ACTAS DEL XI CONGRESO
DR. ANTONIO A. R. MONTEIRO (2011)
2012, Páginas 129–147

EMBEDDING THE UNITARY DUAL OF $GL(n,\mathbb{C})$ INTO STANDARD BEILINSON–BERNSTEIN MODULES

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ABSTRACT. In 1986 David Vogan classified the irreducible unitary representations for a general linear group defined over an Archimedean field. In this manuscript we focus on the case of the complex general linear group and show how to embed the irreducible unitary representations constructed by Vogan into certain standard Beilinson–Bernstein modules. In the generic case our embedding realizes Vogan's classification of the unitary dual in terms of the Beilinson–Bernstein classification for irreducible admissible representations.

1. INTRODUCTION

By definition, the unitary dual of a group is the set of equivalence classes of irreducible unitary representations. A fundamental unsolved problem in the theory of reductive Lie groups is how to classify the unitary dual. However, specific classifications of the unitary dual are known for certain families of reductive Lie groups. For example the unitary dual of the general linear group $GL(n,\mathbb{F})$ where $\mathbb{F}$ is the real numbers, the complex numbers or the quaternions, was classified by Vogan in [12].

For a compact group, it is known that an irreducible representation is finite dimensional and carries an invariant inner product. Suppose G_0 is a reductive Lie group and $K_0 \subseteq G_0$ is a maximal compact subgroup. A classical result of Harish-Chandra [5] shows that the irreducible K_0-submodules in an irreducible unitary representation for G_0 have finite multiplicities. This naturally leads to the following concept: a representation for G_0 is called *admissible* if the multiplicities of all irreducible K_0-subrepresentations are finite. Harish-Chandra's work shows that the unitary dual of G_0 is naturally included in the set of equivalence classes of irreducible admissible representations.

Classifying the irreducible admissible representations has turned out to be much more tractable than the problem of the unitary dual. The first classification, based on the work of Harish-Chandra, was proposed by R. Langlands in the mid 1970s [8]. By now there are several different classifications of irreducible admissible representations, including a geometric version due to A. Beilinson and J. Bernstein [1], that uses a theory of homogeneous sheaves of twisted differential operators (TDOs) on the flag variety of the associated complex Lie algebra.

Typically, the constructions of irreducible unitary representations used in known examples of the unitary dual differ from the constructions of standard modules used in the classification of irreducible admissible representations. Thus it can be an interesting exercise to see how a given classification for the unitary dual relates to the classification of irreducible admissible representations. In her Master's thesis, C. Carreras de Dargoltz investigated the relation between Vogan's classification for the unitary dual of $GL(n,\mathbb{C})$ and the Beilinson–Bernstein classification of irreducible admissible representations. In particular, she showed how to embed the irreducible unitary representations constructed by Vogan for $GL(n,\mathbb{C})$, into certain standard modules used in the Beilinson–Bernstein classification. In the generic case, her result realizes the irreducible unitary modules defined by Vogan as unique irreducible submodules associated to specific classifying modules. However, the problems that can occur are twofold: first of all it can occur that the associated parameter we define isn't antidominant, and secondly, even if this parameter is antidominant, in the singular case it can occur that the standard module isn't a classifying module. The purpose of this manuscript is to state and prove the embedding result established in the thesis. In a future manuscript we will demonstrate the full result which we will frame in the more general context of a problem about how to relate certain unitary representations realized by the cohomological parabolic induction to the Beilinson–Berstein classification.

We now review a few general facts about representations for a reductive Lie group G_0 of Harish-Chandra class [6] with maximal compact subgroup K_0. Let $\mathfrak{g}_0$ denote the Lie algebra of G_0 and let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$. We can define a (complex) representation as a continuous linear action of G_0 in a complete locally convex (complex) topological vector space V. The representation is called *irreducible* if $V \neq \{0\}$ and if the only closed invariant subspaces are $\{0\}$ and V. The representation is said to be *unitary* when V is a Hilbert space and the Lie group acts by unitary (that is: norm preserving) operators. Two unitary representations are said to be *equivalent* if there is an equivariant isometry from one representation onto the other. By definition, *the unitary dual* is the set of equivalence classes of irreducible unitary representations. For a compact Lie group, one knows that the irreducible representations are finite-dimensional and that every finite-dimensional representation has an inner product in which the group acts by unitary operators.

A vector in a G_0-representation is called K_0*-finite* if the span of the corresponding K_0-orbit is finite-dimensional. The G_0-representation V is called *admissible* if the multiplicity of each irreducible K_0-representation in V is finite. As mentioned above, Harish-Chandra has shown than an irreducible unitary representation is admissible. In general, the set of K_0-finite vectors $V_{K_0} \subseteq V$ forms a dense K_0-invariant subspace but when V is admissible then every K_0-finite vector is differentiable (a vector $v \in V$ is called *differentiable* if

$$\lim_{t\to 0} \frac{\exp(t\xi)\cdot v - v}{t}$$

exists for each $\xi \in \mathfrak{g}_0$). It follows that the K_0-module V_{K_0} carries a compatible $\mathfrak{g}$-action obtained via complexifying the differentiation. The resulting $(\mathfrak{g}, K_0)$-module V_{K_0} is called *the Harish-Chandra module of* V. One can prove, for example, that V is an irreducible representation if and only if V_{K_0} is an irreducible $(\mathfrak{g}, K_0)$-module. If V and W are admissible representations with corresponding Harish-Chandra modules V_{K_0} and W_{K_0} then *an infinitesimal morphism* from V to W is a $(\mathfrak{g}, K_0)$-equivariant linear map from V_{K_0} to W_{K_0}. Two admissible representations are called *infinitesimally equivalent* (or just: *equivalent*) if there exists an isomorphism of their underlying Harish-Chandra modules. A fundamental result of Harish-Chandra [5] says that if two irreducible unitary representations have isomorphic Harish-Chandra modules then they are equivalent unitary representations. Thus the unitary dual of G_0 embeds naturally into the set of equivalence classes of irreducible admissible representations.

In this study we will sketch Vogan's classification of the unitary dual for $GL(n,\mathbb{C})$, review the Beilinson–Bernstein classification of irreducible admissible representations, and then relate these two classifications, utilizing the duality theorem of Hecht, Miličić, Schmid and Wolf [7].

2. Vogan's classification

2.1. **Lowest K_0-type.** In this section we consider Vogan's classification for the unitary dual of $G_0 = GL(n,\mathbb{C})$. Details are taken from his article [12]. Let $K_0 = U(n)$ be the maximal compact subgroup of unitary matrices. The classification hinges on the Cartan–Weyl parametrization of the unitary dual for K_0, so we begin with that. Let $\mathfrak{g}_0$ denote the Lie algebra of $GL(n,\mathbb{C})$ and let $\mathfrak{k}_0$ denote the Lie algebra of K_0. Thus $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus i\mathfrak{k}_0$. Later on in this article, we will need to take care to distinguish the complexification of $\mathfrak{k}_0$ from $\mathfrak{g}_0$, but for the time being we ignore this. A finite-dimensional representation of G_0 is called *holomorphic* if the derivative is complex linear on $\mathfrak{g}_0$. There is a natural correspondence, given by extension and restriction, between the irreducible representations of K_0 and the irreducible (finite-dimensional) holomorphic representations of G_0 and we make this identification in what follows.

By definition, *a Borel subgroup* of $GL(n,\mathbb{C})$ is a maximal connected solvable subgroup. One knows every Borel subgroup of $GL(n,\mathbb{C})$ is conjugate to the subgroup B_0 of invertible upper triangular matrices. Let T_0 be the maximal torus of diagonal matrices in K_0. The group of characters of T_0 is naturally identified with the additive group $\mathbb{Z}^n$ of n-tuples of integers. In particular, put $\mathbb{C}^* = GL(1,\mathbb{C})$ and let $(m_1,\ldots,m_n) \in \mathbb{Z}^n$. Then the corresponding character $\chi : T_0 \to \mathbb{C}^*$ is defined by

$$\chi \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & z_n \end{pmatrix} = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}.$$

We write $\chi = (m_1,\ldots,m_n) \in \mathbb{Z}^n$. This character (or weight) is called *dominant* (with respect to the Borel subgroup B_0) if

$$m_1 \geq m_2 \geq \cdots \geq m_n.$$

There is a natural correspondence, given by extension and restriction, between the characters of T_0 and the holomorphic characters of B_0. Now suppose V is an irreducible representation for K_0. Then one knows that there is a unique character of B_0 which appears in the corresponding holomorphic G_0-action on V and that this character is dominant. The character is called *the highest weight in V*. On the other hand, one can prove each dominant weight appears as the highest weight in exactly one irreducible K_0-module. Thus the Cartan–Weyl theory parametrizes the equivalence classes of irreducible K_0-modules with dominant weights. We parametrize the unitary dual $\widehat{K_0}$ of K_0 via the dominant weights.

Vogan uses the Cartan–Weyl classification to define an order on $\widehat{K_0}$ in the following way. Consider the natural form

$$\langle \bullet, \bullet \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$$

defined on the group of characters of T_0 given by

$$\langle (j_1, j_2, \ldots, j_n), (k_1, k_2, \ldots, k_n) \rangle = j_1k_1 + j_2k_2 + \cdots + j_nk_n.$$

Let $\mathfrak{b}_0$ be the Lie algebra of the Borel subgroup B_0 and put

$$\mathfrak{sl}(n,\mathbb{C}) = [\mathfrak{g}_0, \mathfrak{g}_0].$$

Then the adjoint action of K_0 in $\mathfrak{sl}(n,\mathbb{C})$ defines an irreducible representation of K_0. Let 2ρ denote the highest weight of that representation. Thus 2ρ is the sum of the weights for the adjoint action of T_0 in $\mathfrak{b}_0$. According to our identifications

$$2\rho = (n-1, n-3, n-5, \ldots, 1-n).$$

For $\mu \in \widehat{K_0}$ we define

$$\|\mu\| = \langle \mu + 2\rho, \mu + 2\rho \rangle .$$

Now suppose V is an irreducible unitary representation for G_0 and let V_{K_0} be the underlying Harish-Chandra module. Since V is admissible, as a K_0-module

$$V_{K_0} \cong \bigoplus_{\mu \in \widehat{K_0}} m(\mu) V_\mu ,$$

where V_μ is a realization of the irreducible representation with highest weight μ and $m(\mu)$ is a nonnegative integer characterizing the multiplicity of μ in V_{K_0}. We say μ is *a K_0-type of V* if $m(\mu) \neq 0$. A K_0-type μ of V is called *minimal* if

$$\|\mu\| \leq \|\lambda\| \quad \text{for every other } K_0\text{-type } \lambda \text{ of } V.$$

Vogan proves the following.

Theorem 2.1. *Suppose V is an irreducible unitary representation for $GL(n,\mathbb{C})$. Then V has a unique minimal K_0-type and that minimal K_0-type has multiplicity one in V.*

The unique minimal K_0-type of an irreducible unitary representation for $GL(n,\mathbb{C})$ will be called *the lowest K_0-type.*

2.2. **The almost spherical representations of** $GL(n,\mathbb{C})$**.** According to the previous theorem, the unitary dual of $GL(n,\mathbb{C})$ is partitioned by lowest K_0-types. Vogan's parametrization of the irreducible unitary representations of $GL(n,\mathbb{C})$ with lowest K_0-type $\mu = (m_1, \ldots, m_n) \in \widehat{K_0}$ is framed in terms of a classification for a certain type of irreducible unitary representation, called *an almost spherical representation*, which is associated to a specific reductive subgroup, whose definition depends on μ. In case μ has the form

$$\mu = (m, m, \ldots, m)$$

then the lowest K_0-type is a character, the associated reductive subgroup is G_0 and we are looking at Vogan's classification of the irreducible almost spherical representations of $GL(n,\mathbb{C})$. In this section we consider that classification.

By definition, *a parabolic subgroup* of $GL(n,\mathbb{C})$ is a subgroup that contains a Borel subgroup. We can characterize the parabolic subgroups that contain the Borel subgroup B_0 in the following way. We define *a partition of n* to be a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that

$$n_1 + n_2 + \cdots + n_k = n.$$

Each partition defines a reductive subgroup

$$L_0 = GL(n_1,\mathbb{C}) \times GL(n_2,\mathbb{C}) \times \cdots \times GL(n_k,\mathbb{C})$$

embedded along the diagonal. The subgroup P_0 generated by L_0 and B_0 is the associated parabolic subgroup and there is a bijection between the partitions of n and the parabolic subgroups of G_0 that contain B_0. The subgroup L_0 is a maximal reductive subgroup of P_0 and will be called *the associated Levi subgroup.*

We parametrize the unitary characters of $G_0 = GL(n,\mathbb{C})$ in the following way. For $(t,m) \in \mathbb{R} \times \mathbb{Z}$ define $\chi : GL(n,\mathbb{C}) \to \mathbb{C}^*$ by

$$\chi(A) = |\det(A)|^{it} \left(\frac{\det(A)}{|\det(A)|} \right)^m .$$

We call χ *a unitary character of type $m \in \mathbb{Z}$*. Observe that when $A \in U(n)$ then $|\det(A)| = 1$ so there is a family of unitary characters of G_0 which restrict to a given unitary character of K_0.

We define *a spherical representation of* $G_0 = GL(n,\mathbb{C})$ to be a unitary representation for which the trivial K_0-type has non-zero multiplicity (for example: the trivial representation of G_0). A unitary representation V will be called *almost spherical of type* $m \in \mathbb{Z}$ if there exists a unitary character χ of G_0 of type m such that

$$\chi^{-1} \otimes V \text{ is spherical.}$$

In the case of $GL(n,\mathbb{C})$, the concept of an almost spherical is equivalent to the condition that there is a K_0-type in the representation which is a character, but we emphasize that for a general reductive group that isn't the case. At any rate, we now consider Vogan's construction of the irreducible almost spherical representations for $GL(n,\mathbb{C})$. We begin with a partition

$$n_1 + n_2 + \cdots + n_k = n.$$

To avoid some repetitions in the construction [12, Theorem 3.8], we can assume *the partition is decreasing*, that is:

$$n_1 \geq n_2 \geq \cdots \geq n_k.$$

This defines a Levi subgroup

$$L_0 = GL(n_1,\mathbb{C}) \times GL(n_2,\mathbb{C}) \times \cdots \times GL(n_k,\mathbb{C})$$

and a parabolic subgroup P_0, as defined before. We define *an almost spherical character* χ *of type* m for L_0 to be one that can be written as a product

$$\chi = \chi_1 \cdot \chi_2 \cdot \cdots \cdot \chi_k,$$

where

$$\chi_j : GL(n_j,\mathbb{C}) \to \mathbb{C}^*$$

is a unitary character of type m, or else when $n_j = n_{j+1}$ we include the possibility that

$$\chi_j(A) = |\det(A)|^{2t}\,\gamma(A) \quad \text{ and } \quad \chi_{j+1}(B) = |\det(B)|^{-2t}\,\gamma(B)$$

for $0 < t < \frac{1}{2}$, and where

$$\gamma : GL(n_j,\mathbb{C}) \to \mathbb{C}^*$$

is a unitary character of type m. This last possibility corresponds to what is known as *the Stein complementary series* (in particular, the characters χ_j and χ_{j+1} are not unitary). We remark that that character χ uniquely extends to the parabolic subgroup P_0 so we also speak of *an almost spherical character* χ *of type* m for P_0. Vogan uses these characters and the classical normalized parabolic induction to produce the irreducible almost spherical representations of G_0. First we review the classical parabolic induction and then we give a precise statement of Vogan's result.

The normalized parabolic induction incorporates a character that depends on the associated parabolic subgroup P_0. In particular let $\mathfrak{p}_0$ be the Lie algebra of P_0. Then *the nilradical* $\mathfrak{u}_0$ *of* $\mathfrak{p}_0$ is the largest solvable ideal in $[\mathfrak{p}_0,\mathfrak{p}_0]$. It consists of the matrices in $\mathfrak{p}_0$ whose entries are zero in all the blocks defining L_0. The ideal $\mathfrak{u}_0$ is invariant under conjugation by matrices from P_0 and this defines a holomorphic representation

$$\pi : P_0 \to GL(\mathfrak{u}_0)$$

where $GL(\mathfrak{u}_0)$ is the general complex linear group of $\mathfrak{u}_0$. We define *the corresponding half-density character* by

$$\delta(A) = \sqrt{|\det \pi(A)|}, \qquad \text{for } A \in P_0.$$

Again we emphasize we are referring to the determinant of the complex linear transformation $\pi(A)$. We define *the normalized parabolic induction* $I_{p_0}^{G_0}(\chi)$ to be the space of continuous functions

$$f : G_0 \to \mathbb{C} \text{ such that } f(gp) = \chi^{-1}(p)\delta^{-1}(p)f(g), \text{ for } g \in G_0 \text{ and } p \in P_0.$$

Since $G_0 = K_0 \cdot P_0$, the sup-norm on K_0 makes $I(\chi)$ into a Banach representation. When χ is a special spherical character then the Haar integral over K_0 determines a G_0-invariant inner product and the resulting representation is infinitesimally equivalent to an almost spherical representation. These unitary representations are called *the induced almost spherical representations*. Vogan's result is the following.

Theorem 2.2. *Maintain the previous notations.*

(a) *The only isomorphisms among the induced almost spherical representations come from permutations of blocks.*
(b) *The induced almost spherical representations are irreducible.*
(c) *Any irreducible almost spherical representation is equivalent to an induced almost spherical representation.*

2.3. **The parametrization of the unitary dual of** $GL(n,\mathbb{C})$**.** Given $\mu \in \widehat{K_0}$ we now want to describe how Vogan parametrizes the irreducible unitary representations of G_0 with lowest K_0-type μ. Suppose $\mu = (m_1, \ldots, m_n)$. Then we associate a partition of n (and thus a Levi subgroup) to μ in the following way. Let n_1 be the length of the first constant subsequence of the sequence $(m_1, \ldots, m_n)$, n_2 the length of the second constant subsequence, and so on. This defines a partition

$$n = n_1 + n_2 + \cdots + n_k.$$

Thus we have an associated Levi subgroup

$$L_0 = GL(n_1,\mathbb{C}) \times GL(n_2,\mathbb{C}) \times \cdots \times GL(n_k,\mathbb{C}).$$

For example, if $n = 7$ and $\mu = (6,6,5,0,0,0,-3)$ then $n_1 = 2$, $n_2 = 1$, $n_3 = 3$ and $n_4 = 1$. Therefore

$$L_0 = GL(2,\mathbb{C}) \times GL(1,\mathbb{C}) \times GL(3,\mathbb{C}) \times GL(1,\mathbb{C}).$$

What Vogan's classification does is relate the family of irreducible almost spherical representations for L_0 of type μ to the irreducible unitary representations of G_0 with lowest K_0-type μ. In particular, the irreducible almost spherical representations for L_0 of type μ parametrize the irreducible unitary representations of G_0 with lowest K_0-type μ. We now explicitly consider these parameters.

Suppose $n_1 + n_2 + \cdots + n_k = n$ is the partition associated to μ and let L_0 be the corresponding Levi subgroup. As before $B_0 \subseteq G_0$ is the Borel subgroup of upper triangular invertible matrices. Then $B_0 \cap L_0$ is a Borel subgroup of L_0 and the parabolic subgroups of L_0 that contain $B_0 \cap L_0$ are exactly the subgroups of the form $P_0 \cap L_0$ where P_0 is a parabolic subgroup of G_0 that contains B_0. We let D_0 be a parabolic subgroup of L_0 that contains $B_0 \cap L_0$. In a natural way, D_0 induces a parabolic subgroup on each sub-block $GL(n_j,\mathbb{C})$ of L_0 and we write this associated subgroup as $D_0 \cap GL(n_j,\mathbb{C})$. To avoid some repetitions, we can assume that D_0 is such that $D_0 \cap GL(n_j,\mathbb{C})$ is associated to decreasing partitions on each sub-block $GL(n_j,\mathbb{C})$ of L_0. In order to define a special spherical character χ of D_0 of type μ observe that we can think of the sequence $\mu = (m_1, \ldots, m_n)$ as being constant on the corresponding block n_j (this is essentially how we define the partition) and we let q_j denote this integer value. Then we define *an almost spherical character* χ *of* D_0 *of type* $\mu = (m_1, \ldots, m_n)$ to be one that can be written as a product

$$\chi = \chi_1 \cdot \chi_2 \cdot \cdots \cdot \chi_k$$

where each

$$\chi_j : D_0 \cap GL(n_j,\mathbb{C}) \to \mathbb{C}^*$$

is an almost spherical character of type q_j for the parabolic subgroup $D_0 \cap GL(n_j,\mathbb{C})$ of $GL(n_j,\mathbb{C})$. Indeed, the integer values q_j define a character for the maximal compact subgroup

$$U_0 = U(n_1) \times U(n_2) \times \cdots \times U(n_k) \text{ of } L_0$$

and thus we can speak of *the almost spherical representations of* L_0 *of type* μ. In particular, inducing parabolically

$$I_{D_0}^{L_0}(\chi)$$

from an almost spherical character χ of type μ produces an almost spherical representation of L_0 of type μ. These are *the induced almost spherical representations for* L_0.

Vogan's result (generalized from the case of each block) says that these induced almost spherical representations are irreducible and that every irreducible almost spherical representation of L_0 of type μ is isomorphic to one produced in this way. We also know that the only isomorphisms that occur arise from permutations of blocks.

Using this classification of irreducible almost spherical representations, Vogan parametrizes the irreducible unitary representations of $GL(n,\mathbb{C})$ with lowest K_0-type μ as follows.

Theorem 2.3. *Let $\mu \in \widehat{K_0}$ and suppose L_0 is the associated Levi subgroup. Then there exists a one-to-one correspondence between the equivalence classes of irreducible almost spherical representations of type μ for L_0 and the equivalence classes of irreducible unitary representations of G_0 with lowest K_0-type μ.*

When μ has the form $\mu = (m,m,\ldots,m)$ then $L_0 = G_0$ and the representations parametrized are just the irreducible almost spherical for G_0 of type m from the previous subsection. When $L_0 \neq G_0$ then Vogan uses a certain cohomological induction functor to produce irreducible unitary representations for G_0 from the irreducible almost spherical representations for L_0. In the following subsection we consider this functor.

2.4. **Cohomological parabolic induction.** By definition, *a Borel subalgebra* of a complex Lie algebra $\mathfrak{g}$ is a maximal solvable subalgebra and *a parabolic subalgebra* is a subalgebra that contains a Borel subalgebra. In order to define the cohomological parabolic induction we will be interested in the following sort of parabolic subalgebras of $\mathfrak{g}$. Suppose $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{u}$. *A Levi factor* $\mathfrak{l}$ *of* $\mathfrak{q}$ is a complementary subalgebra to $\mathfrak{u}$ in $\mathfrak{q}$ (a Levi factor of a Borel subalgebra is also called *a Cartan subalgebra of* $\mathfrak{g}$). Suppose $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$. We say $\mathfrak{q}$ is *nice* if $\mathfrak{q} \cap \mathfrak{g}_0 = \mathfrak{l}_0$ is the real form of a Levi factor $\mathfrak{l}$ of $\mathfrak{q}$.

Suppose $\mathfrak{g}_0$ is the Lie algebra of $GL(n,\mathbb{C})$ and $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$. Let $\mathfrak{l}_0$ be the Lie algebra of an associated Levi subgroup L_0 as defined in the previous section. We now consider a construction of a nice parabolic subalgebra $\mathfrak{q}$ such that $\mathfrak{q} \cap \mathfrak{g}_0 = \mathfrak{l}_0$. Since $\mathfrak{g}_0$ is a complex Lie algebra (being treated as a real Lie algebra) we can make the following realization of the complexified Lie algebra $\mathfrak{g}$. Introduce the real Lie algebra $\mathfrak{g}_0 \times \mathfrak{g}_0$ and define the multiplication by $i \in \mathbb{C}$ as

$$i \cdot (\xi_1, \xi_2) = (i\xi_1, -i\xi_2).$$

Then the map

$$\mathfrak{g}_0 \to \mathfrak{g}_0 \times \mathfrak{g}_0 \text{ by } \xi \mapsto (\xi, \xi)$$

determines an isomorphism of the complexification $\mathfrak{g}$ with $\mathfrak{g}_0 \times \mathfrak{g}_0$. According to this identification, the Borel subalgebras of $\mathfrak{g}$ are the products of Borel subalgebras of $\mathfrak{g}_0$ and the parabolic subalgebras of $\mathfrak{g}$ are the products of parabolic subalgebras of $\mathfrak{g}_0$. Returning to the previous notations, let $\mathfrak{b}_0 \subseteq \mathfrak{g}_0$ be the Borel subalgebra of upper triangular matrices, let $\mathfrak{l}_0 \subseteq \mathfrak{g}_0$ be a Levi subalgebra of block matrices defined by a partition of n and let $\mathfrak{p}_0 \subseteq \mathfrak{g}_0$ be the corresponding parabolic subalgebra. We define the opposite parabolic subalgebra $\mathfrak{p}_0^{\mathrm{op}} \subseteq \mathfrak{g}_0$ to be the parabolic subalgebra generated by $\mathfrak{l}_0$ and the Borel subalgebra of lower triangular matrices. Then

$$\mathfrak{q} = \mathfrak{p}_0 \times \mathfrak{p}_0^{\mathrm{op}}$$

is a nice parabolic subalgebra and one knows every nice parabolic subalgebra of $\mathfrak{g}$ is G_0-conjugate to one constructed in this way. Given a Levi subgroup L_0 associated to the partition defined by $\mu = (m_1, \ldots, m_n) \in \widehat{K_0}$ then $\mathfrak{q}$ will be called *the corresponding nice parabolic subalgebra.*

The cohomological parabolic induction that Vogan uses is a derived functor construction that produces Harish-Chandra modules for G_0 from Harish-Chandra modules for L_0 (the corresponding maximal compact subgroup for L_0 is $K_0 \cap L_0$, where K_0 is the group of unitary matrices in G_0). In order to give the exact definition we need to introduce a certain associated character of L_0. Let $\mathfrak{u}$ denote the nilradical of $\mathfrak{q}$, let $\mathfrak{u}_0$ denote the nilradical of $\mathfrak{p}_0$, and let $\mathfrak{u}_0^{\mathrm{op}}$ denote the nilradical of $\mathfrak{p}_0^{\mathrm{op}}$. We should emphasize that

$$\mathfrak{u} = \mathfrak{u}_0 \times \mathfrak{u}_0^{\mathrm{op}}$$

is not the complexification of $\mathfrak{u}_0$ in this case, however the notation is convenient because we are running out of letters to use. Let $\mathfrak{h}_0 \subseteq \mathfrak{l}_0$ be the Cartan subalgebra of diagonal matrices and let $\mathfrak{h} \subseteq \mathfrak{g}$ be the complexification of $\mathfrak{h}_0$. Let $\rho(\mathfrak{u})$ denote one half the sum of the roots of $\mathfrak{h}$ in $\mathfrak{u}$. Thus $\rho(\mathfrak{u})$ extends uniquely to a morphism of Lie algebras

$$\rho(\mathfrak{u}) : \mathfrak{l} \to \mathbb{C}.$$

For $g \in L_0$ we define

$$\chi_{2\rho(\mathfrak{u}_0)}(g) = \det(\mathrm{Ad}(g)\mid_{\mathfrak{u}_0}),$$

where we are considering the complex determinant of the holomorphic action. Then

$$\chi_{2\rho(\mathfrak{u}_0^{\mathrm{op}})} = \chi^{-1}_{2\rho(\mathfrak{u}_0)},$$

where $\mathfrak{u}_0^{\mathrm{op}}$ is the nilradical of $\mathfrak{p}_0^{\mathrm{op}}$. Letting $\mathfrak{u}$ denote the nilradical of $\mathfrak{q}$, it follows that the character

$$\chi_{2\rho(\mathfrak{u})}(g) = \det(\mathrm{Ad}(g)\mid_{\mathfrak{u}}) = \left(\frac{\chi_{2\rho(\mathfrak{u}_0)}(g)}{\left|\chi_{2\rho(\mathfrak{u}_0)}(g)\right|}\right)^2$$

for $g \in L_0$, has a well-defined square root. Thus there is a unique continuous character $\chi_{\rho(\mathfrak{u})}$ of L_0 whose complexified derivative is $\rho(\mathfrak{u})$.

Suppose V is a Harish-Chandra module for $(\mathfrak{l}, K_0 \cap L_0)$ and let $\mathfrak{q}^{\mathrm{op}}$ denote the parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{q}$ with respect to $\mathfrak{l}$. Thus

$$\mathfrak{q}^{\mathrm{op}} = \mathfrak{l} \oplus \mathfrak{u}^{\mathrm{op}},$$

where $\mathfrak{u}^{\mathrm{op}} = \mathfrak{u}_0^{\mathrm{op}} \times \mathfrak{u}_0$ is the nilradical of $\mathfrak{q}^{\mathrm{op}}$. We define the following normalized $(\mathfrak{g}, K_0 \cap L_0)$-module

$$\mathrm{M}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q}^{\mathrm{op}})} \left(V \otimes \chi_{\rho(\mathfrak{u})}\right)$$

where $V \otimes \chi_{\rho(\mathfrak{u})}$ is a $\mathfrak{q}^{\mathrm{op}}$-module by trivial extension, $\mathfrak{g}$ acts by left multiplication on $\mathrm{M}(V)$ and $K_0 \cap L_0$ acts by the tensor product of the adjoint action on $U(\mathfrak{g})$ with the given action from $V \otimes \chi_{\rho(\mathfrak{u})}$. The cohomological parabolic induction functors used by Vogan are defined by

$$\Lambda^p(V) = \Gamma^p(\mathrm{M}(V))$$

where Γ^p is the p-th derived functor of the Zuckerman functor from the category of $(\mathfrak{g}, K_0 \cap L_0)$-modules to the category of $(\mathfrak{g}, K_0)$-modules.

Let $\mathfrak{k} \subseteq \mathfrak{g}$ be the complexification of the Lie algebra $\mathfrak{k}_0$ of K_0 and let s denote the dimension of $\mathfrak{k} \cap \mathfrak{u}$. Suppose $\mu \in \widehat{K_0}$, let L_0 be the associated Levi subgroup and let $\mathfrak{q}$ denote the corresponding very nice parabolic subgroup. Then Vogan shows that the cohomological parabolic induction in degree s defines an equivalence between the irreducible almost spherical representations for L_0 of type μ and the irreducible unitary representations for G_0 with lowest K_0-type μ. In particular, we have the following.

Theorem 2.4. *Suppose $\mu = (m_1, \dots, m_n) \in \widehat{K_0}$. Let L_0 be the associated Levi subgroup and let $\mathfrak{q}$ be the corresponding very nice parabolic subalgebra.*

(a) *If V is the Harish-Chandra module of an irreducible almost spherical representation for L_0 of type μ then $\Lambda^s(V)$ is the Harish-Chandra module of an irreducible unitary representation of lowest K_0-type $\mathfrak{u}$.*

(b) *If M is the Harish-Chandra module of an irreducible unitary representation of lowest K_0-type $\mathfrak{u}$ then there exists a unique irreducible almost spherical representation V for L_0 of type μ such that $\Lambda^s(V) \cong M$.*

3. The Beilinson–Bernstein classification

3.1. **Sheaves of twisted differential operators.** The Beilinson–Bernstein classification is built on a theory of sheaves of modules for twisted sheaves of differential operators (TDOs) on the flag variety of a complex reductive Lie algebra, so we begin with that. Most of what we need is spelled out (probably more clearly) in [9].

Since the complex adjoint group $\mathrm{Int}(\mathfrak{g})$ of $\mathfrak{g}$ acts transitively on the set of Borel subalgebras of $\mathfrak{g}$, the resulting homogeneous space X is a smooth algebraic variety, called *the full flag variety of* $\mathfrak{g}$. The TDOs are parametrized by the elements of the dual of a Cartan subalgebra of $\mathfrak{g}$. But in order to make this natural, it is convenient to introduce the abstract Cartan dual. In particular, for $x \in X$ we let $\mathfrak{b}_x$ denote the corresponding Borel subalgebra and $\mathfrak{n}_x$ the nilradical of $\mathfrak{b}_x$. Put

$$\mathfrak{h}_x = \mathfrak{b}_x / \mathfrak{n}_x.$$

If $g_1, g_2 \in \mathrm{Int}(\mathfrak{g})$ are two elements that send $x \in X$ to y then the corresponding isomorphisms

$$g_1 : \mathfrak{h}_x \to \mathfrak{h}_y \quad \text{and} \quad g_2 : \mathfrak{h}_x \to \mathfrak{h}_y$$

are identical. In particular, the stabilizer of x in $\mathrm{Int}(\mathfrak{g})$ acts trivially on $\mathfrak{h}_x$. This allows us to identify the dual spaces $\mathfrak{h}_x^*$ for $x \in X$, in a canonical way. In particular, we define *the abstract Cartan dual* $\mathfrak{h}_{\mathrm{ab}}^*$ to be the space of functions

$$\sigma : X \to \bigcup_{x \in X} \mathfrak{h}_x^* \text{ such that } \sigma(x) \in \mathfrak{h}_x^* \text{ and } \sigma(g \cdot x) = g \cdot \sigma(x).$$

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in a Borel subalgebra $\mathfrak{b}_x$ then the natural projection

$$\mathfrak{h} \to \mathfrak{b}_x / \mathfrak{n}_x$$

coupled with the evaluation at x determines an isomorphism

$$\mathfrak{h}_{\mathrm{ab}}^* \cong \mathfrak{h}^*,$$

called the evaluation of $\mathfrak{h}_{\mathrm{ab}}^*$ in $\mathfrak{h}^*$ at x. This gives us *a well-defined set of roots* $\Delta_{\mathrm{ab}} \subseteq \mathfrak{h}_{\mathrm{ab}}^*$ and *a subset of positive roots* $\Delta_{\mathrm{ab}}^+ \subseteq \Sigma_{\mathrm{ab}}$ corresponding to the roots of $\mathfrak{h}$ in $\mathfrak{b}_x$. Following the development in [9], for each $\lambda \in \mathfrak{h}_{\mathrm{ab}}^*$ we define a corresponding sheaf of TDOs

$$\mathscr{D}_\lambda$$

on the algebraic variety X. As a point of reference, letting $\rho \in \mathfrak{h}^*_{\mathrm{ab}}$ denote one half the sum of the positive roots, we note that

$$\mathscr{D}_{-\rho}$$

is the sheaf of differential operators (with regular coefficients) on X.

Let $Z(\mathfrak{g})$ denote the center of the enveloping $U(\mathfrak{g})$. By definition, *an infinitesimal character* is a homomorphism of algebras

$$\Theta : Z(\mathfrak{g}) \to \mathbb{C}.$$

This is an important invariant associated to an irreducible admissible representation, since one knows $Z(\mathfrak{g})$ acts on an irreducible Harish-Chandra module via an infinitesimal character. If Θ denotes an infinitesimal character we let U_Θ denote the algebra defined as the quotient of $U(\mathfrak{g})$ by the ideal in $U(\mathfrak{g})$ generated by the kernel of Θ. Let W_{ab} denote the Weyl group of $\mathfrak{h}^*_{\mathrm{ab}}$ associated to the root system Δ_{ab}. Then *the Harish-Chandra map* determines a bijection between the set of infinitesimal characters and the orbits of W_{ab} in $\mathfrak{h}^*_{\mathrm{ab}}$. If the $\lambda \in \mathfrak{h}^*_{\mathrm{ab}}$ and the W_{ab}-orbit of λ corresponds to Θ via the Harish-Chandra map we write

$$\Theta = W_{\mathrm{ab}} \cdot \lambda \quad \text{and} \quad \lambda \in \Theta.$$

For $\lambda \in \Theta$ Beilinson and Bernstein prove that

$$\Gamma(X, \mathscr{D}_\lambda) \cong U_\Theta.$$

3.2. **The standard Harish-Chandra modules.** Now suppose G_0 is a reductive Lie group of Harish-Chandra class. To simplify things a bit, we also assume that G_0 is the real form of a connected complex reductive group G (this means, for example, the Cartan subgroups of G_0 are abelian). In particular, when $G_0 = GL(n,\mathbb{C})$, we can identify G_0 with the diagonal in $G = G_0 \times G_0$, where G has the complex Lie algebra $\mathfrak{g}_0 \times \mathfrak{g}_0$ defined previously. We let $K_0 \subseteq G_0$ be a maximal compact subgroup and $K \subseteq G$ the complexification of K_0. The Beilinson–Bernstein classification for irreducible admissible representations classifies the irreducible Harish-Chandra sheaves for $(\mathscr{D}_\lambda, K)$ and relates them to the irreducible Harish-Chandra modules with infinitesimal character $\Theta = W_{\mathrm{ab}} \cdot \lambda$. We now recall a few points about the classification of the irreducible Harish-Chandra sheaves for $(\mathscr{D}_\lambda, K)$.

Fix $x \in X$ and let K_x denote the stabilizer of x. Our assumptions on G_0 imply that an irreducible algebraic representation of K_x is a holomorphic character

$$\chi : K_x \to \mathbb{C}^*.$$

Let $\mathfrak{t}_x$ denote the Lie algebra of K_x and let $\mathfrak{b}_x$ be the Borel subalgebra of $\mathfrak{g}$ corresponding to x. Then the evaluation at x determines a natural correspondence between $\mathfrak{h}^*_{\mathrm{ab}}$ and the homomorphisms of Lie algebras from $\mathfrak{b}_x$ into $\mathbb{C}$. For $\lambda \in \mathfrak{h}^*_{\mathrm{ab}}$ we let

$$\lambda_x : \mathfrak{b}_x \to \mathbb{C}$$

the corresponding homomorphism of Lie algebras. We say a holomorphic character $\chi : K_x \to \mathbb{C}^*$ *is compatible with* λ if

$$d\chi = \lambda_x + \rho_x \,|_{\mathfrak{t}_x},$$

where $d\chi$ is the derivative of χ and ρ_x is the evaluation of ρ at x.

We now consider the construction of the standard Harish-Chandra sheaves for $(\mathscr{D}_\lambda, K)$. Suppose Q is a K-orbit on X. For simplicity of notation we assume each K-orbit Q contains a distinguished point $x \in Q$. Then a holomorphic character $\chi : K_x \to \mathbb{C}^*$ determines a K-homogeneous algebraic line bundle $\mathscr{L}_\chi$ defined over Q. Let $\overline{Q}$ denote the Zariski closure of Q in X. Put

$$\partial Q = \overline{Q} - Q \quad \text{and} \quad U = X - \partial Q.$$

Thus U is Zariski open in X and Q is Zariski closed in U. Let $\mathscr{D}_\chi^Q$ denote the sheaf of differential operators of $\mathscr{L}_\chi$ on Q and suppose χ is compatible with $\lambda \in \mathfrak{h}_{\mathrm{ab}}^*$. Let $\mathscr{D}_\lambda \mid_U$ denote the restriction of the TDO $\mathscr{D}_\lambda$ to U. Then the theory of $\mathscr{D}$-modules defines an exact functor i_+ called *the direct image for $\mathscr{D}$-modules* that relates the category of Harish-Chandra sheaves for $\left(\mathscr{D}_\chi^Q, K\right)$ to the category of Harish-Chandra sheaves for $(\mathscr{D}_\lambda \mid_U, K)$. The direct image in the category of sheaves defines a functor j_* from the category of Harish-Chandra sheaves for $(\mathscr{D}_\lambda \mid_U, K)$ to the category of Harish-Chandra sheaves for $(\mathscr{D}_\lambda, K)$. The object

$$\mathscr{I}_\lambda(Q,\chi) = j_* i_+(\mathscr{L}_\chi)$$

is called *the standard Harish-Chandra sheaf* with the given parameters. One knows that $\mathscr{I}_\lambda(Q,\chi)$ contains a unique irreducible Harish-Chandra subsheaf $\mathscr{J}_\lambda(Q,\chi)$. One can show that every irreducible Harish-Chandra sheaf for $\left(\mathscr{D}_\chi, K\right)$ is isomorphic to one of the form $\mathscr{J}_\lambda(Q,\chi)$ and that

$$\mathscr{J}_\lambda(Q_1,\chi_1) \cong \mathscr{J}_\lambda(Q_2,\chi_2) \Leftrightarrow Q_1 = Q_2 \quad \text{and} \quad \chi_1 = \chi_2.$$

It will be convenient for us to write

$$I_\lambda(Q,\chi) = \Gamma(X, \mathscr{I}_\lambda(Q,\chi))$$

and call this *the standard Harish-Chandra module* with the given parameters. However, the translation of the classification of the irreducible Harish-Chandra sheaves into a classification of irreducible Harish-Chandra modules is a bit subtle, and we need to make some more definitions first. An element $\lambda \in \mathfrak{h}_{\mathrm{ab}}^*$ is called *antidominant* if

$$\overset{\vee}{\alpha}(\lambda) \notin \{1,2,3,4,\ldots\}$$

for each $\alpha \in \Delta_{\mathrm{ab}}^+$, where $\overset{\vee}{\alpha}$ denotes the dual root to α. We note that each Weyl group orbit in $\mathfrak{h}_{\mathrm{ab}}^*$ contains at least one antidominant element. An infinitesimal character $\Theta = W_{\mathrm{ab}} \cdot \lambda$ is called *regular* if the corresponding orbit $W_{\mathrm{ab}} \cdot \lambda$ has the same number of elements as W_{ab}. This is equivalent to the condition that

$$\overset{\vee}{\alpha}(\lambda) \neq 0 \quad \text{for each root } \alpha.$$

An infinitesimal character is called *singular* when it is not regular.

For the irreducible Harish-Chandra modules with regular infinitesimal character we have the following [9].

Theorem 3.1. *Suppose Θ is a regular infinitesimal character and choose $\lambda \in \Theta$ antidominant. Then the Harish-Chandra modules of the form*

$$J_\lambda(Q,\chi) = \Gamma(X, \mathscr{J}_\lambda(Q,\chi))$$

are irreducible and these representations (for λ fixed) parametrize the irreducible admissible representations with infinitesimal character Θ.

For Θ regular and $\lambda \in \Theta$ antidominant the standard Harish-Chandra module

$$I_\lambda(Q,\chi) = \Gamma(X, \mathscr{I}_\lambda(Q,\chi))$$

will be called *the classifying module* with the given parameters. When $I_\lambda(Q,\chi)$ is a classifying module, one knows that $J_\lambda(Q,\chi)$ is the unique irreducible submodule of $I_\lambda(Q,\chi)$.

For the irreducible modules with a singular infinitesimal character, the parametrization is a bit trickier. In particular, for Θ singular and $\lambda \in \Theta$ antidominant, we call

$$I_\lambda(Q,\chi) = \Gamma(X, \mathscr{I}_\lambda(Q,\chi))$$

a classifying module if

$$J_\lambda(Q,\chi) = \Gamma(X, \mathscr{J}_\lambda(Q,\chi)) \text{ is not zero.}$$

When $I_\lambda(Q,\chi)$ is a classifying module one knows that $J_\lambda(Q,\chi)$ is the unique irreducible submodule of $I_\lambda(Q,\chi)$. In general we have the following.

Theorem 3.2. *Suppose Θ is an infinitesimal character and choose $\lambda \in \Theta$ antidominant. Then the irreducible Harish-Chandra modules with infinitesimal character Θ are parametrized by the classifying modules $I_\lambda(Q,\chi)$. In particular, the irreducible Harish-Chandra modules with infinitesimal character Θ are the submodules $J_\lambda(Q,\chi) \subseteq I_\lambda(Q,\chi)$.*

We finish this section with a few special facts about how the Beilinson–Bernstein classification applies to a connected complex reductive Lie group G_0 with compact real form $K_0 \subseteq G_0$. Let $\mathfrak{b}_0 \subseteq \mathfrak{g}_0$ be a Borel subalgebra and let $\mathfrak{h}_0 \subseteq \mathfrak{b}_0$ be a Cartan subalgebra. We assume $\mathfrak{h}_0$ is the Cartan subalgebra of $\mathfrak{b}_0$ normalized by a maximal torus T_0 in K_0. Let Δ_0^+ denote the set of positive roots of $\mathfrak{h}_0$ in $\mathfrak{b}_0$ and let W_0 denote the Weyl group of $\mathfrak{h}_0$ in $\mathfrak{g}_0$. For each $w \in W_0$ we define a set of positive roots $w\Delta_0^+$ and a Borel subalgebra of $\mathfrak{g}_0$

$$w\mathfrak{b}_0 = \mathfrak{h}_0 + \sum_{\alpha\in\Delta_0^+} (\mathfrak{g}_0)^{w\alpha},$$

where $(\mathfrak{g}_0)^{w\alpha}$ represents the corresponding root subspace. Then we can parametrize the K orbits in X in the following way. As before, we identify $\mathfrak{g}$ with $\mathfrak{g}_0 \times \mathfrak{g}_0$. Thus $\mathfrak{g}_0$ is identified with the diagonal in $\mathfrak{g}_0 \times \mathfrak{g}_0$ and G_0 acts on $\mathfrak{g}$ by the formula

$$g\cdot(\xi_1,\xi_2) = (\mathrm{Ad}(g)\xi_1, \mathrm{Ad}(g)\xi_2).$$

Put

$$\mathfrak{b} = \mathfrak{b}_0 \times \mathfrak{b}_0$$

and for each $w \in W_0$ we define a Borel subalgebra $\mathfrak{b}_w$ of $\mathfrak{g}$ by

$$\mathfrak{b}_w = w\mathfrak{b}_0 \times \mathfrak{b}_0.$$

Then the Borel subalgebras $\mathfrak{b}_w$ for $w \in W_0$ naturally parametrize both the K-orbits and the G_0-orbits on X. Choose a point $\mathfrak{b}_w \in X$ and suppose

$$\chi : H_0 \to \mathbb{C}^*$$

is a continuous character. Let $d\chi \in \mathfrak{h}^*$ be the complexification of the differential of χ and let K_w denote the normalizer of $\mathfrak{b}_w$ in K. Then there is a unique holomorphic character of K_w whose derivative is compatible with $d\chi$. We write

$$\lambda_w = d\chi + \rho_w,$$

where ρ_w is one half the sum of the roots of $\mathfrak{h}$ in $\mathfrak{b}_w$. Identify λ_w with $\lambda \in \mathfrak{h}^*_{\mathrm{ab}}$ by the evaluation to the point $\mathfrak{b}_w$. We write $\mathscr{I}(Q_w,\chi)$ to indicate the corresponding standard Harish-Chandra sheaf on X (this notation is unambiguous because in this case λ is uniquely determined by the character χ) and put

$$I(Q_w,\chi) = \Gamma(X, \mathscr{I}(Q_w,\chi))$$

to indicate the corresponding standard Harish-Chandra module.

3.3. **The unitary dual of $GL(n,\mathbb{C})$ and the Beilinson–Bernstein classification.** Because Vogan uses cohomological parabolic induction associated to parabolic subalgebras, we will use some facts about standard Harish-Chandra sheaves on a generalized flag space to relate the two classifications. Some details can be found in [3] and [2]. Suppose $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$. Then the corresponding generalized flag space Y is the quotient of the complex adjoint group $\mathrm{Int}(\mathfrak{g})$ by the normalizer of $\mathfrak{q}$. The points in Y can be identified with the parabolic subalgebras of $\mathfrak{g}$ that are $\mathrm{Int}(\mathfrak{g})$-conjugate to $\mathfrak{q}$. When $\mathfrak{q}$ is the corresponding nice parabolic subalgebras associated to the Levi subgroup defined by a partition $\mu = (m_1,\ldots,m_n) \in \widehat{K_0}$ then one knows that the K-orbit of $\mathfrak{q}$ in Y is closed.

Let L_0 be the associated Levi subgroup and let $\mathfrak{l}$ be the corresponding complexified Lie algebra. Then given an irreducible Harish-Chandra module V associated to an irreducible admissible representation of L_0, we can define a corresponding standard Harish-Chandra sheaf $\mathscr{I}(\mathfrak{q},V)$ on Y [2, Definition as in Section 6]. In the case we are studying, V is the Harish-Chandra module of a representation induced from a character of a real parabolic subgroup of L_0. Because of that, it is not difficult to relate $\mathscr{I}(\mathfrak{q},V)$ to a specific standard Harish-Chandra sheaf on X. We now describe how to do this.

There is a canonical equivariant projection

$$\pi : X \to Y$$

defined by associating a Borel subalgebra $\mathfrak{b}$ to the unique point in Y corresponding to the parabolic subalgebra that contains $\mathfrak{b}$. Let

$$X_{\mathfrak{q}} = \pi^{-1}(\{\mathfrak{q}\})$$

be the fiber in X over Y. Then, in a natural way, $X_{\mathfrak{q}}$ is identified with the flag variety for L_0. Indeed, if $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ contained in $\mathfrak{q}$ then $\mathfrak{b}\cap\mathfrak{l}$ is a Borel subalgebra of $\mathfrak{l}$. In general, there is a unique K-orbit Q in X such that $Q\cap X_{\mathfrak{q}}$ is nonempty and open in $X_{\mathfrak{q}}$. In our case the points in $Q\cap X_{\mathfrak{q}}$ correspond to the Borel subalgebras of $\mathfrak{l}_0$. We now consider specifically how to characterize this particular Q with respect to the parameters previously defined.

Let $\mathfrak{b}_0$ be the Borel subalgebra of upper triangular matrices in $\mathfrak{g}_0$. Recall that $\mu \in \widehat{K_0}$ defines an associated Levi factor L_0. Let $\mathfrak{l}_0$ be the Lie algebra of L_0 and let $\mathfrak{p}_0$ be the parabolic subalgebra with Levi factor $\mathfrak{l}_0$ that contains $\mathfrak{b}_0$. Put

$$\mathfrak{q}^{\mathrm{op}} = \mathfrak{p}_0^{\mathrm{op}} \times \mathfrak{p}_0.$$

Let $\mathfrak{n}_0$ be the nilradical $\mathfrak{b}_0$ and let $\mathfrak{h}_0$ denote the Cartan subalgebra of diagonal matrices. Then the roots of $\mathfrak{h}_0$ in $\mathfrak{n}_0\cap\mathfrak{l}_0$ determine a positive system of roots of $\mathfrak{h}_0$ in $\mathfrak{l}_0$. We define *the μ-related Borel subalgebra*

$$\mathfrak{b}_\mu = \left(\mathfrak{h}_0 \oplus \mathfrak{n}_0 \cap \mathfrak{l}_0 \oplus \mathfrak{u}_0^{\mathrm{op}}\right) \times \mathfrak{b}_0,$$

where $\mathfrak{u}_0^{\mathrm{op}}$ is the nilradical of $\mathfrak{p}_0^{\mathrm{op}}$. *The μ-related K-orbit Q_μ* is the K-orbit of $\mathfrak{b}_\mu$. Note that, according to the parametrization we gave in the previous section, the parameter $w \in W_0$ for Q_μ is the product of the longest element in W_0 with the longest element from the Weyl group for $\mathfrak{l}_0$.

The following proposition is actually valid for any connected complex reductive group G_0 and χ a continuous character, although we only need the result in the more restricted context. Using the above notations we let H_0, B_0, L_0, etc. denote the corresponding connected subgroups of G_0.

Proposition 3.3. *Let $\mu \in \widehat{K_0}$ and maintain the above notations. Suppose D_0 is a parabolic subgroup of L_0 that contains $L_0 \cap B_0$ and let*

$$\chi : D_0 \to \mathbb{C}^*$$

be a continuous character. By restriction, χ determines a character of H_0 and via the construction sketched in the previous section defines a standard Harish-Chandra sheaf $\mathscr{I}(Q_\mu, \chi)$ of $\mathscr{D}_\lambda$-modules defined on X. Let V be the Harish-Chandra module of the parabolically induced representation

$$I_{D_0}^{L_0}(\chi)$$

and let Y be the generalized flag space corresponding to the parabolic subalgebra $\mathfrak{q}^{op}$. Then we can define a standard Harish-Chandra sheaf $\mathscr{I}(\mathfrak{q}^{op}, V)$ on Y.

$$I(\mathfrak{q}^{op}, V) = \Gamma(Y, \mathscr{I}(\mathfrak{q}^{op}, V))$$

and let δ_{D_0} denote the half-density character of D_0. Then we have a natural inclusion of Harish-Chandra modules

$$I(\mathfrak{q}^{op}, V) \subseteq I(Q_\mu, \chi \otimes \delta_{D_0}).$$

Proof: Let $\delta_{B_0 \cap L_0}$ be the half-density character of $B_0 \cap L_0$ and let W be the Harish-Chandra module of

$$I_{B_0 \cap L_0}^{L_0}(\chi \otimes \delta_{B_0 \cap L_0}^{-1} \otimes \delta_{D_0}).$$

Then it follows immediately from the definition of the parabolic induction that we have a natural inclusion of Harish-Chandra modules $V \subseteq W$. This natural inclusion in turn defines a natural inclusion of Harish-Chandra sheaves on Y,

$$\mathscr{I}(\mathfrak{q}^{\mathrm{op}}, V) \subseteq \mathscr{I}(\mathfrak{q}^{\mathrm{op}}, W).$$

Taking global sections and putting $I(y, W) = \Gamma(Y, \mathscr{I}(y, W))$ we obtain a corresponding inclusion of Harish-Chandra modules,

$$I(\mathfrak{q}^{\mathrm{op}}, V) \subseteq I(\mathfrak{q}^{\mathrm{op}}, W),$$

for the pair $(\mathfrak{g}, K_0)$. It remains to prove that we have a natural isomorphism

$$I(Q_\mu, \chi \otimes \delta_{D_0}) \cong I(\mathfrak{q}^{\mathrm{op}}, W).$$

We claim that this result can be deduced by the induction in stages for standard Harish-Chandra sheaves [3, Theorems 4.14 and 5.4]. Indeed, letting $\pi : X \to Y$ denote the canonical projection and letting π_* denote the standard direct image in the category of sheaves one can deduce that

$$\pi_*(\mathscr{I}(Q_\mu, \chi \otimes \delta_{D_0})) \cong \mathscr{I}(\mathfrak{q}^{\mathrm{op}}, W),$$

since the orbit $Q \cap X_{\mathfrak{q}^{\mathrm{op}}}$ is affine and open. Thus the result follows. ■

The next step needed in order to relate the classification of Vogan to the classification of Beilinson–Bernstein is the duality theorem of Hecht, Miličić, Schmid and Wolf [7] as established for a generalized flag variety [4]. The form that is most convenient for us is the one from [10, Theorem 5.3]. To apply this formula we note that since $\mathfrak{q}$ and $\mathfrak{q}^{\mathrm{op}}$ are θ-stable parabolic subalgebras it follows that the dimension of the K-orbit of y is the value s of Theorem 2.4.1. Thus we have the following.

Theorem 3.4. *Maintain the notations from the previous proposition and Theorem 2.4.1. Then the duality theorem defines a natural isomorphism*

$$\Lambda^s(V) \cong I(\mathfrak{q}^{op}, V \otimes \chi_{\rho(\mathfrak{u}^{op})}).$$

In particular we have a natural inclusion of Harish-Chandra modules

$$\Lambda^s(V) \subseteq I(Q_\mu, \chi\otimes\delta_{D_0}\otimes\chi_{\rho(\mathfrak{u}^{op})}).$$

Proof: Recall the $(\mathfrak{g}, K_0\cap L_0)$-module

$$\mathrm{M}(V) = U(\mathfrak{g})\otimes_{U(\mathfrak{q}^{\mathrm{op}})}\left(V\otimes\chi_{\rho(\mathfrak{u})}\right).$$

Let Γ^p denote the p-th derived functor of the Zuckerman functor from the category of $(\mathfrak{g}, K_0\cap L_0)$-modules to the category of $(\mathfrak{g}, K_0)$-modules. Let N denote an irreducible Harish-Chandra module for L_0. Then the duality theorem provides a natural isomorphism

$$I(\mathfrak{q}^{\mathrm{op}}, N) \cong \Gamma^s\left(U(\mathfrak{g})\otimes_{U(\mathfrak{q}^{\mathrm{op}})}\left(N\otimes\chi_{2\rho(\mathfrak{u})}\right)\right) = \Gamma^s\left(\mathrm{M}(N\otimes\chi_{\rho(\mathfrak{u})})\right) = \Lambda^s\left(N\otimes\chi_{\rho(\mathfrak{u})}\right).$$

Putting $N = V\otimes\chi_{\rho(\mathfrak{u}^{\mathrm{op}})}$ and using the fact that $\chi_{\rho(\mathfrak{u}^{\mathrm{op}})} = \chi_{\rho(\mathfrak{u})}^{-1}$ gives the desired result. ■

The previous result is our preliminary embedding result, however it turns out that the value λ for the associated standard modules $I(Q_\mu, \chi\otimes\delta_{D_0}\otimes\chi_{\rho(\mathfrak{u}^{\mathrm{op}})})$ is not necessarily antidominant. The first counterexample we find is in the case of $GL(5,\mathbb{C})$ where $\mu\in\widehat{K_0}$ has the form $(m,m,m,m,m-1)$ and χ_μ is the unitary character of the associated Levi subgroup obtained by trivially extending the unitary character determined by μ from the maximal compact subgroup $U(4)\times U(1)$. What we show here has to do with what happens in the two extreme cases. On the one hand, in the case that $\mu = (m_1,\ldots,m_n)$ is a strictly decreasing sequence (i.e. when Q_μ is the closed K-orbit on X) then we can show the value λ for the associated standard module is antidominant and regular. Thus in this case Vogan's representation coincides with the associated standard Beilinson–Bernstein module (which is a classifying module). Also, in the other extreme case, when μ is the constant sequence (i.e. when Q_μ is the open K-orbit on X) then we can show the value λ for the associated standard module is antidominant. However, in the case of the open orbit, it turns out that $I(Q_\mu, \chi\otimes\delta_{D_0}\otimes\chi_{\rho(\mathfrak{u}^{\mathrm{op}})})$ is not a classifying module when λ is singular (consider the example below). We sum this up in the following theorem.

Theorem 3.5. *Suppose $\mu\in\widehat{K_0}$. Let L_0 be the associated Levi subgroup and let D_0 be a parabolic subgroup of L_0 that contains the Borel subgroup $B_0\cap L_0$ defined previously. Suppose*

$$\chi : D_0 \to \mathbb{C}^*$$

is a special spherical character of type μ. Let V be the Harish-Chandra module of $I_{D_0}^{L_0}(\chi)$ and let Q_μ denote the μ-related K-orbit in the full flag space X. We assume that Q_μ is either the open K-orbit or the closed K-orbit. Then the value λ associated to the standard Harish-Chandra module

$$I(Q_\mu, \chi\otimes\delta_{D_0}\otimes\chi_{\rho(\mathfrak{u}^{op})})$$

is antidominant. In particular, in the case of the closed orbit $I(Q_\mu, \chi\otimes\chi_{\rho(\mathfrak{u}^{op})})$ is a classifying module isomorphic to $\Lambda^s(V)$.

Proof: We want to calculate the numerical values of the positive dual roots on the corresponding parameter λ. Let Δ_0 denote the set of roots of $\mathfrak{h}_0$ in $\mathfrak{g}_0$. Each root $\alpha_0\in\Delta_0$ determines the following two roots of $\mathfrak{h}$ in $\mathfrak{g}$:

$$\alpha : \mathfrak{h}_0\times\mathfrak{h}_0 \to \mathbb{C} \quad \text{defined by} \quad \alpha(H_1,H_2) = \alpha_0(H_1)$$

and

$$\overline{\alpha} : \mathfrak{h}_0\times\mathfrak{h}_0 \to \mathbb{C} \quad \text{defined by} \quad \overline{\alpha}(H_1,H_2) = \overline{\alpha_0(H_2)}.$$

When

$$\lambda_0 : \mathfrak{h}_0 \to \mathbb{C}$$

is a real linear form and λ is the complexification then the values of the corresponding dual roots are given by

$$\overset{\vee}{\alpha}(\lambda) = \frac{1}{2}\left(\lambda_0\left(H_\alpha\right) - i\lambda_0\left(iH_\alpha\right)\right) \quad \text{and} \quad \overset{\vee}{\overline{\alpha}}(\lambda) = \frac{1}{2}\left(\lambda_0\left(H_\alpha\right) + i\lambda_0\left(iH_\alpha\right)\right),$$

where $H_\alpha \in \mathfrak{h}_0$ is the element associated to the dual root $\overset{\vee}{\alpha}_0$. In our case, the matrices H_α are of the form

$$E_{ii} - E_{jj},$$

where $i \neq j$ and the matrices E_{kk} give the standard basis vectors for the diagonal in $\mathfrak{gl}(n,\mathbb{C})$. Now suppose

$$\mu = (m_1, \ldots, m_n)$$

and let $\chi : H_0 \to \mathbb{C}$ be the character

$$\chi \begin{pmatrix} z_1 & 0 & \cdots \\ 0 & \ddots & 0 \\ \vdots & 0 & z_n \end{pmatrix} = \left(\frac{z_1}{|z_1|}\right)^{m_1} \cdots \left(\frac{z_n}{|z_n|}\right)^{m_n}.$$

Write $d\chi : \mathfrak{h} \to \mathbb{C}$ for the corresponding complexification of the derivative and let $\alpha_0 = e_i - e_j$, where $\{e_1, \ldots, e_n\}$ is the dual basis to $\{E_{11}, \ldots, E_{nn}\}$. Then

$$\overset{\vee}{\alpha}(d\chi) = \frac{m_i - m_j}{2} \quad \text{and} \quad \overset{\vee}{\overline{\alpha}}(d\chi) = \frac{m_j - m_i}{2}.$$

Consider the set of positive roots defined by the Borel subalgebra $\mathfrak{b}_\mu$. Using the previous notations, the nilradical of $\mathfrak{b}_\mu$ is

$$\left((\mathfrak{n}_0 \cap \mathfrak{l}_0) \oplus \mathfrak{u}_0^{\mathrm{op}}\right) \times \left((\mathfrak{n}_0 \cap \mathfrak{l}_0) \oplus \mathfrak{u}_0\right).$$

Let $\Delta(\mathfrak{u}_0)$ denote the roots of $\mathfrak{h}_0$ in $\mathfrak{u}_0$. These are some of the roots of the form $e_i - e_j$ where $i < j$. In fact, because of the definition of $\mathfrak{l}_0$ as square blocks where the sequence defined by μ is constant, it follows that for the roots

$$\{\alpha : \alpha_0 \in \Delta(\mathfrak{u}_0)\} \text{ that } \overset{\vee}{\overline{\alpha}}(d\chi) = \frac{m_j - m_i}{2} \leq -\frac{1}{2}.$$

On the other hand let $\Delta(\mathfrak{u}_0^{\mathrm{op}})$ denote the roots of $\mathfrak{h}_0$ in $\mathfrak{u}_0^{\mathrm{op}}$. These are some of roots of the form $e_i - e_j$ where $i > j$. In this case it follows that for the roots

$$\left\{\alpha : \alpha_0 \in \Delta(\mathfrak{u}_0^{\mathrm{op}})\right\} \text{ that } \overset{\vee}{\alpha}(d\chi) = \frac{m_i - m_j}{2} \leq -\frac{1}{2}.$$

Finally, once again because of the definition of $\mathfrak{l}_0$ as square blocks where the sequence defined by μ is constant one can check that

$$\overset{\vee}{\beta}(d\chi) = 0 \quad \text{if } \beta \in \Delta(\mathfrak{n} \cap \mathfrak{l}),$$

where

$$\mathfrak{n} \cap \mathfrak{l} = \mathfrak{n}_0 \cap \mathfrak{l}_0 \times \mathfrak{n}_0 \cap \mathfrak{l}_0$$

is the complexification of $\mathfrak{n}_0 \cap \mathfrak{l}_0$. Assume χ is a unitary character of the form

$$\chi \begin{pmatrix} z_1 & 0 & \cdots \\ 0 & \ddots & 0 \\ \vdots & 0 & z_n \end{pmatrix} = |z_1|^{it_1} \cdots |z_n|^{it_n},$$

with $t_k \in \mathbb{R}$. Then the values $\overset{\vee}{\alpha}(d\chi)$ and $\overset{\vee}{\overline{\alpha}}(d\chi)$ are pure imaginary numbers (possibly zero). We need to consider the effect from the Stein series. Suppose we have consecutive blocks in L_0 of the same size with characters χ_j and χ_{j+1} as defined earlier (corresponding to the Stein series). Suppose $0 < t < \frac{1}{2}$. Then, assuming that the basis vectors E_{ii} and E_{jj} occur in the specified blocks, the possible values of $\overset{\vee}{\alpha}(d\chi)$ and $\overset{\vee}{\overline{\alpha}}(d\chi)$ are $-2t$, $-t$, 0, t or $2t$. At any rate, it follows that if χ is a special spherical character of type μ then for a positive root β (a root of $\mathfrak{h}$ in $\mathfrak{b}_\mu$), the real part of $\overset{\vee}{\beta}(d\chi)$ is less than one-half.

So far we have shown that if χ is a special spherical character of type μ then for a positive root β (a root of $\mathfrak{h}$ in $\mathfrak{b}_\mu$), the real part of $\overset{\vee}{\beta}(d\chi)$ is less than one-half. It remains to consider the effect of the character $\delta_{D_0} \otimes \chi_{\rho(\mathfrak{u}^{\mathrm{op}})}$ and the shift by ρ_μ (i.e. the half sum of the positive roots corresponding to $\mathfrak{b}_\mu$). Let $\rho(\mathfrak{n}\cap\mathfrak{l})$ be the half sum of the roots of $\mathfrak{h}$ in $\mathfrak{n}\cap\mathfrak{l}$. Then, since $\mathfrak{u}^{\mathrm{op}} = \mathfrak{u}_0^{\mathrm{op}} \times \mathfrak{u}_0$, it follows that

$$\rho_\mu = \rho(\mathfrak{n}\cap\mathfrak{l}) + \rho(\mathfrak{u}^{\mathrm{op}}).$$

In particular consider the case of a unitary character $\chi : L_0 \to S^1$ of type μ, λ the associated parameter and $\beta \in \Delta(\mathfrak{n}_\mu)$; we have

$$\overset{\vee}{\beta}(\lambda) = \overset{\vee}{\beta}\left(d\chi + \rho(\mathfrak{u}^{\mathrm{op}}) - \rho_\mu\right) = \overset{\vee}{\beta}\left(d\chi - \rho(\mathfrak{n}\cap\mathfrak{l})\right).$$

We have seen that the real part of $\overset{\vee}{\beta}(d\chi)$ is non-positive. Indeed, in the case of the open orbit $\overset{\vee}{\beta}(d\chi)$ is negative and $\mathfrak{n}\cap\mathfrak{l} = \mathfrak{n}\cap\mathfrak{h} = 0$, which establishes the theorem for this case. On the other hand, for the general case and $\beta \in \Delta(\mathfrak{n}\cap\mathfrak{l})$ one knows that

$$\overset{\vee}{\beta}\left(-\rho(\mathfrak{n}\cap\mathfrak{l})\right) \leq -1,$$

and note that for the closed orbit $\mathfrak{n}\cap\mathfrak{l} = \mathfrak{n}\cap\mathfrak{g} = \mathfrak{n}_\mu$. Let $d\delta_{D_0}$ denote the complexified derivative of δ_{D_0}, where D_0 is a parabolic subgroup as above. Then for each root vector E_{ij} in the nilradical of the Lie algebra we have the factor

$$\begin{pmatrix} z_1 & 0 & \cdots \\ 0 & \ddots & 0 \\ \vdots & 0 & z_n \end{pmatrix} \mapsto |z_i|^{\frac{1}{2}} \left|z_j\right|^{-\frac{1}{2}}$$

appearing in the character $d\delta_{D_0}$. Using the formula above, it follows that

$$\overset{\vee}{\beta}(d\delta_{D_0}) \leq \frac{1}{2}.$$

Thus, putting together the results we have seen, if

$$\lambda = d\chi + d\delta_{D_0} + +\rho(\mathfrak{u}^{\mathrm{op}}) - \rho_\mu$$

then the real part $\overset{\vee}{\beta}(\lambda) < 1$ for $\beta \in \Delta(\mathfrak{n}\cap\mathfrak{l})$. This proves the antidominance result for the closed orbit. ■

Example: The previous theorem embeds a portion of the nonsingular part of the unitary dual of $GL(n,\mathbb{C})$ into the Beilinson–Bernstein classification of irreducible admissible representations. In the singular case, the standard module we define is not necessarily a classifying module. We give an example to illustrate this problem, which can already be observed for $G_0 = GL(2,\mathbb{C})$. Consider the K_0-type

$$\mu = (m,m).$$

Corresponding to this K_0-type we can consider the Stein series representation induced from the character

$$\chi\begin{pmatrix} z_1 & * \\ 0 & z_2 \end{pmatrix} = |z_1|^{\frac{1}{2}}\left(\frac{z_1}{|z_1|}\right)^m |z_2|^{-\frac{1}{2}}\left(\frac{z_2}{|z_2|}\right)^m.$$

Let $\rho \in \mathfrak{h}^*_{\mathrm{ab}}$ denote the half sum of the positive roots and put

$$\mathfrak{b} = \mathfrak{b}_0 \times \mathfrak{b}_0.$$

Using the notation from the previous theorem, in this case $L_0 = GL(2,\mathbb{C})$, $D_0 = B_0$ and $\mathfrak{u}^{\mathrm{op}} = 0$. Let K be the complexification of K_0. Then the K-orbit Q of $\mathfrak{b}$ is open and ρ determines a K-equivariant algebraic line bundle $\mathscr{O}(\rho)$ defined on X. We note that $\mathscr{O}(\rho)$ is a $\mathscr{D}_\lambda$-module for the parameter $\lambda = 0$ and an irreducible Harish-Chandra sheaf. Let $j: Q \to X$ be the inclusion. Then the standard Beilinson–Bernstein module that realizes the Stein series representation in question is given by the global sections of

$$\mathscr{I}(Q,\chi) \cong j_*\left(\mathscr{O}(\rho)\mid_Q\right),$$

where j_* is the standard direct image in the category of sheaves and $\mathscr{O}(\rho)\mid_Q$ is the restriction of $\mathscr{O}(\rho)$ to Q. There is a natural inclusion of sheaves

$$\mathscr{O}(\rho) \to j_*\left(\mathscr{O}(\rho)\mid_Q\right).$$

In particular $\mathscr{J}(Q,\chi) = \mathscr{O}(\rho)$. By the Borel–Weil–Bott Theorem,

$$\Gamma(X, \mathscr{J}(Q,\chi)) = J(Q,\chi) = 0,$$

so that $I(Q,\chi)$ is not a classifying module. To further illustrate, we note that it is easy to identify the correct classifying module (in this case there are only two K-orbits in X). In particular, let $\mathfrak{b}^{\mathrm{op}}$ denote the Borel subalgebra of $\mathfrak{g}$ defined by

$$\mathfrak{b}^{\mathrm{op}} = \mathfrak{b}_0^{\mathrm{op}} \times \mathfrak{b}_0,$$

where $\mathfrak{b}_0^{\mathrm{op}}$ is the Borel subalgebra of $\mathfrak{g}_0$ opposite to $\mathfrak{b}_0$, and let $\mathscr{F}$ denote the quotient of $\mathscr{I}(Q,\chi)$ by $\mathscr{O}(\rho)$. Then $\mathscr{F}$ is supported on the K-orbit S of $\mathfrak{b}^{\mathrm{op}}$ which is Zariski closed in X. It follows that $\mathscr{F}$ is both an irreducible Harish-Chandra sheaf and a standard Harish-Chandra sheaf. That is:

$$\mathscr{F} = \mathscr{J}(S,\chi(\rho)) = \mathscr{I}(S,\chi(\rho)),$$

where

$$\chi(\rho) : H_0 \times H_0 \to \mathbb{C}^*$$

is the corresponding holomorphic character. Applying the long exact sheaf cohomology sequence obtained from the short exact sequence

$$0 \to \mathscr{O}(\rho) \to \mathscr{I}(Q,\chi_{\rho(\mathfrak{b})}) \to \mathscr{I}(S,\chi_{\rho(\mathfrak{b}^{\mathrm{op}})}) \to 0,$$

and using the Borel–Weil–Bott Theorem to conclude that $H^1(X,\mathscr{O}(\rho)) = 0$, we obtain the isomorphism of Harish-Chandra modules,

$$I(Q,\chi) \cong I(S,\chi(\rho)).$$

Thus the correct standard Beilinson–Berstein module for the unitary principal series in question is $I(S,\chi(\rho))$. ■

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