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BESOV REGULARITY OF SOLUTIONS OF THE FRACTIONAL LAPLACIAN

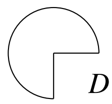
HUGO AIMAR, GASTÓN BELTRITTI, AND IVANA GÓMEZ

ABSTRACT. We review the scope of Dahlke and DeVore method for the analysis of Besov regularity improvement of solutions of PDEs. We sketch some new results concerning the case of the fractional Laplacian.

1. INTRODUCTION

The solution of the Dirichlet problem in the unit disc with a continuous boundary condition f is given by $u(re^{i\theta}) = (P_r * f)(\theta) = \int_0^{2\pi} P_r(\theta-\varphi)f(\varphi)d\theta$, where P_r denotes the Poisson kernel $P_r(\theta) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2}$, $0<r<1$, and $\theta \in (0,2\pi)$. The function $u(re^{i\theta})$ extended as $f(\theta)$ where $r=1$ is continuous in the closed disc $\{|z|\leq 1\}$. In the above, the smoothness of the boundary of the domain does not seem to play a relevant role. Nevertheless as the following example shows, not even the boundedness of u can be expected to be true if the domain has an inner vertex. In fact, the unbounded function $u(r,\theta) = r^{-\frac{2}{3}}\sin\frac{2\theta}{3}$ solves the Dirichlet problem in the domain $D = \{(r\cos\theta, r\sin\theta) : 0<\theta<\frac{3\pi}{2}, 0<r<1\}$ with continuous Dirichlet data

$$f(\theta) = \begin{cases} \sin\frac{2\theta}{3}, & \text{when } 0<\theta<\frac{3\pi}{2} \\ 0, & \text{on the straight line segments of } \partial D. \end{cases}$$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When dealing with diffusions, as usual, an improvement of regularity when time grows could be expected. This is quantitatively true but generally not in a qualitative sense. To be precise, if D is as before and $\Omega = D\times\mathbb{R}^+$, the function $u(r,\theta;t) = J_{-\frac{2}{3}}(r)\sin\frac{2\theta}{3}e^{-t}$, with $J_{-\frac{2}{3}}$ the singular Bessel function, solves $\frac{\partial u}{\partial t} = \Delta u$ in Ω, $u_{|\partial D} = f$ for every $t>0$ and $u(r,\theta;0) = J_{-\frac{2}{3}}(r)\sin\frac{2\theta}{3}$. The order of the singularity, $r^{-\frac{2}{3}}$, in the initial condition close to the origin remains unchanged along the whole process. The parabolic expected smoothing effect only reflects in the factor e^{-t} which reduces the size of the singularity at $r=0$ but not its order.

The above considerations are set to wit for the fact that harmonic functions and temperatures in Lipschitz domains can be far from being smooth in classical senses on and through the boundary of the domain.

Why Besov?

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As it is described in [DD], from the point of view of nonlinear approximation of solutions, Besov becomes a very suitable form of regularity which allows to measure the rate of convergence. Briefly, for $\{\psi_\lambda : \lambda \in \Lambda\}$ a wavelet basis, consider the nonlinear manifold $\mathscr{M}_n = \{S = \sum_{\lambda\in\Lambda} a_\lambda \psi_\lambda : \#(\Lambda) \leq n\}$. The error of approximation of f in the L^p norm by elements in $\mathscr{M}_n$, given by $\sigma_n(f) = \inf_{S\in\mathscr{M}_n} \|f - S\|_p$ satisfies that $\sum_{n=1}^{\infty} \left[n^{\frac{\alpha}{d}} \sigma_n(f)_p \right]^\tau \frac{1}{n}$ is finite if and only if the function f belongs to the Besov space B^α_τ, with $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$ (see [DJP]). Hence the rate of convergence to zero of $\sigma_n(f)$ when n tends to infinity improves when α increases.

So that when we are dealing with a certain family of functions f, the knowledge of as high regularity as possible in the Besov scale measured in terms of the regularity exponent α, would provide faster convergence of nonlinear approximation.

The classical theories of regularity of solutions to PDEs of elliptic and parabolic type, seem to give some hope to the idea that if the family of functions f is the family of solutions of those PDEs, then some Besov regularity improvement can be expected.

That this is actually the case is the main result in [DD], where wavelet techniques are used in an essential and nontrivial way. The main results are contained in the next two statements.

Theorem 1 ([DD])**.** *If D is a Lipschitz bounded domain in $\mathbb{R}^d$, $1 < p < \infty$, $\lambda > 0$, $0 < \alpha < \frac{\lambda d}{d-1}$, and $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$, then*

$$\mathscr{H}(D) \cap B^\lambda_p(D) \subset B^\alpha_\tau(D),$$

where $\mathscr{H}(D)$ is the space of all harmonic functions defined on D.

Theorem 2 ([JK] + Theorem 1)**.** *Let D be a Lipschitz bounded domain in $\mathbb{R}^d$, $1 < p < \infty$, $s > 0$, a function g in $B^s_p(\partial D)$, and u a solution of*

$$\begin{cases} \Delta u = 0, & \textit{in } D \\ u = g, & \textit{in } \partial D. \end{cases}$$

Then $u \in B^\alpha_\tau(D)$, where $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$, and $0 < \alpha < (s + \frac{1}{p})\frac{d}{d-1}$.

2. THE CASE OF TEMPERATURES

The improvement of Besov regularity for harmonic functions on Lipschitz domains obtained by Dahlke and DeVore can be seen as the steady state of a more general result concerning the regularity improvement of temperatures on cylinders based on Lipschitz domains. The theory is developed in [AG1, AG2, AGI2, AGI1].

The next statements gather together the main results of the saga. We shall write Ω to denote the parabolic cylinder $D \times (0,T)$ with $T > 0$ and D a bounded Lipschitz domain in $\mathbb{R}^d$ ($d \geq 2$). Let $\Theta(\Omega)$ denote the space of all temperatures $u = u(x,t)$ in Ω. In other words $\Theta(\Omega) := \{u : \frac{\partial u}{\partial t} = \Delta u \text{ in } \Omega\}$.

Theorem 3. *Let $1 < p < \infty$, $\lambda > 0$, ℓ the largest integer less than $\lambda + d$, $0 < \alpha < \min\{\ell, \frac{\lambda d}{d-1}\}$, and $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$. Then*

$$\Theta(\Omega) \cap L_p((0,T); B^\lambda_p(D)) \subset L_\tau((0,T); B^\alpha_\tau(D)).$$

Moreover, if u is a temperature in Ω we have that

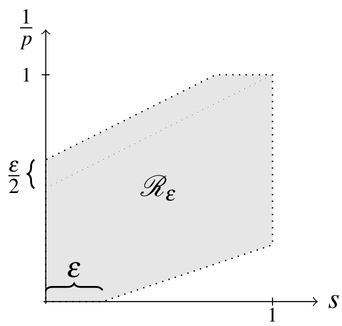
$$\|u\|_{L_\tau((0,T);B^\alpha_\tau(D))} \leq C \|u\|_{L_p((0,T);B^\lambda_p(D))}$$

for some constant C depending on Ω, d, p and λ.

Theorem 4. *Let D be a bounded and Lipschitz domain contained in $\mathbb{R}^d$ and let $T > 0$ be given. Let $\Omega = D \times (0,T)$ be the associated parabolic domain. Then there exists a positive number $\varepsilon \leq 1$ depending only on D such that a solution of the initial-boundary value problem*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, & \text{in } \Omega \\ u(x,t) = f(x), & \text{for } (x,t) \in \partial D \times (0,T) \\ u(x,0) = g(x), & \text{for } x \in D \end{cases}$$

belongs to the parabolic Besov space $\mathbb{B}^{\alpha}_{\tau}(\Omega)$ with $0 < \alpha < \min\{d\frac{p-1}{p}, (s+\frac{1}{p})\frac{d}{d-1}\}$ and $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$, provided that $f \in B^s_p(\partial D)$ and $g \in B_p^{s+\frac{1}{p}}(D)$ for each p and each s with $(s, \frac{1}{p}) \in \mathscr{R}_{\varepsilon}$, given by



$$\frac{1-\varepsilon}{2} < \frac{1}{p} < \frac{1+\varepsilon}{2} \quad \text{and} \quad 0 < s < 1;$$
$$\frac{1+\varepsilon}{2} \leq \frac{1}{p} < 1 \quad \text{and} \quad \frac{2}{p} - 1 - \varepsilon < s < 1;$$
$$0 < \frac{1}{p} \leq \frac{1-\varepsilon}{2} \quad \text{and} \quad 0 < s < \frac{2}{p} + \varepsilon.$$

3. THE CASE OF SOLUTIONS OF THE FRACTIONAL LAPLACIAN

From the Fourier analysis point of view the nonlocal differential operator of order 2σ ($0 < \sigma < 1$), $(-\triangle)^{\sigma}$ is defined as $((-\triangle)^{\sigma} f)^{\wedge}(\xi) = |\xi|^{2\sigma} \widehat{f}(\xi)$, $\xi \in \mathbb{R}^d$. Since this operator is of convolution type with a distribution with support on the whole space $\mathbb{R}^d$, a small and localized perturbation $\widetilde{f}$ of f at any region of the space is registered everywhere by $(-\triangle)^{\sigma}\widetilde{f}$. It is in this sense that the nonlocality of $(-\triangle)^{\sigma}$ can be realized. Hence for D a Lipschitz domain in $\mathbb{R}^d$ the fact that u is a solution of $(-\triangle)^{\sigma} u = 0$ on D has a completely different sense when $0 < \sigma < 1$ than when $\sigma = 1$. The most relevant recent result regarding this theory and its applications is provided by the Dirichlet to Neumann point of view introduced by Caffarelli and Silvestre in [CS]. In the case $\sigma = 1$ the solutions are the harmonic functions in D and this property depends only on the values of u in D. In fact, harmonic functions are those that satisfy a mean value formula for small balls contained in D. Nevertheless, the above mentioned nonlocal properties of $(-\triangle)^{\sigma}$ when $0 < \sigma < 1$ show that a small perturbation $\widetilde{u}$ of u outside D could give that $(-\triangle)^{\sigma}\widetilde{u} \neq 0$ on D. Since mean value formulas are a central tool in Dahlke and DeVore theory and its extensions, those nonlocal behaviours of solutions seem to indicate that no mean value for solutions of $(-\triangle)^{\sigma} u = 0$

can be true. Except, perhaps, if such mean value formulas would themselves be nonlocal. It turns out that this is precisely the case.

The main results in this direction are the following two statements. Their proofs can be found in [ABG].

Theorem 5. *Let $0 < \sigma < 1$ be given. Assume that D is an open set in $\mathbb{R}^d$ on which $(-\triangle)^\sigma f = 0$. Then for every $x \in D$ and every $0 < r < \operatorname{dist}(x, \partial D)$ we have that $f(x) = (\Phi_r * f)(x)$, where $\Phi_r(x) = r^{-d}\Phi\left(\frac{x}{r}\right)$, $\Phi(x) = \int_{y\in\mathbb{R}}\int_{z\in\mathbb{R}^d} \varphi(z,-y)P^a_{|y|}(x-z)|y|^a \; dzdy$, $\varphi_r(x,y) = r^{-(d+1+a)}\varphi\left(\frac{x}{r},\frac{y}{r}\right)$, φ is a $C^\infty(\mathbb{R}^{d+1})$ radial function supported in the unit ball of $\mathbb{R}^{d+1}$ with $\iint_{\mathbb{R}^{d+1}} \varphi(x,y)|y|^a \; dxdy = 1$, and P^a_y is a constant times $y^{1-a}\left(|x|^2+y^2\right)^{-\frac{d+1-a}{2}}$ where $a = 1-2\sigma$.*

Theorem 6. *Let D be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $0 < \sigma < 1$, $1 < p < \infty$, and $0 < \lambda < \frac{d-1}{d}$ be given. Assume that $f \in B^\lambda_p(\mathbb{R}^d)$ and that $(-\triangle)^\sigma f = 0$ on D, then $f \in B^\alpha_\tau(D)$ with $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$ and $0 < \alpha < \lambda\frac{d}{d-1}$.*

Aside from the mean value property in Theorem 5, the analytical tools involved in the proof of Theorem 6 are the characterization of Besov spaces through wavelets ([Mey]), Poincaré inequalities, Calderón maximal functions and Besov regularity ([Cal, DS]).

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INSTITUTO DE MATEMÁTICA APLICADA DEL LITORAL (IMAL), CCT CONICET SANTA FE, COLECTORA RUTA NAC. 168 KM. 472, PARAJE EL POZO, 3000 SANTA FE, ARGENTINA

E-mail, H. Aimar: haimar@santafe-conicet.gov.ar

E-mail, I. Gómez: ivanagomez@santafe-conicet.gov.ar

E-mail, G. Beltritti: gbeltritti@santafe-conicet.gov.ar