varepsilon^\pm(x)-f(x)|)\leq\frac{1}{\varepsilon}\int_{I_\varepsilon^\pm}\varphi(|f(y)-f(x)|)dy,$$

which is the inequality that we wished to obtain. □

Next, we get an inequality that allows us to compare $M^\pm$ with $\mathscr{M}^\pm$.

Lemma 1. *Let $\Phi\in\widehat{\mathscr{I}}\cap\Delta_2$ be a C^1 convex function and let $\Phi'=\varphi$ be such that $A\varphi(t)\leq\varphi(Kt)$ for all $t\geq0$ and some constants $K,A>1$. Then there exists $C>0$ such that*

$$\frac{1}{K}\varphi^{-1}\left(\frac{1}{C}M^\pm(\varphi(|f|))(x)\right)\leq\mathscr{M}^\pm(|f|)(x)\leq\varphi^{-1}(CM^\pm(\varphi(|f|))(x)), \quad (27)$$

where $M^\pm(f)=\sup\limits_{\varepsilon>0}\frac{1}{\varepsilon}\int_{I_\varepsilon^\pm}|f(y)|\,dy$.

Proof. Let φ^{-1} be the generalized inverse of the monotonous function φ which is defined by $\varphi^{-1}(s)=\sup\{t:\varphi(t)\leq s\}$, then

$$t\leq\varphi^{-1}(\varphi(t))\quad\text{for all } t\geq0, \quad (28)$$

and for every $\tilde{\varepsilon}>0$

$$\varphi^{-1}(\varphi(t)-\tilde{\varepsilon})\leq t\quad\text{for all } t\geq0. \quad (29)$$

The condition $A\varphi(t) \le \varphi(Kt)$ for all $t \ge 0$ and some constants $A, K > 1$, implies that $\varphi(0+) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$; therefore, φ^{-1} is a real valued function and $\varphi^{-1} \in \widehat{\mathscr{I}}$. From (17) in Theorem 12 we have

$$|f|^{\pm}_{\varepsilon}(x) \le \varphi^{-1}\left(\frac{C}{\varepsilon}\int_{I^{\pm}_{\varepsilon}} \varphi(|f(y)|)\,dy\right),$$

and since

$$\frac{C}{\varepsilon}\int_{I^{\pm}_{\varepsilon}} \varphi(|f(y)|)\,dy \le CM^{\pm}(\varphi(|f|))(x),$$

then

$$\mathscr{M}^{\pm}(|f|) = \sup_{\varepsilon>0} |f|^{\pm}_{\varepsilon}(x) \le \varphi^{-1}(CM^{\pm}(\varphi(|f|))(x). \tag{30}$$

Now, by (17) in Theorem 12 and the monotonicity of φ we have

$$\frac{1}{\varepsilon}\int_{I^{\pm}_{\varepsilon}} \varphi(|f(y)|)\,dy \le C\varphi(|f|^{\pm}_{\varepsilon}(x)) \le \varphi(\mathscr{M}^{\pm}(|f|)(x), \quad \text{for all } \varepsilon > 0,$$

and therefore

$$M^{\pm}(\varphi(|f|))(x) \le C\varphi(\mathscr{M}^{\pm}(|f|)(x)). \tag{31}$$

As there exist $K, A > 1$ such that $A\varphi(t) \le \varphi(Kt)$ for all $t \ge 0$, then $0 \le \varphi(t) < A\varphi(t) \le \varphi(Kt)$ for all $t \ge 0$ and consequently $0 < \varphi(Kt) - \varphi(t)$ for all $t > 0$. Now, from (29) and taking $0 < \tilde{\varepsilon} = \varphi(Kt) - \varphi(t)$ for all $t > 0$, we get

$$\varphi^{-1}(\varphi(t)) = \varphi^{-1}(\varphi(Kt) - \tilde{\varepsilon}) \le Kt \quad \text{for all } t > 0. \tag{32}$$

From (31), the fact that φ^{-1} is a nondecreasing function and (32), we get

$$\varphi^{-1}\left(\frac{1}{C}M^{\pm}(\varphi(|f|))(x)\right) \le \varphi^{-1}(\varphi(\mathscr{M}^{\pm}(|f|)(x))) \le K\mathscr{M}^{\pm}(|f|)(x). \tag{33}$$

Therefore, from (30) and (33),

$$\frac{1}{K}\varphi^{-1}\left(\frac{1}{C}M^{\pm}(\varphi(|f|))(x)\right) \le \mathscr{M}^{\pm}(|f|)(x) \le \varphi^{-1}(CM^{\pm}(\varphi(|f|))(x)). \qquad \square$$

4.1. **Strong type inequalities for $\mathscr{M}^{\pm}$.**

Theorem 13. *Let $\Phi \in \widehat{\mathscr{I}} \cap \Delta_2$ be a C^1 convex function and let $\Phi' = \varphi$ be such that $A\varphi(t) \le \varphi(Kt)$ for all $t \ge 0$ and for some constants $K, A > 1$. For a function $\theta \in \widehat{\mathscr{I}} \cap \Delta_2$, we have that the function $\theta \circ \varphi^{-1}$ satisfies the ∇_2 condition if and only if there exists a constant $\tilde{C}$ independent of f such that*

$$\int_{\mathbb{R}} \theta(\mathscr{M}^{\pm}(|f|)(x))\,dx \le \tilde{C}\int_{\mathbb{R}} \theta(\tilde{C}|f(x)|)\,dx,$$

for all $f \in L^{\varphi}_{\text{loc}}(\mathbb{R})$.

Proof. $\Leftarrow$) Suppose that $\mathscr{M}^{\pm}(|f|)$ verifies

$$\int_{\mathbb{R}} \theta(\mathscr{M}^{\pm}(|f|)(x))\,dx \le \tilde{C}\int_{\mathbb{R}} \theta(\tilde{C}|f(x)|)\,dx,$$

for all $f \in L^{\varphi}_{\text{loc}}(\mathbb{R})$.

As $\theta \in \widehat{\mathscr{I}} \cap \Delta_2$, there exists $K_1 > 0$ such that

$$\int_{\mathbb{R}} \theta(K\mathscr{M}^{\pm}(|f|)(x))\,dx \le K_1\int_{\mathbb{R}} \theta(K_1|f(x)|)\,dx, \tag{34}$$

for all $f \in L^{\varphi}_{\text{loc}}(\mathbb{R})$.

From (27), (34) and the fact that $M^{\pm}$ is homogeneous, we have

$$\begin{aligned}\int_{\mathbb{R}} \theta\left(\varphi^{-1}\left(\frac{1}{C}M^{\pm}(\varphi(|f|))(x)\right)\right) dx &= \int_{\mathbb{R}} \theta\left(\varphi^{-1}\left(M^{\pm}\left(\frac{1}{C}\varphi(|f|)\right)(x)\right)\right) dx \\ &\leq \int_{\mathbb{R}} \theta(K\mathscr{M}^{\pm}(|f|)(x))\, dx \\ &\leq K_1 \int_{\mathbb{R}} \theta(K_1|f(x)|)\, dx.\end{aligned} \tag{35}$$

Since $t \leq \varphi^{-1}(\varphi(t))$ for all $t \geq 0$, then $K_1|f(x)| \leq \varphi^{-1}(\varphi(K_1|f(x)|))$; now, by the monotonicity of θ and the fact that $\varphi \in \widehat{\mathscr{I}} \cap \Delta_2$, there exists $K_2 > 0$ such that

$$\int_{\mathbb{R}} \theta(K_1|f(x)|)\, dx \leq \int_{\mathbb{R}} \theta(\varphi^{-1}(\varphi(K_1|f(x)|)))\, dx \leq \int_{\mathbb{R}} \theta(\varphi^{-1}K_2(\varphi(|f(x)|)))\, dx, \tag{36}$$

for all $f \in L^{\varphi}_{\mathrm{loc}}(\mathbb{R})$.

Therefore, from (35) and (36), we have

$$\int_{\mathbb{R}} \psi(M^{\pm}(g)(x))\, dx \leq \tilde{C} \int_{\mathbb{R}} \psi(\tilde{C}g(x))\, dx, \tag{37}$$

where $\psi = \theta \circ \varphi^{-1}$, $g = \frac{1}{C}\varphi(|f|)$ for any $f \in L^{\varphi}_{\mathrm{loc}}(\mathbb{R})$ and $\tilde{C} = \max\{K_1, K_2C\}$.

As the inequality (37) holds for any $f \in L^{\varphi}_{\mathrm{loc}}(\mathbb{R})$ being $g = \frac{1}{C}\varphi(|f|)$, we choose $f = \varphi^{-1}(Cg)$ for any nonnegative function $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ and, using the fact that $\varphi(\varphi^{-1}(t)) = t$ provided that $t \in \operatorname{Im}\varphi \cup \{\inf \operatorname{Im}\varphi, \sup \operatorname{Im}\varphi\}$, we obtain

$$\int_{\mathbb{R}} \psi(M^{\pm}(g)(x))\, dx \leq \tilde{C} \int_{\mathbb{R}} \psi(\tilde{C}g(x)),$$

for all nonnegative functions $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ and where $\tilde{C}$ is independent of g. Now, by Theorem 7, we get $\psi = \theta \circ \varphi^{-1} \in \nabla_2$.

$\Rightarrow$) As $\psi = \theta \circ \varphi^{-1} \in \nabla_2$, by Theorem 7, there exists $K_1 > 0$ such that

$$\int_{\mathbb{R}} \psi(M^{\pm}(g)(x))\, dx \leq K_1 \int_{\mathbb{R}} \psi(K_1 g(x))\, dx, \tag{38}$$

for all nonnegative functions $g \in L^1_{\mathrm{loc}}(\mathbb{R})$. By (27) we have

$$\mathscr{M}^{\pm}(|f|)(x) \leq \varphi^{-1}(CM^{\pm}(\varphi(|f|))(x)), \tag{39}$$

and if $K_2 = \max\{C, K_1\}$, both inequalities hold with K_2.

Therefore, from (38), the monotonicity of θ, the homogeneity of $M^{\pm}$ and (39), we have

$$\begin{aligned}\int_{\mathbb{R}} \theta(\mathscr{M}^{\pm}(|f|(x)))\, dx &\leq \int_{\mathbb{R}} \psi(K_2M^{\pm}(\varphi(|f|))(x))\, dx \\ &= \int_{\mathbb{R}} \psi(M^{\pm}(K_2\varphi(|f|))(x))\, dx \\ &\leq K_2 \int_{\mathbb{R}} \psi(K_2^2\varphi(|f(x)|))\, dx \\ &\leq K_3 \int_{\mathbb{R}} \psi(K_3\varphi(|f(x)|))\, dx,\end{aligned} \tag{40}$$

with $K_3 = \max\{K_2, K_2^2\}$.

Since $A\varphi(t) \leq \varphi(Kt)$ for all $t \geq 0$ and for some $A, K > 1$, there exists l such that $K_3 \leq A^l$ and, applying the inequality l times, then

$$K_3\varphi(x) \leq A^l\varphi(x) \leq A^{l-1}\varphi(Kt) \leq \varphi(K^l t).$$

Now

$$K_3 \int_{\mathbb{R}} \psi(K_3\varphi(|f(x)|))\,dx \le K_4 \int_{\mathbb{R}} \psi(\varphi(K_4|f(x)|))\,dx,$$

where $K_4 = \max\{K_3, K^l\}$. By (32) we have $\varphi^{-1}(\varphi(t)) \le Kt$ and since $\theta \circ \varphi^{-1} = \psi$, then $(\psi \circ \varphi)(t) = (\theta \circ \varphi^{-1} \circ \varphi)(t) \le \theta(Kt)$; now

$$K_4 \int_{\mathbb{R}} \psi(\varphi(K_4|f(x)|))\,dx \le K_4 \int_{\mathbb{R}} \theta(KK_4|f(x)|)\,dx \le \tilde{C} \int_{\mathbb{R}} \theta(\tilde{C}|f(x)|)\,dx, \qquad (41)$$

being $\tilde{C} = \max\{K_4, KK_4\}$.

Consequently, from (40) and (41), we get

$$\int_{\mathbb{R}} \theta(\mathscr{M}^{\pm}(|f|(x)))\,dx \le \tilde{C} \int_{\mathbb{R}} \theta(\tilde{C}|f(x)|)\,dx. \qquad \square$$

Remark 14. If $\varphi \in \widehat{\mathscr{I}}$ such that $t^p \le \varphi \le Ct^p$ then $\varphi(Kt) \ge A\varphi(t)$ for all $t \ge 0$ and for any $K > 1$ such that $A = \frac{K^p}{C} > 1$. In consequence, Theorem 13 allows us to consider $\varphi \in \widehat{\mathscr{I}}$ which is not a strictly increasing function and in this case $\tilde{\mu}_{\varepsilon}^{\pm}(f)(x)$ may have more than one element.

We also get sufficient conditions to have a strong type inequality for $\mathscr{M}^{\pm}$ softening the hypothesis of Theorem 13.

Theorem 15. *Let $\Phi \in \widehat{\mathscr{I}} \cap \Delta_2$ be a convex function in C^1 and let $\Phi' = \varphi$ such that $A\varphi(t) \le \varphi(Kt)$ for all $t \ge 0$ and for some constants $K, A > 1$. Then*

$$\int_{\mathbb{R}} \Phi(\mathscr{M}^{\pm}(|f|)(x))\,dx \le C \int_{\mathbb{R}} \Phi(C|f(x)|)\,dx,$$

for all $f \in L^{\varphi}_{\mathrm{loc}}(\mathbb{R})$ and where the constant C is independent of f.

Proof. With the aim of applying Theorem 13, we need to show $\Phi \circ \varphi^{-1} \in \nabla_2$ where $\Phi \in \widehat{\mathscr{I}} \cap \Delta_2$ and a proof of this fact is done in [2]. $\square$

4.2. **Operators $M_p^{\pm}$.** If $\Phi(t) = t^{p+1}$ with $p > 0$ in (27), there exists a positive constant $\tilde{K}$ independent of f such that

$$\frac{1}{\tilde{K}} \left(M^{\pm}(|f|^p)(x)\right)^{\frac{1}{p}} \le \mathscr{M}^{\pm}_{t^{p+1}}(|f|)(x) \le \tilde{K}(M^{\pm}(|f|^p)(x))^{\frac{1}{p}}. \qquad (42)$$

Let $M_p^{\pm}(f)(x) = \left(\sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_{I_{\varepsilon}^{\pm}(x)} |f(t)|^p\,dt\right)^{\frac{1}{p}} = (M^{\pm}(|f|^p)(x))^{\frac{1}{p}}$. The operators $M_p^{\pm}$ are related to one-sided p-averages of a function and they are homogeneous like $M^{\pm}$.

A useful and particularly simple characterization of strong type inequalities involving $M_p^{\pm}$ may be established for this special case employing Theorem 13.

Corollary 16. *Let $\theta \in \widehat{\mathscr{I}}$ and $p > 0$, then there exists $\bar{K} > 0$ such that*

$$\int \theta(M_p^{\pm}(f)(x))\,dx \le \bar{K} \int \theta(\bar{K}|f(x)|)\,dx, \qquad (43)$$

for all $f \in L^p_{\mathrm{loc}}(\mathbb{R})$ if and only if $\theta(t^{1/p}) \in \nabla_2$.

Proof. It follows from Theorem 13 with $\Phi(x) = \frac{x^{p+1}}{p+1}$ because $\Phi \in \widehat{\mathscr{I}} \cap \Delta_2$ is a C^1 convex function such that $A\varphi(t) < \varphi(Kt)$ for all $t \ge 0$ with $A > 1$, $K > A^{\frac{1}{p}}$ and where $\varphi = \Phi'$. $\square$

Remark 17. If (43) holds, then $\|M_p^{\pm}(f)\|_\theta \le C\|f\|_\theta$, where $\|f\|_\theta$ denotes the Luxemburg norm of f defined by

$$\|f\|_\theta = \inf\left\{\lambda > 0 : \int_{\mathbb{R}} \theta\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\},$$

being θ a Young function and $f \in L^\theta(\mathbb{R})$.

Proof. The statement follows straightforwardly from Remark 2 in [2]. □

ACKNOWLEDGMENT

The authors would like to thank the referees for their valuable comments and suggestions.

REFERENCES

[1] S. Favier, S. Acinas, *Maximal inequalities in Orlicz spaces.* Int. J. Math. Anal. (Ruse) **6** (2012), no. 41–44, 2179–2198. MR 2946265.

[2] S. Favier, F. Zó, *Maximal inequalities for a best approximation operator in Orlicz spaces.* Comment. Math. **51** (2011), no. 1, 3–21. MR 2768735.

[3] S. Favier, F. Zó, *A Lebesgue type differentiation theorem for best approximations by constants in Orlicz spaces.* Real Analysis Exchange, **30** (2005), no. 1, 29–42. MR 2126792.

[4] L. Grafakos, *Classical Fourier Analysis.* Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2008. MR 2445437.

[5] E. Hewitt, K. Stromberg, *Real and Abstract Analysis.* Springer-Verlag, Berlin-Heidelberg-New York, 1975. MR 0367121.

[6] V. Kokilashvili, M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces.* World Scientific Publishing Co., Inc., River Edge, NJ, 1991. MR 1156767.

[7] F. J. Martín-Reyes, *New proofs of weighted inequalities for one-sided Hardy-Littlewood maximal functions.* Proc. Amer. Math. Soc. **117** (1993), no. 3, 691–698. MR 1111435.

[8] F. J. Martín-Reyes, P. Ortega Salvador, A. de la Torre, *Weighted inequalities for one-sided maximal functions.* Trans. Amer. Math. Soc. **319** (1990), no. 2, 517–534. MR 0986694.

[9] F. D. Mazzone, F. Zó, *On maximal inequalities arising in best approximation.* JIPAM. J. Inequal. Pure Appl. Math. **10** (2009), no. 2, Article 58, 10 pp. MR 2511951.

[10] P. Ortega Salvador, *Weighted inequalities for one-sided maximal functions in Orlicz spaces.* Studia Math. **131** (1998), no. 2, 101–114. MR 1636403.

[11] P. Ortega Salvador, L. Pick, *Two-weight weak and extra-weak type inequalities for the one-sided maximal operator.* Proc. Royal Soc. Edinburgh. **123** (1993), section A, 1109–1118. MR 1263909.

[12] M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces.* Marcel Dekker, Inc., New York, 1991. MR 1113700.

[13] E. Sawyer, *Weighted inequalities for one-sided Hardy-Littlewood maximal function.* Trans. Amer. Math. Soc. **297** (1986), no. 1, 53–61. MR 0849466.

UNIVERSIDAD NACIONAL DE SAN LUIS, INSTITUTO DE MATEMÁTICA APLICADA SAN LUIS, CONICET AND DEPARTAMENTO DE MATEMÁTICA, (5700) SAN LUIS, ARGENTINA
E-mail: sfavier@unsl.edu.ar

UNIVERSIDAD NACIONAL DE LA PAMPA, DEPARTAMENTO DE MATEMÁTICA DE LA FACULTAD DE CIENCIAS EXACTAS Y NATURALES, (6300) SANTA ROSA, LA PAMPA, ARGENTINA
E-mail: sonia.acinas@gmail.com