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THE ONE-SIDED DYADIC HARDY-LITTLEWOOD MAXIMAL FUNCTION

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ABSTRACT. The problem of characterizing the good weights for the one-sided Hardy-Littlewood maximal operator $M^{++\cdots+}$ in $\mathbb{R}^n$ is open. In order to tackle this problem one possible strategy is the following: (1) define a one-sided dyadic maximal operator $M_d^{++\cdots+}$ with similar properties to the classical dyadic maximal operator M_d, (2) study the weighted inequalities for this one-sided dyadic maximal operator, and (3) control $M^{++\cdots+}$ (in some sense) by $M_d^{++\cdots+}$. This paper is a review of what we know about this issue.

This paper is essentially the talk given by the second author in the Conference "XII Congreso Dr. Antonio Monteiro", Bahía Blanca, May 22-24, 2013, and it is based on the paper [5] and joint work by both authors.

1. THE HARDY-LITTLEWOOD MAXIMAL OPERATOR AND THE DYADIC MAXIMAL OPERATOR: WEIGHTED INEQUALITIES

For a locally integrable function f on $\mathbb{R}^n$, the Hardy-Littlewood maximal function is defined as

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|\,dy,$$

where the supremum is taken over all the cubes Q (with sides parallel to the axes) such that $x \in Q$ and $|Q|$ stands for the measure of Q. It is well known that M controls in some sense the behavior of other operators such as the Hilbert transform

$$Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy$$

and more general singular integrals. Therefore, it is interesting to know the boundedness properties of M. The basic properties are the following: M maps $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, $1 < p \le +\infty$, and $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$ (weak-$L^1(\mathbb{R}^n)$). If one considers weighted Lebesgue spaces it is natural to study the boundedness of the maximal operator M on these spaces. Let us define the spaces we are talking about. A weight v is a nonnegative measurable function defined on $\mathbb{R}^n$ and the corresponding weighted-L^p space is

$$L^p(v) = \left\{ f : \mathbb{R}^n \to \mathbb{R} : ||f||_{L^p(v)} = \left(\int_{\mathbb{R}^n} |f|^p v \right)^{1/p} < \infty \right\}.$$

Muckenhoupt and Sawyer characterized the good weights for M.

Theorem 1.1 ([8, 10]). *Let $1 < p < \infty$ and let u, v be weights on $\mathbb{R}^n$. The following statements are equivalent.*

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(a) *There exists $C>0$ such that for all $f\in L^p(v)$,*

$$\int_{\mathbb{R}^n}(Mf(x))^p u(x)\,dx \le C\int_{\mathbb{R}^n}|f(x)|^p v(x)\,dx.$$

(b) *(Sawyer S_p Condition) There exists $C>0$ such that*

$$\int_Q (M(\sigma\chi_Q)(x))^p u(x)\,dx \le C\int_Q \sigma(x)\,dx < \infty,$$

for all cubes Q, where $\sigma = v^{1-p'}$ and $p+p'=pp'$.

Furthermore, if $u=v$ then the above statements are equivalent to

(c) *(Muckenhoupt A_p condition) There exists $C>0$ such that for all cubes Q*

$$\left(\frac{1}{|Q|}\int_Q u\right)^{1/p}\left(\frac{1}{|Q|}\int_Q u^{1-p'}\right)^{1/p'} \le C.$$

Let us comment briefly the proof of Sawyer's result, that is, the equivalence between (a) and (b). The implication (a)⇒(b) is straightforward. Therefore, all we have to review is the proof of (b)⇒(a). The main ingredients are the following.

(1) Sawyer solves the same problem for the dyadic maximal operator which is defined as

$$M_d f(x) = \sup_{x\in Q,\, Q\,\text{dyadic}} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the supremum is taken over all the dyadic cubes Q with $x\in Q$. He proves that M_d applies $L^p(v)$ into $L^p(u)$ if and only if the pair (u,v) satisfies $S_{p,d}$ condition, that is, there exists $C>0$ such that

$$\int_Q (M_d(\sigma\chi_Q)(x))^p u(x)\,dx \le C\int_Q \sigma(x)\,dx < \infty$$

for all dyadic cubes Q.

(2) The Hardy-Littlewood maximal operator is controlled in a certain sense by the dyadic maximal operator. This is a result by Fefferman and Stein [3] and it says that

$$M_k f(x) \le \frac{C_n}{|Q(0,2^{k+2})|}\int_{Q(0,2^{k+2})} (\tau_{-t}\circ M_d\circ \tau_t) f(x)\,dt,$$

where $\tau_t g(x) = g(x-t)$,

$$Q(0,2^{k+2}) = \{x=(x_1,\dots,x_n)\in\mathbb{R}^n : |x|_\infty = \max_i |x_i| \le 2^{k+1}\},$$

and M_k is the truncated maximal operator

$$M_k f(x) = \sup_{x\in Q,\ell(Q)\le 2^k} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes Q with side length smaller than or equal to 2^k and $x\in Q$.

(3) The pair of weights (u,v) satisfies S_p condition if and only if for all $t\in\mathbb{R}^n$ the pair of weights $(\tau_t\circ u, \tau_t\circ v)$ satisfies $S_{p,d}$ condition with a constant independent of t.

2. THE ONE-SIDED SETTING

2.1. **The one-sided Hardy-Littlewood maximal function in dimension 1.** For a locally integrable function f on $\mathbb{R}$, the one-sided Hardy-Littlewood maximal functions are defined as

$$M^+f(x)=\sup_{h>0}\frac{1}{h}\int_x^{x+h}|f(y)|\,dy \qquad \text{and} \qquad M^-f(x)=\sup_{h>0}\frac{1}{h}\int_{x-h}^{x}|f(y)|dy.$$

These maximal functions control singular integrals whose kernels have support in $(-\infty,0)$ or in $(0,\infty)$ (see [1]). Sawyer [11] characterized the good weights for these operators (see also [6, 2]).

Theorem 2.1 ([11])**.** *Let $1<p<\infty$ and let u,v be weights on $\mathbb{R}^n$. The following are equivalent:*

(a) *There exists $C>0$ such that for all $f\in L^p(v)$*

$$\int_{\mathbb{R}}(M^+f(x))^p u(x)\,dx\le C\int_{\mathbb{R}}|f(x)|^p v(x)\,dx.$$

(b) *(S_p^+ Condition) There exists $C>0$ such that*

$$\int_I (M^+(\sigma\chi_I)(x))^p u(x)\,dx\le C\int_I \sigma(x)\,dx<\infty$$

for all intervals $I=(a,b)$ such that $\int_{-\infty}^a u(x)\,dx>0$, where $\sigma=v^{1-p'}$ and $p+p'=pp'$.

Furthermore, if $u=v$ then the above statements are equivalent to

(c) *(A_p^+ condition) There exists $C>0$ such that for all $b\in\mathbb{R}$ and all $h>0$*

$$\frac{1}{h}\left(\int_{b-h}^{b}u\right)^{1/p}\left(\int_b^{b+h}u^{1-p'}\right)^{1/p'}\le C.$$

2.2. **The one-sided Hardy-Littlewood maximal function in dimension $n>1$.** A natural generalization of M^+ in $\mathbb{R}^n$ is the following: for $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, we define

$$M^{++\cdots+}f(x_1,x_2,\dots,x_n)=\sup_{h>0}\frac{1}{h^n}\int_{Q_x(h)}|f(y)|\,dy,$$

where $Q_x(h)=[x_1,x_1+h)\times[x_2,x_2+h)\times\cdots\times[x_n,x_n+h)$. Let us notice that in $\mathbb{R}^n$ there are 2^n one-sided maximal operators of this kind. The weighted inequalities for $M^{++\cdots+}$ in $\mathbb{R}^n$ have not been characterized. The only positive result is the characterization of the good weights for the weak type (p,p) inequality of M^{++} in dimension $n=2$ (see [4]).

Theorem 2.2 ([4])**.** *Let $1<p<\infty$ and let u,v be weights on $\mathbb{R}^2$. The following are equivalent:*

(a) *There exists $C>0$ such that for all $\lambda>0$ and all $f\in L^p(v)$*

$$\int_{\{x\in\mathbb{R}^2:M^{++}f(x)>\lambda\}}u(x)dx\le\frac{C}{\lambda^p}\int_{\mathbb{R}^2}|f(x)|^p v(x)dx.$$

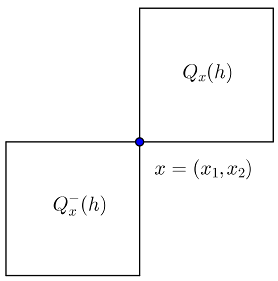
(b) *(A_p^{++} condition in $\mathbb{R}^2$)*

$$\sup_{x\in\mathbb{R}^2}\sup_{h>0}\frac{1}{h^2}\left(\int_{Q_x^-(h)}u\right)^{1/p}\left(\int_{Q_x(h)}v^{1-p'}\right)^{1/p'}<\infty,$$

where $Q_x^-(h)=[x_1-h,x_1)\times[x_2-h,x_2)$ (see the next figure).

For $p = 1$ the result holds with A_1^{++} meaning that there exists $C > 0$ such that for all $h > 0$

$$\frac{1}{h^2} \int_{Q_x^-(h)} u \leq Cv(x) \quad \text{a.e. } x = (x_1, x_2).$$



As we have noted, the problem of characterizing the good weights for $M^{++\cdots+}$ in $\mathbb{R}^n$ is open. In order to tackle this problem, keeping in mind the proofs of the two-sided Hardy-Littlewood maximal operator in $\mathbb{R}^n$, we consider the following possible strategy:

- Step 1: Define a one-sided dyadic maximal operator $M_d^{++\cdots+}$ with properties similar to the classical dyadic operator M_d.
- Step 2: Study the weighted inequalities for this one-sided dyadic maximal operator.
- Step 3: Control $M^{++\cdots+}$ by averages of $\tau_{-t} \circ M_d^{++\cdots+} \circ \tau_t$.

The rest of the paper is devoted to present the progress we have made.

3. ONE-SIDED DYADIC MAXIMAL OPERATORS

We note that a one-sided dyadic maximal operator was previously studied in dimension 1. Consider in the real line the one-sided operator

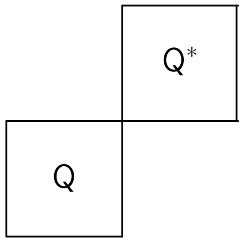
$$N^+ f(x) = \sup_{x \in I,\ I \text{ dyadic}} \frac{1}{|I^*|} \int_{I^*} |f(y)| dy,$$

where the supremum is taken over all dyadic intervals I such that $x \in I$ and if $I = [a, a+h)$ then $I^* = [a+h, a+2h)$. Note that $I \cup I^*$ is not necessarily dyadic. This operator was studied in [7]; it was proved there that N^+ is pointwise equivalent to M^+. This fact was useful to study one-sided fractional maximal functions and, consequently, to study one-sided fractional integrals (Riemann-Liouville and Weyl fractional integrals).

Ombrosi [9] generalized this operator to $\mathbb{R}^n$, $n > 1$, defining

$$N^{+\cdots+} f(x) = \sup_{x \in Q,\ Q \text{ dyadic}} \frac{1}{|Q^*|} \int_{Q^*} |f(y)| dy,$$

where Q is dyadic and $Q^* = Q + \overrightarrow{h}$, $\overrightarrow{h} = (l, \ldots, l)$, and l is the side length of Q (see the next figure).



Weighted weak type inequalities were studied in [9]. We point out that this dyadic operator does not satisfy the inequality $N^{+\cdots+}f \le CM_d f$ for some constant independent of f, being M_d the classical dyadic maximal operator defined in Section 1. We think that a good one-sided dyadic maximal operator $M_d^{+\cdots+}$ should satisfy

$$M_d^{+\cdots+}f \le CM_d f \qquad \text{and} \qquad M_d^{+\cdots+}f \le CM^{+\cdots+}f.$$

Furthermore it should control in some sense the (non-dyadic) one-sided maximal operator $M^{+\cdots+}$. In $\mathbb{R}$ (dimension 1) we have succeeded in defining one-sided dyadic maximal operators with these properties. They are defined as follows:

$$M_d^+ f(x) = \sup_{I \text{ dyadic},\ x\in I^-,} \frac{1}{|I^+|}\int_{I^+} |f(y)|dy$$

and

$$M_d^- f(x) = \sup_{I \text{ dyadic},\ x\in I^+,} \frac{1}{|I^-|}\int_{I^-} |f(y)|dy,$$

where I^- and I^+ are the two halves of I (note that now $I = I^- \cup I^+$ is dyadic). It is clear that

$$M_d^+ f, M_d^- f \le 2M_d f, \qquad M_d f \le M_d^+ f + M_d^- f, \qquad M_d^+ f, M_d^- f \le 2M^+ f.$$

Furthermore, if for all $k \in \mathbb{Z}$ we consider the truncated operators

$$N_k^+ f(x) = \sup_{0<h\le 2^k} \frac{1}{h}\int_{x+h}^{x+2h} |f(y)|dy, \qquad M_k^+ f(x) = \sup_{0<h\le 2^k} \frac{1}{h}\int_{x}^{x+h} |f(y)|dy,$$

then we have (see [5]) that

$$M_k^+ f(x) \le CN_k^+ f(x) \le \frac{C}{2^{k+4}}\int_0^{2^{k+4}} (\tau_{-t}\circ M_d^+ \circ \tau_t) f(x)\, dt,$$

where $\tau_t f(x) = f(x-t)$. All the above inequalities mean that M_d^+ seems to be the good candidate for the dyadic one-sided maximal operator. In the same paper [5] we have characterized the good weights for M_d^+.

Theorem 3.1 ([5]). *Let $1 < p < \infty$, $u, v \ge 0$, $\sigma = v^{1-p'}$. The following statements are equivalent.*

(a) *There exists $C > 0$ such that for all $f \in L^p(v)$*

$$\int_{\mathbb{R}} (M_d^+ f(x))^p u(x)\, dx \le C\int_{\mathbb{R}} |f(x)|^p v(x)\, dx.$$

(b) *($S_{p,d}^+$ condition) There exists $C > 0$ such that*

$$\int_I (M_d^+(\sigma\chi_{I^+})(x))^p u(x)\, dx \le C\int_{I^+} \sigma(x)\, dx < \infty,$$

for all dyadic intervals I such that $\int_{I^-} u > 0$.

Furthermore, if $u = v$ then the above statements are equivalent to

(c) *($A^+_{p,d}$ condition) There exists $C>0$ such that*

$$\left(\int_{I^-} u\right)^{1/p}\left(\int_{I^+} u^{1-p'}\right)^{1/p'} \leq C|I|$$

for all dyadic intervals I.

We have also proved in [5] that if we consider different weights u and v then the $A^+_{p,d}$ condition

$$\left(\int_{I^-} u\right)^{1/p}\left(\int_{I^+} v^{1-p'}\right)^{1/p'} \leq C|I|$$

characterizes the weak type (p,p) inequality. In the case $p=1$ the condition is replaced by the natural one.

As a consequence of these results, we obtain a new proof of the characterizations of the good weights for M^+ in $\mathbb{R}$.

In the next section we search for some generalization of M^+_d to greater dimensions.

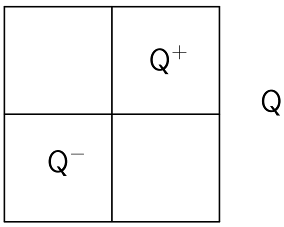
4. THE DYADIC ONE-SIDED HARDY-LITTLEWOOD MAXIMAL FUNCTION IN DIMENSION GREATER THAN 1

We shall work in $\mathbb{R}^2$ but everything we are going to say is valid in $\mathbb{R}^n$ for all $n\geq 1$.

The first candidate for the one-sided dyadic maximal operator is the following one:

$$M^{++}_d f(x) = \sup_{Q \text{ dyadic},\ x\in Q^-} \frac{1}{|Q^+|}\int_{Q^+}|f(y)|dy,$$

where if $Q=Q_x(2h)$ is dyadic then Q^- is the cube $Q_x(h)$ and $Q^+ = Q^- + (h,h)$ (see the next figure and observe that Q^- and Q^+ are also dyadic cubes).



Note that

- this operator generalizes the operator M^+_d defined in $\mathbb{R}$ ($n=1$),
- it satisfies $M^{++}_d f \leq CM_d f$ and $M^{++}_d f\leq CM^{++}f$,
- but we have not been able to control the truncations of M^{++} by averages of $\tau_{-t}\circ M^{++}_d \circ \tau_t f$.

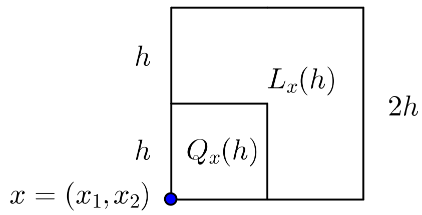
Nevertheless, we can consider different one-sided dyadic maximal operators. Before stating the definition, we remind that the one-sided Hardy-Littlewood maximal function in $\mathbb{R}^2$ is defined as

$$M^{++}f(x) = \sup_{0<h}\frac{1}{h^2}\int_{Q_x(h)}|f(y)|dy\,.$$

If we take the maximal operator

$$N^{++}f(x) = \sup_{0<h} \frac{1}{3h^2} \int_{L_x(h)} |f(y)|dy,$$

where $L_x(h) = Q_x(2h) \setminus Q_x(h)$ (see the next figure), we observe that N^+ is pointwise equivalent to M^{++}.

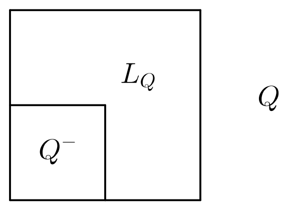


This fact suggests the following definition.

Definition 4.1. *For a locally integrable function f we define the following dyadic one-sided Hardy-Littlewood maximal operator.*

$$\widetilde{M}_d^{++} f(x) = \sup_{Q \text{ dyadic},\ x \in Q^-} \frac{1}{|L_Q|} \int_{L_Q} |f(y)|dy,$$

where $L_Q = Q \setminus Q^-$.



$$L_Q = Q \setminus Q^-$$

We point out the following properties.

(i) This operator generalizes the operator M_d^+ defined in $\mathbb{R}$ $(n = 1)$.

(ii) We are able to control the truncations of M^{++} by averages of $\tau_{-t} \circ \widetilde{M}_d^{++} \circ \tau_t f$. More precisely, if

$$N_k^{++} f(x) = \sup_{0<h\le 2^k} \frac{1}{3h^2} \int_{L_x(h)} |f(y)|dy$$

and

$$M_k^{++} f(x) = \sup_{0<h\le 2^k} \frac{1}{h^2} \int_{Q_x(h)} |f(y)|dy$$

then we have that

$$M_k^{++} f(x) \le C N_k^{++} f(x) \quad \le \frac{C}{(2^{k+4})^2} \int_{(0,2^{k+4}] \times (0,2^{k+4}]} (\tau_{-t} \circ \widetilde{M}_d^{++} \circ \tau_t) f(x)\, dt\,.$$

(iii) It satisfies $\widetilde{M}_d^{++} f \le C M_d f$ for some constant independent of f.

(iv) However, it has a serious downside: it does not satisfy $\widetilde{M}_d^{++} f \leq CM^{++} f$ with a constant independent of f.

Property (ii) implies that if we know the good weights for $\widetilde{M}_d^{++}$ we can obtain some good weights for M^{++}. We are able to characterize the weights for $\widetilde{M}_d^{++}$.

Theorem 4.2. *Let $1 < p < \infty$ and let u and v be two weights. The following are equivalent.*

(a) *There exists $C > 0$ such that for all $f \in L^p(v)$*

$$\int_{\mathbb{R}^2} (\widetilde{M}_d^{++} f(x))^p u(x)\, dx \leq C \int_{\mathbb{R}^2} |f(x)|^p v(x)\, dx.$$

(b) *There exists $C > 0$ such that*

$$\int_Q (\widetilde{M}_d^{++}(\sigma \chi_{L_Q})(x))^p u(x)\, dx \leq C \int_{L_Q} \sigma(x)\, dx < \infty$$

for all dyadic cubes Q with $\int_{Q^-} u > 0$.

Furthermore, if $u = v$ then the above conditions are equivalent to (c).

(c) *There exists $C > 0$ such that for all dyadic cubes Q,*

$$\frac{1}{|Q|} \left(\int_{Q^-} u \right)^{1/p} \left(\int_{L_Q} u^{1-p'} \right)^{1/p'} \leq C.$$

This result together with property (b) allows us to prove the following sufficient condition for the boundedness of M^{++}.

Theorem 4.3. *Let$1 < p < \infty$ and let u be a weight. If u satisfies $(A)_p^+$, that is,*

$$\sup_{x \in \mathbb{R}^2} \sup_{h > 0} \frac{1}{h^2} \left(\int_{Q_x(h)} u \right)^{1/p} \left(\int_{L_x(h)} u^{1-p'} \right)^{1/p'} < \infty$$

then there exists $C > 0$ such that for all $f \in L^p(u)$,

$$\int_{\mathbb{R}^2} (M^{++} f(x))^p u(x)\, dx \leq C \int_{\mathbb{R}^2} |f(x)|^p u(x)\, dx.$$

We shall say that u satisfies $(A)_1^+$ if there exists $C > 0$ such that for all cubes Q,

$$\frac{1}{|Q^-|} \int_{Q^-} u \leq Cu(x), \quad \text{a.e. } x \in L_Q.$$

It is very easy to see that $(A)_1^+ \subset (A)_p^+$.

Now, consider the maximal operator

$$\widetilde{M}^{--} f(x) = \sup_{Q \text{ cube}\,:\, x \in L_Q} \frac{1}{|Q^-|} \int_{Q^-} |f|$$

defined on locally integrable functions f. As usual, we are able to prove that

$$\widetilde{M}^{--} f < \infty \quad \text{a.e. } \; 0 < \delta < 1 \quad \Longrightarrow \quad (\widetilde{M}^{--} f)^\delta \in (A)_1^+.$$

This implication provides nontrivial examples of good weights for the operator M^{++}. The results of this section will appear elsewhere.

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