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ON THE CONSTRUCTION OF VARIATIONAL INTEGRATORS FOR OPTIMAL CONTROL OF NONHOLONOMIC MECHANICAL SYSTEMS

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ABSTRACT. In this paper we derive variational integrators for optimal control problems of nonholonomic mechanical systems. We rewrite the system as a constrained second-order variational problem, that is, as a problem where the Lagrangian and constraints are defined in terms of the position, velocity and the acceleration of the system. Instead of discretizing directly the equations of motion, we discretize the corresponding Hamilton's principle of critical action to derive a geometric integrator. We use the classical Lagrange multipliers method for constrained problems to derive this numerical scheme. An optimal control problem of a nonholonomic particle is given to illustrate the contents of the work.

1. INTRODUCTION

Nonholonomic mechanics is a traditional topic in Mathematics and Engineering Sciences, due to its applications in robotics and motion planning, among others. The introduction of new geometric tools has permitted a fast development in the last years, and nowadays nonholonomic mechanics is a very active research topic within the area known as Geometric Mechanics. There is a need to develop adapted numerical methods for this kind of mechanical systems, therefore the discrete versions of nonholonomic mechanics have attracted a lot attention in recent years [4, 5, 10, 11, 15, 17, 18], in some cases, incorporating the more sophisticated but natural language of Lie groupoids [13].

Variational integrators are a class of integration methods for Lagrangian systems, where the integrator is derived from the discretization of Hamilton's principle of critical action rather than a direct discretization of the corresponding Euler-Lagrange equations. A summary of the basic theory is given in Marsden and West [16]. The integrators derived in this way naturally preserve (or nearly preserve) the quantities that are preserved in the continuous framework, the symplectic form, the total energy and, in presence of symmetries which are invariant under the discretization process, the linear and/or angular momentum.

The aim of this work is to design a variational integrator to solve an optimal control problem for a controlled nonholonomic mechanical system. We see that this problem will be equivalent to solve a discrete higher-order constrained variational problem.

It is well known (see [3]) that a fully actuated optimal control problem is equivalent to solve an optimization problem for a second-order Lagrangian. If the system is underactuated, that is, the number of control inputs is less than the dimension of the configuration space; the optimal control problem is equivalent to solving an optimization problem for a second-order Lagrangian subject to second-order constraints (see [6]).

Discrete Mechanics and Optimal Control (DMOC) is a new research area within Geometric Mechanics started some years ago (see for instance [16, 20] and references therein). In our paper, we show the application of discrete mechanics and variational integrators to

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optimal control problems of nonholonomic mechanical systems. Moreover the problem, solved in this way, can be consider as a new application of our recent paper [8].

The structure of this work is the following: In Section 2 we introduce discrete mechanics and variational integrators for first and second-order systems subject to first and second-order constraints, respectively. In Section 3 we present the mathematical structure of an optimal control problem of nonholonomic mechanical systems. We show how to solve this problem in the continuous case and construct the discrete version of the problem to derive a variational integrator for this system. Finally we give an illustrative example.

All the manifolds are real, second countable and C^∞. The maps and the structures are assumed to be C^∞. Sum over repeated indices is understood.

2. Variational integrators for mechanical systems with constraints

2.1. **Discrete mechanics and variational integrators.** Let Q be a n-dimensional differentiable manifold with local coordinates (q^i), $1 \leq i \leq n$, the configuration space of a mechanical system. Denote by TQ its tangent bundle with induced local coordinates $(q^i, \dot{q}^i)$. Given a Lagrangian function $L: TQ \to \mathbb{R}$, its Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad 1 \leq i \leq n. \tag{1}$$

These equations determine a system of implicit second-order differential equations. If we assume that the Lagrangian is regular, that is, the $n \times n$ matrix $\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)$ is non-degenerate, the local existence and uniqueness of solutions is guaranteed for initial conditions.

Variational integrators (see [16] for details) are derived from a discrete variational principle. These integrators also retain some of the main geometric properties of the continuous systems, such as simplecticity, momentum conservation (as long as the symmetry survives the discretization procedure) and a good behavior of the energy associated to the system (see [12] and references therein). In what follows we introduce the construction of this type of variational integrators.

A *discrete Lagrangian* is a differentiable function $L_d\colon Q \times Q \to \mathbb{R}$, which may be considered as an approximation of the action integral defined by a continuous Lagrangian $L\colon TQ \to \mathbb{R}$. That is, given a time step $h > 0$ small enough,

$$L_d(q_0, q_1) \approx \int_0^h L(q(t), \dot{q}(t))\, dt,$$

where $q(t)$ is the unique solution of the Euler-Lagrange equations for L with boundary conditions $q(0) = q_0$ and $q(h) = q_1$.

We construct the grid $\{t_k = kh \mid k = 0, \ldots, N\}$, with $Nh = T$ and define the discrete path space $\mathscr{P}_d(Q) := \{q_d : \{t_k\}_{k=0}^N \to Q\}$. We identify a discrete trajectory $q_d \in \mathscr{P}_d(Q)$ with its image $q_d = \{q_k\}_{k=0}^N$, where $q_k := q_d(t_k)$. The discrete action $\mathscr{A}_d : \mathscr{P}_d(Q) \to \mathbb{R}$ along this sequence is calculated by summing the discrete Lagrangian on each adjacent pair and defined by

$$\mathscr{A}_d(q_d) = \mathscr{A}_d(q_0, \ldots, q_N) := \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}). \tag{2}$$

We would like to point out that the discrete path space is isomorphic to the smooth product manifold which consists on $N+1$ copies of Q, the discrete action inherits the smoothness of the discrete Lagrangian and the tangent space $T_{q_d}\mathscr{P}_d(Q)$ at q_d is the set of maps $v_{q_d} : \{t_k\}_{k=0}^N \to TQ$ such that $\tau_Q \circ v_{q_d} = q_d$ which will be denoted by $v_{q_d} = \{(q_k, v_k)\}_{k=0}^N$, where $\tau_Q : TQ \to Q$ is the canonical projection.

We know that for any product manifold $Q_1 \times Q_2$, $T^*_{(q_1,q_2)}(Q_1 \times Q_2) \simeq T^*_{q_1}Q_1 \oplus T^*_{q_2}Q_2$, for $q_1 \in Q_1$ and $q_2 \in Q_2$ where T^*Q denotes the cotangent bundle of a differentiable manifold Q. Therefore, any covector $\alpha \in T^*_{(q_1,q_2)}(Q_1 \times Q_2)$ admits an unique decomposition $\alpha = \alpha_1 + \alpha_2$ where $\alpha_i \in T^*_{q_i}Q_i$, for $i = 1,2$. Thus, given a discrete Lagrangian L_d we have the following decomposition

$$dL_d(q_0,q_1) = D_1L_d(q_0,q_1) + D_2L_d(q_0,q_1).$$

where $D_1L_d(q_0,q_1) \in T^*_{q_0}Q$ and $D_2L_d(q_0,q_1) \in T^*_{q_1}Q$.

The discrete variational principle, or Cadzow's principle, states that the solutions of the discrete system determined by L_d must extremize the action sum given fixed points q_0 and q_N. Extremizing $\mathscr{A}_d$ over q_k with $1 \leq k \leq N-1$, we obtain the following system of difference equations

$$D_1L_d(q_k,q_{k+1}) + D_2L_d(q_{k-1},q_k) = 0. \tag{3}$$

These equations are usually called discrete Euler-Lagrange equations. Given a solution $\{q^*_k\}_{k\in\mathbb{Z}}$ of eq. (3) and assuming the regularity hypothesis (the matrix $(D_{12}L_d(q_k,q_{k+1}))$ is regular), it is possible to define a (local) discrete flow $\Upsilon_{L_d} : \mathscr{U}_k \subset Q \times Q \to Q \times Q$, by $\Upsilon_{L_d}(q_{k-1},q_k) = (q_k,q_{k+1})$ from (3), where $\mathscr{U}_k$ is a neighborhood of the point (q^*_{k-1},q^*_k).

2.2. **Variational integrators for systems with constraints.** In this subsection we recall some basic elements on discrete mechanics for systems with constraints using discrete variational calculus (see [2]). The solutions of the discrete Euler-Lagrange equations for systems with constraints are the critical sequences of a discrete action sum subjected to some constraint functions.

Consider a continuous mechanical system subject to constraint functions which is determined by a Lagrangian $L : TQ \to \mathbb{R}$ and a constraint submanifold $\mathscr{M}$ of TQ locally defined by the vanishing of m independent constraints $\Phi^\alpha : TQ \to \mathbb{R}$ with $1 \leq \alpha \leq m$. That is, $\text{span}\{D\Phi^\alpha\}$ is of maximum rank at each point.

Let us consider the augmented Lagrangian $\widetilde{L} : TQ \times \mathbb{R}^m \to \mathbb{R}$ given by

$$\widetilde{L}(q,\dot{q},\lambda) := L(q,\dot{q}) + \lambda_\alpha \Phi^\alpha(q,\dot{q}).$$

The Euler-Lagrange equations for $\widetilde{L}$ are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} + \dot{\lambda}_\alpha \left(\frac{\partial \Phi^\alpha}{\partial \dot{q}^i}\right) + \lambda_\alpha \left(\frac{d}{dt}\left(\frac{\partial \Phi^\alpha}{\partial \dot{q}^i}\right) - \frac{\partial \Phi^\alpha}{\partial q^i}\right) = 0, \quad 1 \leq i \leq n,$$
$$\Phi^\alpha(q^i(t),\dot{q}^i(t)) = 0, \quad 1 \leq \alpha \leq m,$$

where λ_α, with $\alpha = 1,\dots,m$, are the Lagrange multipliers.

In order to discretize this system, the velocity space TQ is substituted by the cartesian product $Q \times Q$ and then, the Lagrangian L is replaced by a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$. In the same way, we discretize $\mathscr{M}$ as a discrete constraint submanifold $\mathscr{M}_d \subset Q \times Q$ defined, locally, by the vanishing of m independent constraint functions $\Phi^\alpha_d : Q \times Q \to \mathbb{R}$, $1 \leq \alpha \leq m$.

The discrete constrained variational problem is defined by

$$\begin{cases} \text{ext } \mathscr{A}_d(q_0,q_1,\dots,q_N), & \text{with } q_0 \text{ and } q_N \text{ fixed,} \\ \text{subject to } \Phi^\alpha_d(q_k,q_{k+1}) = 0, & 1 \leq \alpha \leq m \text{ and } 0 \leq k \leq N-1, \end{cases} \tag{4}$$

where $\mathscr{A}_d$ was defined in equation (2).

We consider the augmented discrete Lagrangian $\widetilde{L}_d : Q \times Q \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\widetilde{L}_d(q,\bar{q},\lambda) = L_d(q,\bar{q}) + \lambda_\alpha \Phi^\alpha_d(q,\bar{q}).$$

From the Lagrange multiplier theorem [1], we deduce that the solutions of the constrained problem (4) coincide with the solutions of the discrete variational problem

$$\begin{cases} \text{ext } \tilde{\mathscr{A}_d}\ (q_0, q_1, \ldots, q_N, \lambda^0, \lambda^1, \ldots, \lambda^{N-1}), \text{ with } q_0 \text{ and } q_N \text{ fixed },\\ q_k \in Q, \quad \lambda_k \in \mathbb{R}^m, \quad k = 0, \ldots, N-1, \quad q_N \in Q, \end{cases} \tag{5}$$

where

$$\tilde{\mathscr{A}_d}\ (q_0, q_1, \ldots, q_N, \lambda^0, \lambda^1, \ldots, \lambda^{N-1}) := \sum_{k=0}^{N-1} \widetilde{L}_d(q_k, q_{k+1}, \lambda^k),$$

and λ^k is a n-vector with components λ^k_α with $1 \leq \alpha \leq m$.

Therefore, applying standard discrete variational calculus one can deduce that the solutions of problem (4) verify the following set of difference equations

$$D_1L_d(q_k, q_{k+1}) + D_2L_d(q_{k-1}, q_k) + \lambda^k_\alpha D_1\Phi^\alpha_d(q_k, q_{k+1}) + \lambda^{k-1}_\alpha D_2\Phi^\alpha_d(q_{k-1}, q_k) = 0, \tag{6}$$

for $1 \leq k \leq N-1$, together with the discrete constraints Φ^α_d,

$$\Phi^\alpha_d(q_k, q_{k+1}) = 0\,, \qquad 1 \leq \alpha \leq m \quad \text{and} \quad 0 \leq k \leq N-1.$$

Here $D_i\Phi^\alpha_d$ with $i = 1, 2$ are defined following the definition of D_1L_d and D_2L_d.

For all functions $G \in C^\infty(Q \times Q)$ we denote by $D_{12}G$ the $n \times n$-matrix $\left(\dfrac{\partial^2 G}{\partial q^A \partial \bar{q}^B}\right)$ where $(q^A, \bar{q}^A)$ are local coordinates on the product manifold $Q \times Q$, with $A = 1, \ldots, n$. Then, if the matrix

$$\begin{pmatrix} D_{12}L_d(q, \bar{q}) + \lambda_\alpha D_{12}\Phi^\alpha_d(q, \bar{q}) & \dfrac{\partial \Phi^\alpha_d}{\partial q}(q, \bar{q}) \\ \left(\dfrac{\partial \Phi^\alpha_d}{\partial \bar{q}}(q, \bar{q})\right)^T & \mathbf{0}_{m\times m} \end{pmatrix}_{(n+m)\times(n+m)}$$

is regular, by the implicit function theorem, if we have a point $(q^*_{k-1}, q^*_k, q^*_{k+1}, \lambda^{k-1}_*, \lambda^k_*)$ satisfying eq. (6), there exists a neighborhood $\mathscr{U}_k \subset \mathscr{M}_d \times R^m$ of the point $(q^*_{k-1}, q^*_k, \lambda^{k-1}_*)$, and a unique local application

$$\begin{array}{rrcl} \Upsilon_d : & \mathscr{U}_k & \longrightarrow & \mathscr{M}_d \times \mathbb{R}^m \\ & (q_{k-1}, q_k, \lambda^{k-1}) & \longmapsto & (q_k, q_{k+1}, \lambda^k)\,, \end{array}$$

where $(q_{k-1}, q_k, q_{k+1}, \lambda^{k-1}, \lambda^k)$ satisfies eq. (6). Thus,

$$\Upsilon_d(q_{k-1}, q_k, \lambda^{k-1}) = (q_k, q_{k+1}, \lambda^k)$$

is a discrete second-order flow of equation (6).

2.3. **Discrete second-order mechanics for systems with second-order constraints.** In what follows we denote by $T^{(2)}Q$ the second-order tangent bundle of a manifold Q defined as

$$T^{(2)}Q := \{w \in TTQ : T\tau_Q(w) = \tau_{TQ}(w)\},$$

where $\tau_Q : TQ \to Q$ and $\tau_{TQ} : TTQ \to TQ$ are the canonical bundle projections, locally given by $\tau_Q(q, v) = q$ and $\tau_{TQ}(q, v, \dot{q}, \dot{v}) = (q, v)$. Locally, $T^{(2)}Q$ is given by the points $(q, v, \dot{q}, \dot{v}) \in TTQ$ such that $\dot{q} = v$.

Let us consider a second-order Lagrangian mechanical system with Lagrangian $\bar{L} : T^{(2)}Q \longrightarrow \mathbb{R}$ and a second-order constraint submanifold $\mathscr{M} \subset T^{(2)}Q$, locally defined by the vanishing of m independent second-order constraints functions $\Phi^{\alpha}: T^{(2)}Q \to \mathbb{R}$. The Euler-Lagrange equations for the second-order constrained problem are

$$
\begin{aligned}
0 &= \frac{\partial \bar{L}}{\partial q^i} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}^i}\right) - \dot{\lambda}_\alpha \frac{\partial \Phi^\alpha}{\partial \dot{q}^i} + \lambda_\alpha \frac{d}{dt}\left(\frac{\partial \Phi^\alpha}{\partial \dot{q}^i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \bar{L}}{\partial \ddot{q}^i}\right) \\
&\quad + \ddot{\lambda}_\alpha \frac{\partial \Phi^\alpha}{\partial \ddot{q}^i} + 2\dot{\lambda}_\alpha \frac{d}{dt}\left(\frac{\partial \Phi^\alpha}{\partial \ddot{q}^i}\right) + \lambda_\alpha \frac{d^2}{dt^2}\left(\frac{\partial \Phi^\alpha}{\partial \ddot{q}^i}\right), \\
0 &= \Phi^\alpha(q^i, \dot{q}^i, \ddot{q}^i), \quad 1 \le \alpha \le m, \quad 1 \le i \le n.
\end{aligned}
$$

A natural discrete space substituting the second-order tangent bundle $T^{(2)}Q$ is $Q \times Q \times Q$, and therefore a discretization of this system consists on a discrete Lagrangian function $\bar{L}_d : Q \times Q \times Q \to \mathbb{R}$ and a constraint submanifold $\mathscr{M}_d$ of $Q \times Q \times Q$ which is locally defined by the vanishing of m independent constraint functions $\Phi_d^\alpha : Q \times Q \times Q \to \mathbb{R}$.

Fixing q_0, q_1 and q_{N-1}, q_N for some integer $N > 4$, we consider the discrete sequences on Q, $(q_0, q_1, \dots, q_N) \subset Q^{N+1}$ verifying the discrete constraints

$$\Phi_d^\alpha(q_{k-1}, q_k, q_{k+1}) = 0, \quad \forall k = 1, \dots, N-1.$$

Define the discrete action sum by

$$\widetilde{\mathscr{A}_d}(q_0, \dots, q_N) := \sum_{k=0}^{N-2} \bar{L}_d(q_k, q_{k+1}, q_{k+2}).$$

We are looking for solutions of the following constrained discrete variational problem with constraints

$$
\left\{
\begin{array}{l}
\text{ext } \widetilde{\mathscr{A}_d}(q_0, q_1, \dots, q_N), \quad \text{with } q_0, q_1 \text{ and } q_{N-1}, q_N \text{ fixed} \\
\\
\text{subject to } \Phi^\alpha(q_k, q_{k+1}, q_{k+2}) = 0, \quad \text{with } 1 \le \alpha \le m \text{ and } 0 \le k \le N-2.
\end{array}
\right. \tag{7}
$$

As in the previous section, consider the augmented Lagrangian $\widehat{L}_d : Q \times Q \times Q \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\widehat{L}_d(x, y, z, \lambda) = \bar{L}_d(x, y, z) + \lambda_\alpha \Phi_d^\alpha(x, y, z),$$

where (x, y, z) are local coordinates on $Q \times Q \times Q$. The solutions of the problem (7) coincide with the solutions of the following unconstrained problem

$$
\left\{
\begin{array}{l}
\text{extremize } \overline{\mathscr{A}}_d\ (q_0, q_1, \dots, q_N, \lambda^0, \lambda^1, \dots, \lambda^{N-2}), \ \ q_0, q_1 \text{ and } q_{N-1}, q_N \text{ fixed} \\
q_k \in Q, \quad \lambda^k \in \mathbb{R}^m, \quad k = 0, \dots, N;
\end{array}
\right. \tag{8}
$$

where

$$\overline{\mathscr{A}}_d\ (q_0, q_1, \dots, q_N, \lambda^0, \lambda^1, \dots, \lambda^{N-2}) := \sum_{k=0}^{N-2} \widehat{L}_d(q_k, q_{k+1}, q_{k+2}, \lambda^k),$$

and λ^k is a n-vector with components λ^k_α for $1 \le \alpha \le m$.

Hence, the extremality conditions are

$$
\begin{aligned}
0 &= D_3\widetilde{L}_d(q_{k-2}, q_{k-1}, q_k) + D_2\widetilde{L}_d(q_{k-1}, q_k, q_{k+1}) \\
&\quad + D_1\widetilde{L}_d(q_k, q_{k+1}, q_{k+2}) + \lambda^{k-2}_\alpha D_3\Phi_d^\alpha(q_{k-2}, q_{k-1}, q_k) \\
&\quad + \lambda^{k-1}_\alpha D_2\Phi_d^\alpha(q_{k-1}, q_k, q_{k+1}) + \lambda^k_\alpha D_1\Phi_d^\alpha(q_k, q_{k+1}, q_{k+2}), \quad 2 \le k \le N-2; \\
0 &= \Phi_d^\alpha(q_{k-2}, q_{k-1}, q_k); \\
0 &= \Phi_d^\alpha(q_{k-1}, q_k, q_{k+1}); \\
0 &= \Phi_d^\alpha(q_k, q_{k+1}, q_{k+2}).
\end{aligned}
$$

Here, if $F \in \mathscr{C}^\infty(Q\times Q\times Q), D_iF$ and $D_{ij}F$ with $i,j=1,2,3$ are defined analogously to D_1G and $D_{12}G$ for $G\in C^\infty(Q\times Q)$.

If the point $(q_0,q_1,q_2,q_3,q_4,\lambda^0_\alpha,\lambda^1_\alpha,\lambda^2_\alpha)$ is a solution of the previous equations and the matrix

$$\begin{pmatrix} D_{13}\widetilde{L}_d(x,y,z)+\lambda_\alpha D_{13}\Phi^\alpha_d(x,y,z) & D_3\Phi^\alpha_d(x,y,z) \\ (D_1\Phi^\alpha_d(x,y,z))^T & \mathbf{0}\end{pmatrix} \tag{9}$$

is regular for all (x,y,z) and $\lambda_\alpha\in\mathbb{R}$, $1\leq\alpha\leq m$, there exists a (local) unique application

$$\begin{array}{rccc}\Upsilon_d: & \overline{\mathscr{M}}_d\times\mathbb{R}^{2m} & \longrightarrow & \overline{\mathscr{M}}_d\times\mathbb{R}^{2m}\\ & (q_0,q_1,q_2,q_3,\lambda^0_\alpha,\lambda^1_\alpha) & \longmapsto & (q_1,q_2,q_3,q_4,\lambda^1_\alpha,\lambda^2_\alpha)\end{array}$$

which univocally determines q_4 and λ^2_α, $1\leq\alpha\leq m$, from the initial conditions $(q_0,q_1,q_2,q_3,\lambda^0_\alpha,\lambda^1_\alpha)$. Here, $\overline{\mathscr{M}}_d$ denotes the submanifold of $Q^4:=Q\times Q\times Q\times Q$

$$\overline{\mathscr{M}}_d=\{(q_0,q_1,q_2,q_3)\in Q^4 \mid \Phi^\alpha_d(q_0,q_1,q_2)=0,\ \Phi^\alpha_d(q_1,q_2,q_3)=0,\ 1\leq\alpha\leq m\}.$$

Remark 1. *It is possible to show that, under regularity assumption, this discrete second-order flow preserves a natural symplectic structure defined on $Q^4\times\mathbb{R}^{2m}$.*

Let us consider the discrete 1-forms Θ^+_d,Θ^-_d on $Q^4\times\mathbb{R}^{2m}$ given by

$$\begin{aligned}\Theta^+_d(q_0,q_1,q_2,q_3,\lambda^0,\lambda^1) = -\sum_{i=0}^{1}\Bigg(\sum_{j=1}^{i+1} & D_jL_d(q_{i-j+1},q_{i-j+2},q_{i-j+3})\\ & +\lambda^{i-j+1}_\alpha D_j\Phi^\alpha_d(q_{i-j+1},q_{i-j+2},q_{i-j+3})\Bigg)\,dq_i,\end{aligned}$$

$$\begin{aligned}\Theta^-_d(q_{N-3},q_{N-2},q_{N-1},q_N,\lambda^{N-3},\lambda^{N-2}) = \sum_{i=N-1}^{N}\Bigg(\sum_{j=i-N+3}^{3} & D_jL_d(q_{i-j+1},q_{i-j+2},q_{i-j+3})\\ & +\lambda^{i-j+1}_\alpha D_j\Phi^\alpha_d(q_{i-j+1},q_{i-j+2},q_{i-j+3})\Bigg)\,dq_i.\end{aligned}$$

These forms are called Poincaré-Cartan 1-forms, and they give rise to the discrete Poincaré-Cartan 2-form Ω_d on $Q^4\times\mathbb{R}^{2m}$, given by $\Omega_d=-d\Theta^+_d=-d\Theta^-_d$; it is easy to see that this form is symplectic. Then,

$$\begin{aligned}\Omega_d(q_0,q_1,q_2,q_3,\lambda^0,\lambda^1) &= \left[D_{21}L_d(q_0,q_1,q_2)+\lambda^0_\alpha D_{21}\Phi^\alpha_d(q_0,q_1,q_2)\right]dq_1\wedge dq_0\\ &+\left[D_{31}L_d(q_0,q_1,q_2)+\lambda^0_\alpha D_{31}\Phi^\alpha_d(q_0,q_1,q_2)\right]dq_2\wedge dq_0\\ &+D_1\Phi^\alpha_d(q_0,q_1,q_2)d\lambda^0\wedge dq_0\\ &+\left[D_{11}L_d(q_1,q_2,q_3)+\lambda^1_\alpha D_{11}\Phi^\alpha_d(q_1,q_2,q_3)\right]dq_0\wedge dq_1\\ &+\left[D_{12}L_d(q_0,q_1,q_2)+\lambda^0_\alpha D_{12}\Phi^\alpha_d(q_0,q_1,q_2)\right]dq_0\wedge dq_1\\ &+\left[D_{31}L_d(q_1,q_2,q_3)+\lambda^1_\alpha D_{31}\Phi^\alpha_d(q_1,q_2,q_3)\right]dq_2\wedge dq_1\\ &+\left[D_{32}L_d(q_0,q_1,q_2)+\lambda^0_\alpha D_{32}\Phi^\alpha_d(q_0,q_1,q_2)\right]dq_2\wedge dq_1\\ &+D_1\Phi^\alpha_d(q_1,q_2,q_3)d\lambda^1\wedge dq_1+D_2\Phi^\alpha_d(q_0,q_1,q_2)d\lambda^0\wedge dq_1.\end{aligned}$$

The canonical inclusion $j:\overline{\mathscr{M}}_d\times\mathbb{R}^{2m}\hookrightarrow Q^4\times\mathbb{R}^{2m}$ gives rise to the 2-form $\Omega_{\overline{\mathscr{M}}_d}=j^*\Omega_d$. From the definition of Υ_d it is easy to see that

$$(\Upsilon_d\mid_{\overline{\mathscr{M}}_d\times\mathbb{R}^{2m}})^*\Omega_{\overline{\mathscr{M}}_d}=\Omega_{\overline{\mathscr{M}}_d}.$$

Remark 2. *Let us consider an action of a Lie group G on the manifold Q that naturally induces an action on $Q^4 \times \mathbb{R}^{2m}$ given by*

$$g \cdot (q_0, q_1, q_2, q_3, \lambda^0, \lambda^1) = (g \cdot q_0, g \cdot q_1, g \cdot q_2, g \cdot q_3, \lambda^0, \lambda^1) \text{ for all } g \in G.$$

It is easy to see that this action admits the discrete momentum maps

$$J_d^{\pm} : Q^4 \times \mathbb{R}^{2m} \to \mathfrak{g}^*$$

defined as

$$\begin{aligned} J_d^{\pm}(q_0, q_1, q_2, q_3, \lambda^0, \lambda^1) &: \mathfrak{g} \to \mathbb{R} \\ &\xi \mapsto \langle \Theta_d^{\pm}(q_0, q_1, q_2, q_3, \lambda^0, \lambda^1), \xi_{(Q^4\times\mathbb{R}^{2m})}(q_0, q_1, q_2, q_3, \lambda^0, \lambda^1) \rangle \end{aligned}$$

where $\xi_{(Q^4\times\mathbb{R}^{2m})}$ is the infinitesimal generator corresponding to an element of the Lie algebra $\xi \in \mathfrak{g}$; that is, $\xi_{Q^4}(q_0, q_1, q_2, q_3, \lambda^0, \lambda^1) = (\xi_Q(q_0), \xi_Q(q_1), \xi_Q(q_2), \xi_Q(q_3), 0, 0)$.

If G is a symmetry of the discrete mechanical system with constraints Φ_d^{α}, that is, L_d and Φ_d^{α} are G-invariant by the action $g \cdot (x, y, z, \lambda) = (g \cdot x, g \cdot y, g \cdot z, \lambda)$, it is easy to see that $J_d^+ = J_d^-$ and $J_d := J_d^+ = J_d^-$ satisfies

$$\left\langle J_d\left(\Upsilon_d(q_0, q_1, q_2, q_3, \lambda^0, \lambda^1)\right), \xi \right\rangle = \left\langle J_d(q_0, q_1, q_2, q_3, \lambda^0, \lambda^1), \xi \right\rangle.$$

Thus, the flow Υ_d preserves the momentum map.

3. Optimal control of nonholonomic mechanical systems

In this section we consider a particular class of mechanical control systems, nonholonomic mechanical control systems [3, 19]. We start with the configuration space for this class of mechanical control systems which is an n-dimensional differentiable manifold with local coordinates $(q^i), 1 \leq i \leq n = \dim(Q)$. Linear constraints in the velocities are given by equations of the form

$$\varphi^a(q^i, \dot{q}^i) = \mu_i^a(q)\dot{q}^i = 0, \quad 1 \leq a \leq m,$$

depending, in general, on configuration coordinates and their velocities. From an intrinsic point of view, the linear constraints are defined by a regular distribution $\mathscr{D}$ on Q of rank $n - m$ such that the annihilator of $\mathscr{D}$ is locally given at each point $q \in Q$ by

$$\mathscr{D}_q^o = \text{span}\{\mu^a(q) = \mu_i^a dq^i; 1 \leq a \leq m\},$$

where the differential 1-forms $\mu^a(q)$ are independent at each $q \in Q$.

Now we consider a Riemannian metric $\mathscr{G}$ specifying the kinetic energy of the mechanical system. The metric is locally determined by the matrix $M = (g_{ij})_{1\leq i,j\leq n}$, where

$$g_{ij} = \mathscr{G}\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right).$$

Denote also by $\flat_{\mathscr{G}} : TQ \to T^*Q$ the corresponding induced vector bundle isomorphism and by $\#_{\mathscr{G}} : T^*Q \to TQ$ the inverse isomorphism. We can construct the Levi-Civita connection $\nabla^{\mathscr{G}}$ on Q as the unique affine connection which is torsion-less and metric with respect to $\mathscr{G}$. It is determined by the standard formula

$$\begin{aligned} 2\mathscr{G}(\nabla_X^{\mathscr{G}} Y, Z) &= X(\mathscr{G}(Y,Z)) + Y(\mathscr{G}(X,Z)) - Z(\mathscr{G}(X,Y)) \\ &\quad + \mathscr{G}(X, [Z,Y]) + \mathscr{G}(Y, [Z,X]) - \mathscr{G}(Z, [Y,X]) \end{aligned}$$

for all vector fields $X, Y, Z \in \mathfrak{X}(Q)$; here $\mathfrak{X}(Q)$ denotes the Lie algebra of vector fields on Q.

Fixing a potential function $V : Q \to \mathbb{R}$, the mechanical system is defined by the mechanical Lagrangian $L : TQ \longrightarrow \mathbb{R}$,

$$L(v_q) = \frac{1}{2}\mathscr{G}(v_q, v_q) - V(q), \qquad \text{where } v_q \in T_qQ. \tag{10}$$

Using some basic tools of Riemannian geometry, we write the equations of motion of the unconstrained system as

$$\nabla^{\mathscr{G}}_{\dot{c}(t)}\dot{c}(t) + \text{grad}_{\mathscr{G}}V(c(t)) = 0; \tag{11}$$

where $t \mapsto (q^1(t), \ldots, q^n(t))$ is the local representation of c and, $\text{grad}_{\mathscr{G}}V$ is the vector field on Q characterized by

$$\mathscr{G}(\text{grad}_{\mathscr{G}}V, X) = X(V), \quad \text{for every } X \in \mathfrak{X}(Q).$$

In local coordinates, $\text{grad}_{\mathscr{G}}V(c(t)) = g^{ij}\dfrac{\partial V}{\partial q^j}$, where (g^{ij}) is the inverse matrix of (g_{ij}).

The Lagrange-d'Alembert principle states that the equations of motion for a controlled nonholonomic control system determined by $(L, \mathscr{D})$ are given by

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(q(t), \dot{q}(t))\right) - \frac{\partial L}{\partial q^i}(q(t), \dot{q}(t)) &= \lambda_\alpha(t)\mu_i^\alpha(q(t)) + u^a(t)\theta_a(q(t)), \\ \mu_i^\alpha(q(t))\dot{q}^i(t) &= 0,\end{aligned} \tag{12}$$

where θ_a are independent differential 1-forms at each point of Q, with $m+1 \leq a \leq n$; $(u_{m+1}, \ldots, u_n) \in U \subset \mathbb{R}^{n-m}$ are the control inputs and $\lambda_\alpha, 1 \leq \alpha \leq m$ are the Lagrange multipliers. Using the Levi-Civita connection the intrinsic equations of motion for the controlled nonholonomic control problem are given by

$$\begin{aligned}&\nabla^{\mathscr{G}}_{\dot{c}(t)}\dot{c}(t) + \text{grad}_{\mathscr{G}}V(c(t)) - u^a(t)Y_a(c(t)) \in \mathscr{D}^{\perp}_{\dot{c}(t)}, \\ &\dot{c}(t) \in \mathscr{D}_{c(t)}.\end{aligned} \tag{13}$$

Here, we denote by $Y_a = \sharp_{\mathscr{G}}\theta_a$, $\mathscr{D} = \text{span}\{Y_a\}$, $m+1 \leq a \leq n$, and $\mathscr{D}^{\perp}$ denotes the orthogonal complement of $\mathscr{D}$ respect to the tangent bundle decomposition $TQ = \mathscr{D} \oplus \mathscr{D}^{\perp}$.

An alternative way of writing equations (13) is the following one:

$$\begin{aligned}&\mathscr{C}^{ab}\mathscr{G}_{c(t)}\left(\nabla^{\mathscr{G}}_{\dot{c}(t)}\dot{c}(t) + \text{grad}_{\mathscr{G}}V(c(t)), Y_b(c(t))\right) = u^a, \quad m+1 \leq a \leq n; \\ &\dot{c}(t) \in \mathscr{D}_{c(t)},\end{aligned} \tag{14}$$

where $\mathscr{C}_{ab} = \mathscr{G}(Y_a, Y_b)$ and $\mathscr{C}^{ab}$ is the inverse matrix of $\mathscr{C}_{ab}$.

In local coordinates, defining the n^3 functions Γ^k_{ij} (Christoffel symbols for ∇) by

$$\nabla_{\frac{\partial}{\partial q^i}}\frac{\partial}{\partial q^j} = \Gamma^k_{ij}\frac{\partial}{\partial q^k},$$

we may rewrite the nonholonomic control equations as

$$\begin{aligned}Y_b^{k'}(c(t))g_{kk'}(c(t))\left(\ddot{q}^k(t) + \Gamma^k_{ij}(c(t))\dot{q}^i(t)\dot{q}^j(t) + g^{ki}(c(t))\frac{\partial V}{\partial q^i}(c(t))\right) &= \mathscr{C}_{ab}u^a(t), \\ \mu_i^a(c(t))\dot{q}^i(t) &= 0.\end{aligned}$$

Given a cost function $C : \mathscr{D} \times U \to \mathbb{R}$, a solution of the optimal control problem consists in finding a trajectory $(c(t), u(t))$ satisfying equations (14) given initial and final boundary conditions $(c(t_0), \dot{c}(t_0))$ and $(c(t_f), \dot{c}(t_f))$, respectively, extremizing the cost functional

$$\mathscr{A} = \int_{t_0}^{t_f} C(c(t), \dot{c}(t), u(t))\, dt,$$

where $\dot{c}(t) \in \mathscr{D}_{c(t)}$.

It is well know that this optimal control problem is equivalent to the following second-order problem: Extremize the functional

$$\widetilde{\mathscr{A}} = \int_{t_0}^{t_f} \widetilde{L}(c(t), \dot{c}(t), \ddot{c}(t))\, dt$$

subject to $\dot{c}(t) \in \mathscr{D}_{c(t)}$, where

$$\widetilde{L}(c(t), \dot{c}(t), \ddot{c}(t)) = C\left(c(t), \dot{c}(t), \mathscr{C}_{ab}\mathscr{G}\left(\nabla^{\mathscr{G}}_{\dot{c}(t)}\dot{c}(t) + \mathrm{grad}_{\mathscr{G}} V(c(t)), Y^b\right)\right).$$

3.1. **Variational integrators for nonholonomic mechanical control systems.** In this subsection, we show that the discrete variational approach with constraints of a second-order problem is an appropriate framework for the treatment of the discrete versions of optimal control problems for nonholonomic mechanical system considered before. The main application is the construction of geometric numerical integrators for this type of control systems.

A possible discretization of the nonholonomic Euler-Lagrange equations with controls (14) is

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + h(u^a)_k \flat_{\mathscr{G}}(Y_a)\mid_{q^k} = 0\,, \quad 1 \leq k \leq N-1\,; \tag{15}$$

where h is the fixed time step and L_d is a discretization of the Lagrangian L. Using the same ideas as before, we can rewrite equation (15) as

$$-\mathscr{C}^{ab}(q_k)\langle D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}), Y_b\mid_{q_k}\rangle = h(u^a)_k, \quad m+1 \leq a \leq n \tag{16}$$

$$\mu_i^a(q_k)\left(\frac{q_{k+1} - q_k}{h}\right) = 0\,, \tag{17}$$

where now we are assuming that Q is a vector space.

Remark 3. *In general it is not necessary to assume that Q is a vector space. Assuming that Q is equipped with a Riemannian metric $\widetilde{\mathscr{G}}$, (not necessarily equal to the prescribed Riemannian metric $\mathscr{G}$) and denote by $\exp_{q_0} : \mathscr{U} \subset T_{q_0}Q \to Q$ the exponential mapping defined on a neighborhood $\mathscr{U}$ of T_qQ such that $\mathscr{U} \mapsto \exp_{q_0}(\mathscr{U})$ is a diffeomorphism.*

If $\tau = h(\exp_{q_0})^{-1}$, then $\tau(q_0, q_1) = h(\exp_{q_0})^{-1}(q_1) \in T_{q_0}Q$ and therefore $\tau(q_0, q_1) = hv_{q_0} \in T_{q_0}Q$ gives rise to a discretization mapping.

Now we can discretize the constraint equations (17) *as*

$$\mu_i^a(q_k)\frac{\tau(q_0, q_1)}{h} = 0.$$

If $Q = \mathbb{R}^n$ we take the Euclidean metric and the discretization would be

$$\frac{\tau(q_0, q_1)}{h} = \frac{q_1 - q_0}{h}.$$

For simplicity we use this discretization but it is straightforward to use another mapping τ.

The optimal control problem is determined prescribing the discrete cost functional

$$\mathscr{A}_d = \sum_{k=1}^{N-1} hC(q_k, q_{k+1}, (u^a)_k)$$

with initial and final boundary points q_0, q_1 and q_{N-1}, q_N, respectively.

Since the control variables appear explicitly in (15), the previous discrete nonholonomic optimal control problem is equivalent to the second-order discrete variational problem with constraints determined by

$$\bar{L}_d(q_{k-1},q_k,q_{k+1}) = C\left(q_k,q_{k+1},-\frac{1}{h}\mathscr{C}^{ab}(q_k)\langle D_2L_d(q_{k-1},q_k)+D_1L_d(q_k,q_{k+1}),Y_a\,|_{q_k}\rangle\right),$$

$$\mu^a(q_k)\left(\frac{q_{k+1}-q_k}{h}\right)=0.$$

As in the previous section, we define $\widehat{L}_d : Q\times Q\times Q\times\mathbb{R}^m\to\mathbb{R}$ as

$$\widehat{L}_d(q_k,q_{k+1},q_{k+2},\lambda_{\alpha,k}) = \bar{L}_d(q_k,q_{k+1},q_{k+2})+\lambda_{a,k}\mu^a(q_{k+1})\left(\frac{q_{k+2}-q_{k+1}}{h}\right),$$

and from the results deduced in the previous section we can derive the discrete equations of motion which give rise to a geometric integrator for the proposed optimal control problem. Since this integrator has been obtained from a variational point of view, it automatically preserves the symplectic structure and momentum map (if it is the case) for optimal control problems of nonholonomic mechanical systems (see [7, 8] and references therein). The discrete equations of motion are

$$\begin{aligned}
0 &= \mu^a(q_{k-1})\left(\frac{q_k-q_{k-1}}{h}\right),\\
0 &= \mu^a(q_{k+1})\left(\frac{q_{k+2}-q_{k+1}}{h}\right),\\
0 &= \mu^a(q_k)\left(\frac{q_{k+1}-q_k}{h}\right),\\
0 &= D_3\bar{L}_d(q_{k-2},q_{k-1},q_k)+D_2\bar{L}_d(q_{k-1},q_k,q_{k+1})+D_1\bar{L}_d(q_k,q_{k+1},q_{k+2})\\
&\quad+\frac{\lambda_a^{k-1}}{h}\left(-\mu_i^a(q_k)+\frac{\partial\mu_i^a}{\partial q_k}\Big|_{q_k}(q^i_{k+1}-q^i_k)\right)+\frac{\lambda_a^{k-2}}{h}\mu_i^a(q_{k-1}).
\end{aligned}$$

3.2. **An illustrative example: The nonholonomic particle.** The following typical example will illustrate some of the constructions of the previous sections. It corresponds to a discretization of the nonholonomic particle in $\mathbb{R}^3$ with the Euclidean metric described by

$$L(x,y,z,\dot{x},\dot{y},\dot{z})=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

and the nonholonomic constraint $\Phi=\dot{z}-y\dot{x}=0$, which is related to the following distribution:

$$\mathscr{D}=\operatorname{span}\left\{\frac{\partial}{\partial x}+y\frac{\partial}{\partial z},\frac{\partial}{\partial y}\right\}.$$

By introducing controls we obtain the following controlled nonholonomic problem

$$\begin{aligned}
&\ddot{x}+y\ddot{z}=u_1,\\
&\ddot{y}=u_2,\\
&\dot{z}-y\dot{x}=0.
\end{aligned}$$

If we consider the cost function $\mathscr{C}=\dfrac{u_1^2+u_2^2}{2}$, we obtain the following second-order variational problem with second-order constraints: Extremize

$$\widetilde{\mathscr{A}}=\int_{t_0}^{t_f}\bar{L}\left(q^A(t),\dot{q}^A(t),\ddot{q}^A(t)\right)dt,$$

subject to the second-order constraints given by

$$\Phi(q^A, \dot{q}^A, \ddot{q}^A) = \dot{z} - y\dot{x} = 0.$$

Here $\bar{L} : T^{(2)}Q \to \mathbb{R}$ is defined by

$$\bar{L}(q^A, \dot{q}^A, \ddot{q}^A) = \frac{(\ddot{x} + y\ddot{z})^2}{2} + \frac{\ddot{y}^2}{2}.$$

The equations of motion are given by the following system of fourth order ordinary differential equations

$$\begin{aligned}
0 &= x^{(iv)} + z^{(iv)} + 2\dddot{z}\dot{y} + \ddot{z}\ddot{y} - \dot{\lambda}y - \lambda\dot{y},\\
0 &= y^{(iv)} + \ddot{z}(\ddot{x} + \ddot{z}y) + \lambda\dot{x},\\
0 &= \dot{\lambda} + \ddot{y}(\ddot{x} + \ddot{z}y) + 2\dot{y}(\dddot{x} + \dddot{z}y + \ddot{z}\dot{y}) + y(x^{(iv)} + z^{(iv)} + 2\dddot{z}\dot{y} + \ddot{z}\ddot{y}),\\
\dot{z} &= y\dot{x}.
\end{aligned}$$

Now, we discretize the previous constrained problem. Denoting by

$$\Delta^2[q_k] = \left(\frac{q_{k+2} - 2q_{k+1} + q_k}{h^2}\right),$$

we obtain a constrained second-order discrete variational problem determined by

$$\begin{aligned}
\bar{L}_d(q_k, q_{k+1}, q_{k+2}) &= \left(\Delta^2[x_k] + y_{k+1}\Delta^2[z_k]\right)^2 + \Delta^2[y_k],\\
\Phi_d(q_k, q_{k+1}, q_{k+2}) &= \frac{z_{k+2} - z_k}{2h} - y_{k+1}\frac{x_{k+2} - x_k}{2h} = 0.
\end{aligned}$$

Fixing initial and final conditions, the discrete algorithm is given by the solutions of

$$\begin{aligned}
y_{k+1}\lambda^{k+1} - \lambda^{k-1}y_{k-1} &= -\frac{4}{h}\left(\Delta^2[x_k] + y_{k+1}\Delta^2[z_k] - 2\Delta^2[x_{k-1}] - 2y_k\Delta^2[z_{k-1}]\right.\\
&\qquad \left. + \Delta^2[x_{k-2}] + y_{k-1}\Delta^2[z_{k-2}]\right),\\
\lambda^k(x_{k+1} - x_{k-1}) &= 4h\left(\Delta^2[x_{k-1}] + y_k\Delta^2[z_{k-1}]\right)\Delta^2[z_{k-1}],\\
\lambda^{k+1} &= \lambda^{k-1} - \frac{4}{h}\left(\Delta^2[x_k] + y_{k+1}\Delta^2[z_k] - 2\Delta^2[x_{k-1}] - 2y_k\Delta^2[z_{k-1}]\right.\\
&\qquad \left. + \Delta^2[x_{k-2}] + y_{k-1}\Delta^2[z_{k-2}]\right),\\
z_{k+1} &= z_{k-1} + y_k(x_{k+1} - x_{k-1}).
\end{aligned}$$

Finally, given a second order Lagrangian L the energy function associated with this system is given by (see [14])

$$E(q, \dot{q}, \ddot{q}) = \left(\frac{\partial \bar{L}}{\partial \dot{q}^A} - \frac{d}{dt}\frac{\partial \bar{L}}{\partial \ddot{q}^A}\right)\dot{q}^A + \frac{\partial \bar{L}}{\partial \ddot{q}^A} - \bar{L}(q, \dot{q}, \ddot{q}).$$

For the nonholonomic particle the second-order energy associated with the augmented Lagrangian is given by

$$E(q, \dot{q}, \ddot{q}, \lambda) = (-\lambda y - \dddot{x} - \dot{y}\ddot{z} - y\dddot{z})\dot{x} - \dddot{y}\dot{y} + (\lambda - \dot{y}(\ddot{x} + y\ddot{z}) + y(\dddot{x} + \dot{y}\ddot{z} + y\dddot{z}))\dot{z}.$$

A possible discretization of the higher-order energy is given by

$$
\begin{aligned}
E_d = &\left[-\lambda_0\left(\frac{y_0+y_1}{2}\right)-\left(\frac{x_3-3x_2+3x_1-x_0}{h^3}\right)-\left(\frac{y_1+y_0}{2}\right)\left(\frac{z_3-3z_2+3z_1-z_0}{h^3}\right)\right.\\
&\left.-\left(\frac{y_1-y_0}{h}\right)\left(\frac{z_0-2z_1+z_2}{h^2}\right)\right]\left(\frac{x_1-x_0}{h}\right)-\left(\frac{y_1-y_0}{h}\right)\left(\frac{y_3-3y_2+3y_1-y_0}{h^3}\right)\\
&+\left[\lambda_0-\left(\frac{y_1-y_0}{h}\right)\left(\frac{x_0-2x_1+x_2}{h^2}+\frac{y_1+y_0}{2}\left(\frac{z_2-2z_1+z_0}{h^2}\right)\right)\right.\\
&+\left(\frac{y_1+y_0}{2}\right)\left(\frac{x_3-3x_2+3x_1-3x_0}{h^3}+\left(\frac{y_1-y_0}{h}\right)\left(\frac{z_2-2z_1+z_0}{h^2}\right)\right)\\
&\left.+\left(\frac{y_1+y_0}{2}\right)\left(\frac{z_3-3z_2+3z_1-z_0}{h^3}\right)\right]\left(\frac{z_1-z_0}{h}\right).
\end{aligned}
$$

It is well know, since the integrator is variational, that this discrete energy function presents a good behavior under the solutions of the discrete second-order Euler-Lagrange equations (see [4] and [6] for example).

Conclusions and future work

We have defined, from a discretization of Hamilton's principle, variational integrators for second-order mechanical systems with second-order constraints. As an interesting application, we have used our techniques to construct a variational integrator to solve an optimal control problem for nonholonomic mechanical systems. Finally, we show how to apply the theory in a particular example: the optimal control of a nonholonomic particle. In unpublished work we establish a geometric and intrinsic description for optimal control problems for nonholonomic mechanical systems, and the geometric derivation of new numerical schemes. Also we want to study the numerical behavior of the proposed numerical methods in different simulations and the extension of backward error analysis techniques for these methods.

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